13

Tensors and general relativity

13.1 Points and their coordinates

We use coordinates to label the physical points of a spacetime and the mathematical points of an abstract object. For example, we may label a point on a sphere by its latitude and longitude with respect to a polar axis and meridian. If we use a different axis and meridian, our coordinates for the point will change, but the point remains as it was. **Physical and mathematical points exist independently of the coordinates we use to talk about them. When we change our system of coordinates, we change our labels for the points, but the points remain as they were.**

At each point p, we can set up various coordinate systems that assign unique coordinates $x^i(p)$ and $x'^i(p)$ to p and to points near it. For instance, polar coordinates (θ, ϕ) are unique for all points on a sphere—except the north and south poles which are labeled by $\theta = 0$ and $\theta = \pi$ and all $0 \le \phi < 2\pi$. By using a second coordinate system with $\theta' = 0$ and $\theta' = \pi$ on the equator in the (θ, ϕ) system, we can assign unique coordinates to the north and south poles in that system. Embedding simplifies labeling. In a 3-dimensional euclidian space and in the 4-dimensional Minkowski spacetime in which the sphere is a surface, each point of the sphere has unique coordinates, (x, y, z) and (t, x, y, z).

We will use coordinate systems that represent the points of a space or spacetime uniquely and smoothly at least in local patches, so that the maps

$$\begin{aligned} x^i &= x^i(p) = x^i(p(x')) = x^i(x') \\ x'^i &= x'^i(p) = x'^i(p(x)) = x'^i(x) \end{aligned} \tag{13.1}$$

are well defined, differentiable, and one to one in the patches. We'll often group the n coordinates x^i together and write them collectively as x without superscripts. Since the coordinates $x(p)$ label the point p, we sometimes will call them "the point x." But p and x are different. The point p is unique with infinitely many coordinates x, x', x'', ... in infinitely many coordinate systems.

We begin this chapter by noticing carefully how things change as we change our coordinates. Our goal is to write physical theories so their equations look the same in all systems of coordinates as Einstein taught us.

13.2 Scalars

A **scalar** is a quantity B that is the same in all coordinate systems

$$B' = B. \tag{13.2}$$

If it also depends upon the coordinates of the spacetime point $p(x) = p(x')$, then it is a **scalar field**, and

$$B'(x') = B(x). \tag{13.3}$$

13.3 Contravariant vectors

By the chain rule, the change in dx'^i due to changes in the unprimed coordinates is

$$dx'^i = \sum_k \frac{\partial x'^i}{\partial x^k}\, dx^k. \tag{13.4}$$

This transformation defines **contravariant vectors**: a quantity A^i is a component of a contravariant vector if it transforms like dx^i

$$A'^i = \sum_k \frac{\partial x'^i}{\partial x^k}\, A^k. \tag{13.5}$$

The coordinate differentials dx^i form a contravariant vector. A contravariant vector $A^i(x)$ that depends on the coordinates is a **contravariant vector**

field and transforms as

$$A'^i(x') = \sum_k \frac{\partial x'^i}{\partial x^k} A^k(x). \tag{13.6}$$

13.4 Covariant vectors

The chain rule for partial derivatives

$$\frac{\partial}{\partial x'^i} = \sum_k \frac{\partial x^k}{\partial x'^i} \frac{\partial}{\partial x^k} \tag{13.7}$$

defines **covariant vectors**: a quantity C_i that transforms as

$$C'_i = \sum_k \frac{\partial x^k}{\partial x'^i} C_k \tag{13.8}$$

is a **covariant vector**. A covariant vector $C_i(x)$ that depends on the coordinates and transforms as

$$C'_i(x') = \sum_k \frac{\partial x^k}{\partial x'^i} C_k(x) \tag{13.9}$$

is a **covariant vector field**.

Example 13.1 (Gradient of a scalar) The derivatives of a scalar field $B'(x') = B(x)$ form a covariant vector field because

$$\frac{\partial B'(x')}{\partial x'^i} = \frac{\partial B(x)}{\partial x'^i} = \sum_k \frac{\partial x^k}{\partial x'^i} \frac{\partial B(x)}{\partial x^k}, \tag{13.10}$$

which shows that the gradient $\partial B(x)/\partial x^k$ fits the definition (13.9) of a covariant vector field. □

13.5 Tensors

Tensors are structures that transform like products of vectors. A rank-zero tensor is a scalar. A rank-one tensor is a covariant or contravariant vector. Second-rank tensors are distinguished by how they transform under changes of coordinates:

$$\begin{aligned} \text{covariant} \quad F'_{ij} &= \frac{\partial x^k}{\partial x'^i} \frac{\partial x^l}{\partial x'^j} F_{kl} \\ \text{contravariant} \quad M'^{ij} &= \frac{\partial x'^i}{\partial x^k} \frac{\partial x'^j}{\partial x^l} M^{kl} \\ \text{mixed} \quad N'^i_j &= \frac{\partial x'^i}{\partial x^k} \frac{\partial x^l}{\partial x'^j} N^k_l . \end{aligned} \tag{13.11}$$

We can define tensors of higher rank by extending these definitions to quantities with more indices. The rank of a tensor also is called its order and its degree.

If $S(x)$ is a scalar field, then its derivatives with respect to the coordinates are covariant vectors (13.10) and tensors

$$V_i = \frac{\partial S}{\partial x^i}, \quad T_{ik} = \frac{\partial^2 S}{\partial x^i \partial x^k}, \quad \text{and} \quad U_{ik\ell} = \frac{\partial^3 S}{\partial x^i \partial x^k \partial x^\ell}. \tag{13.12}$$

Example 13.2 (Rank-2 tensors) If A_k and B_ℓ are covariant vectors, and C^m and D^n are contravariant vectors, then the product $A_k\, B_\ell$ is a second-rank covariant tensor; $C^m\, D^n$ is a second-rank contravariant tensor; and $A_k\, C^m$, $A_k\, D^n$, $B_k\, C^m$, and $B_k\, D^n$ are second-rank mixed tensors. □

Since the transformation laws that define tensors are linear, any linear combination (with constant coefficients) of tensors of a given rank and kind is a tensor of that rank and kind. Thus if F_{ij} and G_{ij} are both second-rank covariant tensors, so is their sum $H_{ij} = F_{ij} + G_{ij}$.

13.6 Summation convention and contractions

An index that appears in the same monomial once as a covariant subscript and once as a contravariant superscript, is a dummy index that is summed over

$$A_i\, B^i \equiv \sum_{i=1}^{n} A_i\, B^i \tag{13.13}$$

usually from 0 to 3. Such a sum in which an index is repeated once covariantly and once contravariantly is a **contraction**. The **rank** of a tensor is the number of its uncontracted indices.

Although the product $A_k\, C^\ell$ is a mixed second-rank tensor, the contraction $A_k\, C^k$ is a scalar because

$$A'_k\, C'^k = \frac{\partial x^\ell}{\partial x'^k} \frac{\partial x'^k}{\partial x^m} A_\ell\, C^m = \frac{\partial x^\ell}{\partial x^m} A_\ell\, C^m = \delta^\ell_m A_\ell\, C^m = A_\ell\, C^\ell. \tag{13.14}$$

Similarly, the doubly contracted product $F^{ik}F_{ik}$ is a scalar.

Example 13.3 (Kronecker delta) The summation convention and the chain rule imply that

$$\frac{\partial x'^i}{\partial x^k}\frac{\partial x^k}{\partial x'^\ell} = \frac{\partial x'^i}{\partial x'^\ell} = \delta^i_\ell = \begin{cases} 1 & \text{if } i = \ell \\ 0 & \text{if } i \neq \ell. \end{cases} \tag{13.15}$$

The repeated index k has disappeared in this contraction. The **Kronecker delta** δ^i_j is a mixed second-rank tensor; it transforms as

$$\delta'^i_j = \frac{\partial x'^i}{\partial x^k}\frac{\partial x^l}{\partial x'^j}\,\delta^k_l = \frac{\partial x'^i}{\partial x^k}\frac{\partial x^k}{\partial x'^j} = \frac{\partial x'^i}{\partial x'^j} = \delta^i_j \tag{13.16}$$

and is **invariant** under changes of coordinates. □

13.7 Symmetric and antisymmetric tensors

A covariant tensor is **symmetric** if it is independent of the order of its indices. That is, if $S_{ik} = S_{ki}$, then S is symmetric. Similarly a contravariant tensor $S^{k\ell m}$ is symmetric if permutations of its indices k, ℓ, m leave it unchanged. The metric of spacetime $g_{ik}(x) = g_{ki}(x)$ is a symmetric rank-2 covariant tensor because it is an inner product of two tangent basis vectors.

A covariant or contravariant tensor is **antisymmetric** if it changes sign when any two of its indices are interchanged. The Maxwell field strength $F_{k\ell}(x) = -F_{\ell k}(x)$ is an antisymmetric rank-2 covariant tensor.

If $T^{ik}\,\epsilon_{ik} = 0$ where $\epsilon_{12} = -\epsilon_{21} = 1$ is antisymmetric, then $T^{12} - T^{21} = 0$. Thus $T^{ik}\,\epsilon_{ik} = 0$ means that the tensor T^{ik} is symmetric.

13.8 Quotient theorem

Suppose that the product $B\,A$ of a quantity B with unknown transformation properties with all tensors A a given rank and kind is a tensor. Then B must be a tensor.

The simplest example is when $B_i A^i$ is a scalar for all contravariant vectors A^i

$$B'_i A'^i = B_j A^j. \tag{13.17}$$

Then since A^i is a contravariant vector

$$B'_i A'^i = B'_i \frac{\partial x'^i}{\partial x^j} A^j = B_j A^j \tag{13.18}$$

or

$$\left(B'_i \frac{\partial x'^i}{\partial x^j} - B_j \right) A^j = 0. \tag{13.19}$$

Since this equation holds for all vectors A, we may promote it to the level of a vector equation

$$B'_i \frac{\partial x'^i}{\partial x^j} - B_j = 0. \tag{13.20}$$

Multiplying both sides by $\partial x^j / \partial x'^k$ and summing over j, we get

$$B'_i \frac{\partial x'^i}{\partial x^j} \frac{\partial x^j}{\partial x'^k} = B_j \frac{\partial x^j}{\partial x'^k} \tag{13.21}$$

which shows that the unknown quantity B_i transforms as a covariant vector

$$B'_k = \frac{\partial x^j}{\partial x'^k} B_j. \tag{13.22}$$

The quotient rule works for tensors A and B of arbitrary rank and kind. The proof in each case is similar to the one given here.

13.9 Tensor equations

Maxwell's homogeneous equations (12.45) relate the derivatives of the field-strength tensor to each other as

$$0 = \partial_i F_{jk} + \partial_k F_{ij} + \partial_j F_{ki}. \tag{13.23}$$

They are generally covariant **tensor equations** (sections 13.19 & 13.20). They follow from the Bianchi identity (12.71)

$$dF = ddA = 0. \tag{13.24}$$

Maxwell's inhomegneous equations (12.46) relate the derivatives of the field-strength tensor to the current density j^i and to the square root of the modulus g of the determinant of the metric tensor g_{ij} (section 13.12)

$$\frac{\partial(\sqrt{g}\, F^{ik})}{\partial x^k} = \mu_0 \sqrt{g}\, j^i. \tag{13.25}$$

They are generally covariant tensor equations. We'll write them as the divergence of a contravariant vector in section 13.29, derive them from an action principle in section 13.31, and write them as invariant forms in section 14.7.

If we can write a physical law in one coordinate system as a tensor equation $G^{j\ell}(x) = 0$, then in any other coordinate system the corresponding tensor equation $G'^{ik}(x') = 0$ is valid because

$$G'^{ik}(x') = \frac{\partial x'^i}{\partial x^j} \frac{\partial x'^k}{\partial x^\ell} G^{j\ell}(x) = 0. \tag{13.26}$$

Physical laws also remain the same if expressed in terms of invariant forms. **A theory written in terms of tensors or forms has equations that are true in all coordinate systems if they are true in any coordinate system**. Only such generally covariant theories have a chance at being right because we can't be sure that our particular coordinate system is the chosen one. One can make a theory the same in all coordinate systems by applying the principle of stationary action (section 13.31) to an action that is invariant under all coordinate transformations.

13.10 Comma notation for derivatives

Commas are used to denote derivatives. If $f(\theta, \phi)$ is a function of θ and ϕ, we can write its derivatives with respect to these coordinates as

$$f_{,\theta} = \partial_\theta f = \frac{\partial f}{\partial \theta} \quad \text{and} \quad f_{,\phi} = \partial_\phi f = \frac{\partial f}{\partial \phi}. \tag{13.27}$$

And we can write its double derivatives as

$$f_{,\theta\theta} = \frac{\partial^2 f}{\partial \theta^2}, \quad f_{,\theta\phi} = \frac{\partial^2 f}{\partial \theta \partial \phi}, \quad \text{and} \quad f_{,\phi\phi} = \frac{\partial^2 f}{\partial \phi^2}. \tag{13.28}$$

If we use indices $i, k, \ldots$ to label the coordinates x^i, x^k, then we can write the derivatives of a scalar f as

$$f_{,i} = \partial_i f = \frac{\partial f}{\partial x^i} \quad \text{and} \quad f_{,ik} = \partial_k \partial_i f = \frac{\partial^2 f}{\partial x^k \partial x^i} \tag{13.29}$$

and those of tensors T^{ik} and F_{ik} as

$$T^{ik}_{,j\ell} = \frac{\partial^2 T^{ik}}{\partial x^j \partial x^\ell} \quad \text{and} \quad F_{ik,j\ell} = \frac{\partial^2 F_{ik}}{\partial x^j \partial x^\ell} \tag{13.30}$$

and so forth.

Semicolons are used to denote covariant derivatives (section 13.15).

13.11 Basis vectors and tangent vectors

A point $p(x)$ in a space or spacetime with coordinates x is a scalar (13.3) because it is the same point $p'(x') = p(x') = p(x)$ in any other system of coordinates x'. Thus its derivatives with respect to the coordinates

$$\frac{\partial p(x)}{\partial x^i} = e_i(x) \tag{13.31}$$

form a **covariant vector** $e_i(x)$

$$e'_i(x') = \frac{\partial p'(x')}{\partial x'^i} = \frac{\partial p(x)}{\partial x'^i} = \frac{\partial x^k}{\partial x'^i}\frac{\partial p(x)}{\partial x^k} = \frac{\partial x^k}{\partial x'^i}\, e_k(x). \tag{13.32}$$

Small changes dx^i in the coordinates (in any fixed system of coordinates) lead to small changes in the point $p(x)$

$$dp(x) = e_i(x)\, dx^i. \tag{13.33}$$

The covariant vectors $e_i(x)$ therefore form a basis (1.49) for the space or spacetime at the point $p(x)$. These **basis vectors** $e_i(x)$ are tangent to the curved space or spacetime at the point x and so are called **tangent vectors**. Although complex and fermionic manifolds may be of interest, the manifolds, points, and vectors of this chapter are assumed to be real.

13.12 Metric tensor

A **Riemann manifold** of dimension d is a space that locally looks like d-dimensional euclidian space $\mathbb{E}^d$ and that is smooth enough for the derivatives (13.31) that define tangent vectors to exist. The surface of the Earth, for example, looks flat at distances less than a kilometer.

Just as the surface of a sphere can be embedded in flat 3-dimensional space, so too every Riemann manifold can be embedded without change of shape (isometrically) in a euclidian space $\mathbb{E}^n$ of suitably high dimension (Nash, 1956). In particular, every Riemann manifold of dimension $d = 3$ (or 4) can be isometrically embedded in a euclidian space of at most $n = 14$ (or 19) dimensions, $\mathbb{E}^{14}$ or $\mathbb{E}^{19}$ (Günther, 1989).

The euclidian dot products (example 1.15) of the tangent vectors (13.31) define the metric of the manifold

$$g_{ik}(x) = e_i(x) \cdot e_k(x) = \sum_{\alpha=1}^{n} e_i^\alpha(x)\, e_k^\alpha(x) = e_k(x) \cdot e_i(x) = g_{ki}(x) \tag{13.34}$$

which is symmetric, $g_{ik}(x) = g_{ki}(x)$. Here $1 \le i, k \le d$ and $1 \le \alpha \le n$.

The dot product of this equation is the dot product of the n-dimensional euclidian embedding space $\mathbb{E}^n$.

Because the tangent vectors $e_i(x)$ are covariant vectors, the metric tensor transforms as a covariant tensor if we change coordinates from x to x'

$$g'_{ik}(x') = \frac{\partial x^j}{\partial x'^i} \frac{\partial x^\ell}{\partial x'^k} \, g_{j\ell}(x). \tag{13.35}$$

The squared distance ds^2 between two nearby points is the dot product of the small change $dp(x)$ (13.33) with itself

$$\begin{aligned} ds^2 &= dp(x) \cdot dp(x) = (e_i(x)\, dx^i) \cdot (e_i(x)\, dx^i) \\ &= e_i(x) \cdot e_i(x)\, dx^i dx^k = g_{ik}(x)\, dx^i dx^k. \end{aligned} \tag{13.36}$$

So by measuring the distances ds between nearby points, one can determine the metric $g_{ik}(x)$ of a Riemann space.

Example 13.4 (The sphere S^2 in $\mathbb{E}^3$) In polar coordinates, a point $\boldsymbol{p}$ on the 2-dimensional surface of a sphere of radius R has coordinates $\boldsymbol{p} = R(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ in an embedding space $\mathbb{E}^3$. The tangent space $\mathbb{E}^2$ at $\boldsymbol{p}$ is spanned by the tangent vectors

$$\begin{aligned} \boldsymbol{e}_\theta &= \boldsymbol{p}_{,\theta} = \frac{\partial \boldsymbol{p}}{\partial \theta} = R\,(\cos\theta\cos\phi, \cos\theta\sin\phi, -\sin\theta) \\ \boldsymbol{e}_\phi &= \boldsymbol{p}_{,\phi} = \frac{\partial \boldsymbol{p}}{\partial \phi} = R\,(-\sin\theta\sin\phi, \sin\theta\cos\phi, 0). \end{aligned} \tag{13.37}$$

The dot products of these tangent vectors are easy to compute in the embedding space $\mathbb{E}^3$. They form the metric tensor of the sphere

$$g_{ik} = \begin{pmatrix} g_{\theta\theta} & g_{\theta\phi} \\ g_{\phi\theta} & g_{\phi\phi} \end{pmatrix} = \begin{pmatrix} \boldsymbol{e}_\theta \cdot \boldsymbol{e}_\theta & \boldsymbol{e}_\theta \cdot \boldsymbol{e}_\phi \\ \boldsymbol{e}_\phi \cdot \boldsymbol{e}_\theta & \boldsymbol{e}_\phi \cdot \boldsymbol{e}_\phi \end{pmatrix} = \begin{pmatrix} R^2 & 0 \\ 0 & R^2 \sin^2\theta \end{pmatrix}. \tag{13.38}$$

Its determinant is $\det(g_{ik}) = R^4 \sin^2\theta$. Since $\boldsymbol{e}_\theta \cdot \boldsymbol{e}_\phi = 0$, the squared infinitesimal distance (13.36) is

$$ds^2 = \boldsymbol{e}_\theta \cdot \boldsymbol{e}_\theta \, d\theta^2 + \boldsymbol{e}_\phi \cdot \boldsymbol{e}_\phi \, d\phi^2 = R^2 d\theta^2 + R^2 \sin^2\theta \, d\phi^2. \tag{13.39}$$

We change coordinates from the angle θ to a radius $r = R\sin\theta/a$ in which a is a dimensionless scale factor. Then $R^2 d\theta^2 = a^2 dr^2/\cos^2\theta$, and $\cos^2\theta = 1 - \sin^2\theta = 1 - a^2 r^2/R^2 = 1 - kr^2$ where $k = (a/R)^2$. In these coordinates, the squared distance (13.39) is

$$ds^2 = \frac{a^2}{1 - kr^2} \, dr^2 + a^2 r^2 \, d\phi^2 \tag{13.40}$$

and the r, ϕ metric of the sphere and its inverse are

$$g_{ik} = a^2 \begin{pmatrix} (1-kr^2)^{-1} & 0 \\ 0 & r^2 \end{pmatrix} \quad \text{and} \quad g^{ik} = a^{-2} \begin{pmatrix} 1-kr^2 & 0 \\ 0 & r^{-2} \end{pmatrix}. \tag{13.41}$$

The sphere is a **maximally symmetric space** (section 13.24). □

Example 13.5 (Graph paper) Imagine a piece of slightly crumpled graph paper with horizontal and vertical lines. The lines give us a two-dimensional coordinate system (x^1, x^2) that labels each point $p(x)$ on the paper. The vectors $e_1(x) = \partial_1 p(x)$ and $e_2(x) = \partial_2 p(x)$ define how a point moves $dp(x) = e_i(x)\, dx^i$ when we change its coordinates by dx^1 and dx^2. The vectors $e_1(x)$ and $e_2(x)$ span a different tangent space at the intersection of every horizontal line with every vertical line. Each tangent space is like the tiny square of the graph paper at that intersection. We can think of the two vectors $e_i(x)$ as three-component vectors in the three-dimensional embedding space we live in. The squared distance between any two nearby points separated by $dp(x)$ is $ds^2 \equiv dp^2(x) = e_1^2(x)(dx^1)^2 + 2e_1(x)\cdot e_2(x)\, dx^1 dx^2 + e_2^2(x)(dx^2)^2$ in which the inner products $g_{ij} = e_i(x)\cdot e_j(x)$ are defined by the euclidian metric of the embedding euclidian space $\mathbb{R}^3$. □

But our universe has time. A **semi-euclidian** spacetime $\mathbb{E}^{(p,d-p)}$ of dimension d is a flat spacetime with a dot product that has p minus signs and $q = d - p$ plus signs. A **semi-riemannian** manifold of dimension d is a spacetime that locally looks like a **semi-euclidian** spacetime $\mathbb{E}^{(p,d-p)}$ and that is smooth enough for the derivatives (13.31) that define its tangent vectors to exist.

Every semi-riemannian manifold can be embedded without change of shape (isometrically) in a semi-euclidian spacetime $\mathbb{E}^{(u,n-u)}$ for sufficiently large u and n (Greene, 1970; Clarke, 1970). Every physically reasonable (globally hyperbolic) semi-riemannian manifold with 1 dimension of time and 3 dimensions of space can be embedded without change of shape (isometrically) in a flat semi-euclidian spacetime of 1 temporal and at most 19 spatial dimensions $\mathbb{E}^{(1,19)}$ (Müller and Sánchez, 2011; Aké et al., 2018).

The semi-euclidian dot products of the tangent vectors of a semi-riemannian manifold of d dimensions define its metric as

$$g_{ik}(x) = e_i(x)\cdot e_k(x) = -\sum_{\alpha=1}^{u} e_i^\alpha(x)\, e_k^\alpha(x) + \sum_{\alpha=u+1}^{n} e_i^\alpha(x)\, e_k^\alpha(x) \tag{13.42}$$

for $0 \le i, k \le d-1$. The metric (13.42) is symmetric $g_{ik}(x) = g_{ki}$. In an extended summation convention, the dot product (13.42) is $g_{ik}(x) = e_{i\alpha}(x)\, e^{\alpha}_{k}(x)$.

The squared pseudo-distance or **line element** ds^2 between two nearby points is the inner product of the small change $dp(x)$ (13.33) with itself

$$\begin{aligned} ds^2 &= dp(x) \cdot dp(x) = (e_i(x)\, dx^i) \cdot (e_i(x)\, dx^i) \\ &= e_i(x) \cdot e_i(x)\, dx^i dx^k = g_{ik}(x)\, dx^i dx^k. \end{aligned} \tag{13.43}$$

Thus measurements of line elements ds^2 determine the metric $g_{ik}(x)$ of the spacetime.

Some Riemann spaces have natural embeddings in semi-euclidian spaces. One example is the hyperboloid H^2.

Example 13.6 (The hyperboloid H^2) If we embed a hyperboloid H^2 of radius R in a semi-euclidian spacetime $\mathbb{E}^{(1,2)}$, then a point $\boldsymbol{p} = (x, y, z)$ on the 2-dimensional surface of H^2 obeys the equation $R^2 = x^2 - y^2 - z^2$ and has polar coordinates $\boldsymbol{p} = R(\cosh\theta, \sinh\theta\cos\phi, \sinh\theta\sin\phi)$. The tangent vectors are

$$\begin{aligned} \boldsymbol{e}_\theta &= \boldsymbol{p}_{,\theta} = \frac{\partial \boldsymbol{p}}{\partial \theta} = R\,(\sinh\theta, \cosh\theta\cos\phi, \cosh\theta\sin\phi) \\ \boldsymbol{e}_\phi &= \boldsymbol{p}_{,\phi} = \frac{\partial \boldsymbol{p}}{\partial \phi} = R\,(0, -\sinh\theta\sin\phi, \sinh\theta\cos\phi). \end{aligned} \tag{13.44}$$

The line element $d\boldsymbol{p}^2 = ds^2$ between nearby points is

$$ds^2 = \boldsymbol{e}_\theta \cdot \boldsymbol{e}_\theta\, d\theta^2 + \boldsymbol{e}_\phi \cdot \boldsymbol{e}_\phi\, d\phi^2. \tag{13.45}$$

The metric and line element (13.45) are

$$R^2 \begin{pmatrix} 1 & 0 \\ 0 & \sinh^2\theta \end{pmatrix} \quad \text{and} \quad ds^2 = R^2\, d\theta^2 + R^2\, \sinh^2\theta\, d\phi^2. \tag{13.46}$$

We change coordinates from the angle θ to a radius $r = R\sinh\theta/a$ in which a is a dimensionless scale factor. Then in terms of the parameter $k = (a/R)^2$, the metric and line element (13.46) are (exercise 13.7)

$$a^2 \begin{pmatrix} (1 + r^2)^{-1} & 0 \\ 0 & r^2 \end{pmatrix} \quad \text{and} \quad ds^2 = a^2 \left(\frac{dr^2}{1 + kr^2} + r^2\, d\phi^2 \right) \tag{13.47}$$

which describe one of only three maximally symmetric (section 13.24) two-dimensional spaces. The other two are the sphere S^2 (13.40) and the plane. □

13.13 Inverse of metric tensor

The metric g_{ik} is a nonsingular matrix (exercise 13.4), and so it has an inverse g^{ik} that satisfies

$$g^{ik} g_{k\ell} = \delta^i_\ell = g'^{ik} g'_{k\ell} \tag{13.48}$$

in all coordinate systems. The inverse metric g^{ik} is a rank-2 contravariant tensor (13.11) because the metric $g_{k\ell}$ is a rank-2 covariant tensor (13.35). To show this, we combine the transformation law (13.35) with the definition (13.48) of the inverse of the metric tensor

$$\delta^i_\ell = g'^{ik} g'_{k\ell} = g'^{ik} \, \frac{\partial x^r}{\partial x'^k} \, \frac{\partial x^s}{\partial x'^\ell} \, g_{rs} \tag{13.49}$$

and multiply both sides by

$$g^{tu} \, \frac{\partial x'^\ell}{\partial x^t} \, \frac{\partial x'^v}{\partial x^u}. \tag{13.50}$$

Use of the Kronecker-delta chain rule (13.15) now leads (exercise 13.5) to

$$g'^{iv}(x') = \frac{\partial x'^i}{\partial x^t} \, \frac{\partial x'^v}{\partial x^u} \, g^{tu}(x) \tag{13.51}$$

which shows that the inverse metric g^{ik} transforms as a rank-2 contravariant tensor.

The contravariant vector A^i associated with any covariant vector A_k **is defined as** $A^i = g^{ik} \, A_k$ which ensures that A^i transforms contravariantly (exercise 13.6). This is called **raising an index**. It follows that the covariant vector corresponding to the contravariant vector A^i is $A_k = g_{ki} \, A^i = g_{ki} \, g^{i\ell} \, A_\ell = \delta^\ell_k \, A_\ell = A_k$ which is called **lowering an index**. These definitions apply to all tensors, so $T^{ik\ell} = g^{ij} g^{km} g^{\ell n} T_{jmn}$, and so forth.

Example 13.7 (Making scalars) Fully contracted products of vectors and tensors are scalars. Two contravariant vectors A^i and B^k contracted with the metric tensor form the scalar $g_{ik} A^i B^k = A_k B^k$. Similarly, $g^{ik} A_i B_k = A^k B_k$. Derivatives of scalar fields with respect to the coordinates are covariant vectors $S_{,i}$ (example 13.1) and covariant tensors $S_{,ik}$ (section 13.5). If S is a scalar, then $S_{,i}$ is a covariant vector, $g^{ik} S_{,k}$ is a contravariant vector, and the contraction $g^{ik} S_{,i} \, S_{,k}$ is a scalar. □

In what follows, I will often use *space* to mean either *space* or *spacetime.*

13.14 Dual vectors, cotangent vectors

Since the inverse metric g^{ik} is a rank-2 contravariant tensor, **dual vectors**

$$e^i = g^{ik} e_k \tag{13.52}$$

are contravariant vectors. They are orthonormal to the tangent vectors e_ℓ because

$$e^i \cdot e_\ell = g^{ik} e_k \cdot e_\ell = g^{ik}\, g_{k\ell} = \delta^i_\ell . \tag{13.53}$$

Here and throughout these sections, the dot product is that (13.34) of euclidian space $\mathbb{E}^d$ (or $\mathbb{E}^n$) or that (13.42) of semi-euclidian space $\mathbb{E}^{(p,d-p)}$ (or $\mathbb{E}^{(u,n-u)}$. The dual vectors e^i are called **cotangent vectors** or **tangent covectors**. The tangent vector e_k is the sum $e_k = g_{ki}\, e^i$ because

$$e_k = g_{ki}\, e^i = g_{ki}\, g^{i\ell}\, e_\ell = \delta^\ell_k\, e_\ell = e_k . \tag{13.54}$$

The definition (13.52) of the dual vectors and their orthonormality (13.53) to the tangent vectors imply that their inner products are the matrix elements of the inverse of the metric tensor

$$e^i \cdot e^\ell = g^{ik}\, e_k \cdot e^\ell = g^{ik}\, \delta^\ell_k = g^{i\ell} . \tag{13.55}$$

The outer product of a tangent vector with its cotangent vector $P = e_k e^k$ (summed over the dimensions of the space) is both a projection matrix P from the embedding space onto the tangent space and an identity matrix for the tangent space because $P e_i = e_i$. Its transpose $P^{\mathsf{T}} = e^k e_k$ is both a projection matrix P from the embedding space onto the cotangent space and an identity matrix for the cotangent space because $P^{\mathsf{T}} e^i = e^i$. So

$$P = e_k\, e^k = I_t \quad \text{and} \quad P^{\mathsf{T}} = e^k e_k = I_{ct} . \tag{13.56}$$

Details and examples are in the file tensors.pdf in Tensors_and_general_relativity at `github.com/kevinecahill`.

13.15 Covariant derivatives of contravariant vectors

The **covariant derivative** $D_\ell V^k$ of a contravariant vector V^k is a derivative of V^k that transforms like a mixed rank-2 tensor. An easy way to make such a derivative is to note that the invariant description $V(x) = V^i(x)\, e_i(x)$ of a contravariant vector field $V^i(x)$ in terms of tangent vectors $e_i(x)$ is a scalar.

Its derivative

$$\frac{\partial V}{\partial x^\ell} = \frac{\partial V^i}{\partial x^\ell}\, e_i + V^i\, \frac{\partial e_i}{\partial x^\ell} \tag{13.57}$$

is therefore a covariant vector. And the inner product of that covariant vector $V_{,\ell}$ with a contravariant tangent vector e^k is a mixed rank-2 tensor

$$\begin{aligned} D_\ell V^k &= e^k \cdot V_{,\ell} = e^k \cdot \left(V^i_{,\ell} e_i + e_{i,\ell} V^i\right) = \delta^k_i V^i_{,\ell} + e^k \cdot e_{i,\ell} V^i \\ &= V^k_{,\ell} + e^k \cdot e_{i,\ell}\, V^i. \end{aligned} \tag{13.58}$$

The inner product $e^k \cdot e_{i,\ell}$ is usually written as

$$e^k \cdot e_{i,\ell} = e^k \cdot \frac{\partial e_i}{\partial x^\ell} \equiv \Gamma^k{}_{i\ell} \tag{13.59}$$

and is variously called an **affine connection** (it relates tangent spaces lacking a common origin), a **Christoffel connection**, and a **Christoffel symbol** of the second kind. The covariant derivative itself often is written with a semicolon, thus

$$D_\ell V^k = V^k_{;\ell} = V^k_{,\ell} + e^k \cdot e_{i,\ell}\, V^i = V^k_{,\ell} + \Gamma^k{}_{i\ell}\, V^i. \tag{13.60}$$

Example 13.8 (Covariant derivatives of cotangent vectors) Using the identity

$$0 = \delta^k_{i,\ell} = (e^k \cdot e_i)_{,\ell} = e^k_{,\ell} \cdot e_i + e^k \cdot e_{i,\ell} \tag{13.61}$$

and the projection matrix (13.56), we find that

$$D_\ell e^k = e^k_{,\ell} + e^k \cdot e_{i,\ell}\, e^i = e^k_{,\ell} - e^k_{,\ell} \cdot e_i\, e^i = e^k_{,\ell} - e^k_{,\ell} = 0 \tag{13.62}$$

the covariant derivatives of cotangent vectors vanish. □

Under general coordinate transformations, $D_\ell V^k$ transforms as a rank-2 mixed tensor

$$\left(D_\ell V^k\right)'(x') = \left(V^k_{;\ell}\right)'(x') = \frac{\partial x'^k}{\partial x^p}\frac{\partial x^m}{\partial x'^\ell}\, V^p_{;m}(x) = x'^k_{,p} x^m_{,\ell'}\, V^p_{;m}(x). \tag{13.63}$$

Tangent basis vectors e_i are derivatives (13.31) of the spacetime point p with respect to the coordinates x^i, and so $e_{i,\ell} = e_{\ell,i}$ because partial derivatives commute

$$e_{i,\ell} = \frac{\partial e_i}{\partial x^\ell} = \frac{\partial^2 p}{\partial x^\ell \partial x^i} = \frac{\partial^2 p}{\partial x^i \partial x^\ell} = e_{\ell,i}. \tag{13.64}$$

Thus the affine connection (13.59) is symmetric in its lower indices

$$\Gamma^k{}_{i\ell} = e^k \cdot e_{i,\ell} = e^k \cdot e_{\ell,i} = \Gamma^k{}_{\ell i}. \tag{13.65}$$

Although the covariant derivative $V^i_{;\ell}$ (13.60) is a rank-2 mixed tensor, the affine connection $\Gamma^k{}_{i\ell}$ transforms inhomogeneously (exercise 13.8)

$$\begin{aligned}\Gamma'^k{}_{i\ell} &= e'^k \cdot \frac{\partial e'_i}{\partial x'^\ell} = \frac{\partial x'^k}{\partial x^p}\,\frac{\partial x^m}{\partial x'^\ell}\,\frac{\partial x^n}{\partial x'^i}\,\Gamma^p{}_{nm} + \frac{\partial x'^k}{\partial x^p}\,\frac{\partial^2 x^p}{\partial x'^\ell \partial x'^i} \\ &= x'^k_{,p}\, x^m_{,\ell'}\, x^n_{,i'}\, \Gamma^p{}_{nm} + x'^k_{,p}\, x^p_{,\ell' i'}\end{aligned} \tag{13.66}$$

and so is not a tensor. Its variation $\delta\Gamma^k{}_{i\ell} = \Gamma'^k_{i\ell} - \Gamma^k{}_{i\ell}$ is a tensor, however, because the inhomogeneous terms in the difference cancel.

Since the Levi-Civita connection $\Gamma^k{}_{i\ell}$ is symmetric in i and ℓ, in four-dimensional spacetime, there are 10 Γ's for each k, or 40 in all. The 10 correspond to 3 rotations, 3 boosts, and 4 translations.

13.16 Covariant derivatives of covariant vectors

The derivative of the scalar $V = V_k\, e^k$ is the covariant vector

$$V_{,\ell} = (V_k\, e^k)_{,\ell} = V_{k,\ell}\, e^k + V_k\, e^k_{,\ell}. \tag{13.67}$$

Its inner product with the covariant vector e_i transforms as a rank-2 covariant tensor. Thus using again the identity (13.61), we see that the covariant derivative of a covariant vector is

$$\begin{aligned}D_\ell V_i = V_{i;\ell} &= e_i \cdot V_{,\ell} = e_i \cdot \left(V_{k,\ell}\, e^k + V_k\, e^k_{,\ell}\right) = \delta^k_i\, V_{k,\ell} + e_i \cdot e^k_{,\ell}\, V_k \\ &= V_{i,\ell} - e_{i,\ell} \cdot e^k\, V_k = V_{i,\ell} - \Gamma^k{}_{i\ell}\, V_k.\end{aligned} \tag{13.68}$$

$D_\ell V_i$ transforms as a rank-2 covariant tensor because it is the inner product of a covariant tangent vector e_i with the derivative $V_{,\ell}$ of a scalar. Note that $\Gamma^k_{i\ell}$ appears with a minus sign in $V_{i;\ell}$ and a plus sign in $V^k_{;\ell}$.

Example 13.9 (Covariant derivatives of tangent vectors) Using again the projection matrix (13.56), we find that

$$D_\ell e_i = e_{i\ell} = e_{i,\ell} - e_{i,\ell} \cdot e^k e_k = e_{i,\ell} - e_{i,\ell} = 0 \tag{13.69}$$

that covariant derivatives of tangent vectors vanish. □

13.17 Covariant derivatives of tensors

Tensors transform like products of vectors. So we can make the derivative of a tensor transform covariantly by using Leibniz's rule (5.49) to differentiate products of vectors and by turning the derivatives of the vectors into their covariant derivatives (13.60) and/or (13.68).

Example 13.10 (Covariant derivative of a rank-2 contravariant tensor) An arbitrary rank-2 contravariant tensor T^{ik} transforms like the product of two contravariant vectors $A^i B^k$. So its derivative $\partial_\ell T^{ik}$ transforms like the derivative of the product of the vectors $A^i B^k$

$$\partial_\ell (A^i B^k) = (\partial_\ell A^i) B^k + A^i \partial_\ell B^k. \tag{13.70}$$

By using twice the formula (13.60) for the covariant derivative of a contravariant vector, we can convert these two ordinary derivatives $\partial_\ell A^i$ and $\partial_\ell B^k$ into tensors

$$\begin{aligned} D_\ell (A^i B^k) = (A^i B^k)_{;\ell} &= (A^i_{,\ell} + \Gamma^i{}_{j\ell} A^j) B^k + A^i (B^k_{,\ell} + \Gamma^k{}_{j\ell} B^j) \\ &= (A^i B^k)_{,\ell} + \Gamma^i{}_{j\ell} A^j B^k + \Gamma^k{}_{j\ell} A^i B^j. \end{aligned} \tag{13.71}$$

Thus the covariant derivative of a rank-2 contravariant tensor is

$$D_\ell T^{ik} = T^{ik}{}_{;\ell} = T^{ik}{}_{,\ell} + \Gamma^i{}_{j\ell} T^{jk} + \Gamma^k{}_{j\ell} T^{ij}. \tag{13.72}$$

It transforms as a rank-3 tensor with one covariant index. □

Example 13.11 (Covariant derivative of a rank-2 mixed tensor) A rank-2 mixed tensor $T^i{}_k$ transforms like the product $A^i B_k$ of a contravariant vector A^i and a covariant vector B_k. Its derivative $\partial_\ell T^i{}_k$ transforms like the derivative of the product of the vectors $A^i B_k$

$$\partial_\ell (A^i B_k) = (\partial_\ell A^i) B_k + A^i \partial_\ell B_k. \tag{13.73}$$

We can make these derivatives transform like tensors by using the formulas (13.60) and (13.68)

$$\begin{aligned} D_\ell (A^i B_k) = (A^i B_k)_{;\ell} &= (A^i_{,\ell} + \Gamma^i{}_{j\ell} A^j) B_k + A^i (B_{k,\ell} - \Gamma^j{}_{k\ell} B_j) \\ &= (A^i B_k)_{,\ell} + \Gamma^i{}_{j\ell} A^j B_k - \Gamma^j{}_{k\ell} A^i B_j. \end{aligned} \tag{13.74}$$

Thus the covariant derivative of a mixed rank-2 tensor is

$$D_\ell T^i{}_k = T^i{}_{k;\ell} = T^i{}_{k,\ell} + \Gamma^i{}_{j\ell} T^j{}_k - \Gamma^j{}_{k\ell} T^i{}_j. \tag{13.75}$$

It transforms as a rank-3 tensor with two covariant indices. □

Example 13.12 (Covariant derivative of a rank-2 covariant tensor) A rank-2 covariant tensor T_{ik} transforms like the product $A_i B_k$ of two covariant vectors A_i and B_k. Its derivative $\partial_\ell T_{ik}$ transforms like the derivative of the product of the vectors $A_i B_k$

$$\partial_\ell (A_i B_k) = (\partial_\ell A_i) B_k + A_i \partial_\ell B_k. \tag{13.76}$$

We can make these derivatives transform like tensors by twice using the formula (13.68)

$$\begin{aligned} D_\ell(A_i\, B_k) = (A_i\, B_k)_{;\ell} &= A_{i;\ell}\, B_k + A_i\, B_{k\ell} \\ &= (A_{i,\ell} - \Gamma^j{}_{i\ell}\, A_j) B_k + A_i (B_{k,\ell} - \Gamma^j{}_{k\ell}\, B_j) \quad (13.77) \\ &= (A_i\, B_k)_{,\ell} - \Gamma^j{}_{i\ell}\, A_j\, B_k - \Gamma^j{}_{k\ell}\, A_i\, B_j. \end{aligned}$$

Thus the covariant derivative of a rank-2 covariant tensor T_{ik} is

$$D_\ell\, T_{ik} = T_{ik;\ell} = T_{ik,\ell} - \Gamma^j{}_{i\ell}\, T_{jk} - \Gamma^j{}_{k\ell}\, T_{ij}. \quad (13.78)$$

It transforms as a rank-3 covariant tensor.

Another way to derive the same result is to note that the scalar form of a rank-2 covariant tensor T_{ik} is $T = e^i \otimes e^k\, T_{ik}$. So its derivative is a covariant vector

$$T_{,\ell} = e^i \otimes e^k\, T_{ik,\ell} + e^i_{,\ell} \otimes e^k\, T_{ik} + e^i \otimes e^k_{,\ell}\, T_{ik}. \quad (13.79)$$

Using the projector $P_t = e^j e_j$ (13.56), the duality $e^i \cdot e_n = \delta^i_n$ of tangent and cotangent vectors (13.53), and the relation $e_j \cdot e^k_{,\ell} = -\, e^k \cdot e_{j,\ell} = -\Gamma^k_{j\ell}$ (13.59 & 13.61), we can project this derivative onto the tangent space and find after shuffling some indices

$$\begin{aligned} (e^n e_n \otimes e^j e_j)\, T_{,\ell} &= e^i \otimes e^k\, T_{ik,\ell} + e^n \otimes e^k\, (e_n \cdot e^i_{,\ell}) T_{ik} + e^i \otimes e^j (e_j \cdot e^k_{,\ell})\, T_{ik} \\ &= e^i \otimes e^k\, T_{ik,\ell} - e^n \otimes e^k\, \Gamma^i{}_{n\ell}\, T_{ik} - e^i \otimes e^j\, \Gamma^k{}_{j\ell}\, T_{ik} \\ &= (e^i \otimes e^k) \left(T_{ik,\ell} - \Gamma^j{}_{i\ell}\, T_{jk} - \Gamma^j{}_{k\ell}\, T_{ij} \right) \end{aligned}$$

which again gives us the formula (13.78). □

As in these examples, covariant derivatives are **derivations**:

$$D_k(AB) = (AB)_{;k} = A_{;k}\, B + A\, B_{;k} = (D_k A)\, B + A\, D_k B. \quad (13.80)$$

The rule for a general tensor is to treat every contravariant index as in (13.60) and every covariant index as in (13.68). The covariant derivative of a mixed rank-4 tensor, for instance, is

$$T^{ab}_{xy;k} = T^{ab}_{xy,k} + T^{jb}_{xy} \Gamma^a{}_{jk} + T^{am}_{xy} \Gamma^b{}_{mk} - T^{ab}_{jy} \Gamma^j{}_{xk} - T^{ab}_{xm} \Gamma^m{}_{yk}. \quad (13.81)$$

13.18 The covariant derivative of the metric tensor vanishes

The metric tensor is the inner product (13.42) of tangent basis vectors

$$g_{ik} = e_i^\alpha \, \eta_{\alpha\beta} \, e_k^\beta \tag{13.82}$$

in which α and β are summed over the dimensions of the embedding space. Thus by the product rule (13.77), the covariant derivative of the metric

$$D_\ell \, g_{ik} = g_{ik;\ell} = D_\ell \, (e_i^\alpha \, \eta_{\alpha\beta} \, e_k^\beta) = (D_\ell \, e_i^\alpha) \, \eta_{\alpha\beta} \, e_k^\beta + e_i^\alpha \, \eta_{\alpha\beta} \, D_\ell \, e_k^\beta = 0 \tag{13.83}$$

vanishes because covariant derivatives of tangent vectors vanish (13.69), $D_\ell \, e_i^\alpha = e_{i;\ell}^\alpha = 0$ and $D_\ell \, e_k^\beta = e_{k;\ell}^\beta = 0$.

13.19 Covariant curls

Because the connection $\Gamma^k_{i\ell}$ is symmetric (13.65) in its lower indices, the covariant curl of a covariant vector V_i is simply its ordinary curl

$$V_{\ell;i} - V_{i;\ell} = V_{\ell,i} - V_k \, \Gamma^k{}_{\ell i} - V_{i,\ell} + V_k \, \Gamma^k{}_{i\ell} = V_{\ell,i} - V_{i,\ell}. \tag{13.84}$$

Thus the Faraday field-strength tensor $F_{i\ell} = A_{\ell,i} - A_{i,\ell}$ being the curl of the covariant vector field A_i is a generally covariant second-rank tensor.

13.20 Covariant derivatives and antisymmetry

The covariant derivative (13.78) $A_{i\ell;k}$ is $A_{i\ell;k} = A_{i\ell,k} - A_{m\ell} \, \Gamma^m{}_{ik} - A_{im} \, \Gamma^m{}_{\ell k}$. If the tensor A is antisymmetric $A_{i\ell} = -A_{\ell i}$, then by adding together the three cyclic permutations of the indices $i\ell k$, we find that the antisymmetry of the tensor and the symmetry (13.65) of the affine connection $\Gamma^m{}_{ik} = \Gamma^m{}_{ki}$ conspire to cancel the terms with Γs

$$\begin{aligned} A_{i\ell;k} + A_{ki;\ell} + A_{\ell k;i} = {} & A_{i\ell,k} - A_{m\ell} \, \Gamma^m{}_{ik} - A_{im} \, \Gamma^m{}_{\ell k} \\ & + A_{ki,\ell} - A_{mi} \, \Gamma^m{}_{k\ell} - A_{km} \, \Gamma^m{}_{i\ell} \\ & + A_{\ell k,i} - A_{mk} \, \Gamma^m{}_{\ell i} - A_{\ell m} \, \Gamma^m{}_{ki} \\ = {} & A_{i\ell,k} + A_{ki,\ell} + A_{\ell k,i} \end{aligned} \tag{13.85}$$

an identity named after Luigi Bianchi (1856–1928).

The Maxwell field-strength tensor $F_{i\ell}$ is antisymmetric by construction ($F_{i\ell} = A_{\ell,i} - A_{i,\ell}$), and so Maxwell's homogeneous equations

$$\begin{aligned} \tfrac{1}{2} \, \epsilon^{ijk\ell} \, F_{jk,\ell} &= F_{jk,\ell} + F_{k\ell,j} + F_{\ell j,k} \\ &= A_{k,j\ell} - A_{j,k\ell} + A_{\ell,kj} - A_{k,\ell j} + A_{j,\ell k} - A_{\ell,jk} = 0 \end{aligned} \tag{13.86}$$

are tensor equations valid in all coordinate systems.

13.21 What is the affine connection?

We insert the identity matrix (13.56) of the tangent space in the form $e^j\, e_j$ into the formula (13.59) for the affine connection $\Gamma^k{}_{i\ell} = e^k \cdot e_{i,\ell}$. In the resulting combination $\Gamma^k{}_{i\ell} = e^k \cdot e^j\ e_j \cdot e_{\ell,i}$ we recognize $e^k \cdot e^j$ as the inverse (13.55) of the metric tensor $e^k \cdot e^j = g^{kj}$. Repeated use of the relation $e_{i,k} = e_{k,i}$ (13.64) then leads to a formula for the affine connection

$$\begin{aligned}\Gamma^k{}_{i\ell} &= e^k \cdot e_{i,\ell} = e^k \cdot e^j\ e_j \cdot e_{i,\ell} = e^k \cdot e^j\ e_j \cdot e_{\ell,i} = \tfrac{1}{2}\, g^{kj} \left(e_j \cdot e_{i,\ell} + e_j \cdot e_{\ell,i}\right) \\ &= \tfrac{1}{2}\, g^{kj} \left((e_j \cdot e_i)_{,\ell} - e_{j,\ell} \cdot e_i + (e_j \cdot e_\ell)_{,i} - e_{j,i} \cdot e_\ell\right) \\ &= \tfrac{1}{2}\, g^{kj} \left(g_{ji,\ell} + g_{j\ell,i} - e_{j,\ell} \cdot e_i - e_{j,i} \cdot e_\ell\right) \\ &= \tfrac{1}{2}\, g^{kj} \left(g_{ji,\ell} + g_{j\ell,i} - e_{\ell,j} \cdot e_i - e_{i,j} \cdot e_\ell\right) \\ &= \tfrac{1}{2}\, g^{kj} \left(g_{ji,\ell} + g_{j\ell,i} - (e_i \cdot e_\ell)_{,j}\right) = \tfrac{1}{2}\, g^{kj} \left(g_{ji,\ell} + g_{j\ell,i} - g_{i\ell,j}\right)\end{aligned} \tag{13.87}$$

in terms of the inverse of the metric tensor and a combination of its derivatives. The metric g_{ik} determines the affine connection $\Gamma^k{}_{i\ell}$.

The affine connection with all lower indices is

$$\Gamma_{ni\ell} = g_{nk}\Gamma^k{}_{i\ell} = \tfrac{1}{2} \left(g_{ni,\ell} + g_{n\ell,i} - g_{i\ell,n}\right). \tag{13.88}$$

13.22 Parallel transport

The movement of a vector along a curve on a manifold so that its length and direction in successive tangent spaces do not change is called **parallel transport**. In parallel transport, a vector $V = V^k\, e_k = V_k\, e^k$ may change $dV = V_{,\ell}\, dx^\ell$, but the projection of the change $P\, dV = e^i e_i\, dV = e_i e^i\, dV$ into the tangent space must vanish, $P\, dV = 0$. In terms of its contravariant components $V = V^k e_k$, this condition for parallel transport is just the vanishing of its covariant derivative (13.60)

$$\begin{aligned}0 &= e^i dV = e^i V_{,\ell} dx^\ell = e^i (V^k e_k)_{,\ell} dx^\ell = e^i \left(V^k_{,\ell} e_k + V^k e_{k,\ell}\right) dx^\ell \\ &= \left(\delta^i_k V^k_{,\ell} + e^i \cdot e_{k,\ell}\, V^k\right) dx^\ell = \left(V^i_{,\ell} + \Gamma^i{}_{k\ell}\, V^k\right) dx^\ell.\end{aligned} \tag{13.89}$$

In terms of its covariant components $V = V_k e^k$, the condition of parallel transport is just the vanishing of its covariant derivative (13.68)

$$\begin{aligned} 0 = e_i dV = e_i V_{,\ell} dx^\ell = e_i (V_k e^k)_{,\ell} dx^\ell = e_i \left(V_{k,\ell} e^k + V_k e^k_{,\ell} \right) dx^\ell \\ = \left(\delta^k_i V_{k,\ell} + e_i \cdot e^k_{,\ell} V^k \right) dx^\ell = \left(V_{i,\ell} - \Gamma^k{}_{i\ell} V_k \right) dx^\ell. \end{aligned} \tag{13.90}$$

If the curve is $x^\ell(u)$, then these conditions (13.89 & 13.90) for parallel transport are

$$\frac{dV^i}{du} = V^i_{,\ell} \frac{dx^\ell}{du} = -\Gamma^i{}_{k\ell} V^k \frac{dx^\ell}{du} \quad \text{and} \quad \frac{dV_i}{du} = V_{i,\ell} \frac{dx^\ell}{du} = \Gamma^k{}_{i\ell} V_k \frac{dx^\ell}{du}. \tag{13.91}$$

Example 13.13 (Parallel transport on a sphere) We parallel-transport the vector $\boldsymbol{v} = \boldsymbol{e}_\phi = (0, 1, 0)$ up from the equator along the line of longitude $\phi = 0$. Along this path, the vector $\boldsymbol{v} = (0, 1, 0) = \boldsymbol{e}_\phi$ is constant, so $\partial_\theta \boldsymbol{v} = 0$ and so both $e^\theta \cdot \boldsymbol{e}_{\phi,\theta} = 0$ and $e^\phi \cdot \boldsymbol{e}_{\phi,\theta} = 0$. Thus $D_\theta v^k = v^k_{;\theta} = 0$ between the equator and the north pole. As $\theta \to 0$ along the meridian $\phi = 0$, the vector $\boldsymbol{v} = (0, 1, 0)$ approaches the vector $\boldsymbol{e}_\theta$. We then parallel-transport $\boldsymbol{v} = \boldsymbol{e}_\theta$ down from the north pole along the line of longitude $\phi = \pi/2$ to the equator. Along this path, the vector $\boldsymbol{v} = \boldsymbol{e}_\theta / r = (0, \cos\theta, -\sin\theta)$ obeys the parallel-transport condition (13.90) because its θ-derivative is

$$\boldsymbol{v}_{,\theta} = r^{-1} \boldsymbol{e}_{,\theta} = (0, \cos\theta, -\sin\theta)_{,\theta} = -(0, \sin\theta, \cos\theta) = -\hat{\boldsymbol{r}}|_{\phi=\pi/2}. \tag{13.92}$$

So $\boldsymbol{v}_{,\theta}$ is perpendicular to the tangent vectors $\boldsymbol{e}_\theta$ and $\boldsymbol{e}_\phi$ along the curve $\phi = \pi/2$. Thus $e^k \cdot \boldsymbol{v}_{,\theta} = 0$ for $k = \theta$ and $k = \phi$ and so $\boldsymbol{v}_{;\theta} = 0$, along the meridian $\phi = \pi/2$. When $\boldsymbol{e}_\theta$ reaches the equator, it is $\boldsymbol{e}_\theta = (0, 0, -1)$. Finally, we parallel-transport $\boldsymbol{v}$ along the equator back to the starting point $\phi = 0$. Along this path, the vector $\boldsymbol{v} = (0, 0, -1) = e_\theta$ is constant, so $\boldsymbol{v}_{,\phi} = 0$ and $\boldsymbol{v}_{;\phi} = 0$. The change from $\boldsymbol{v} = (0, 1, 0)$ to $\boldsymbol{v} = (0, 0, -1)$ is due to the curvature of the sphere. □

13.23 Curvature

To find the curvature at a point $p(x)$, we parallel-transport a vector V_i along a curve $x^\ell(u)$ that runs around a tiny square about the point $p(x) = p(x_0)$ as u runs from 0 to 1. We then measure the change in the vector

$$\Delta V_i = \oint \Gamma^k{}_{i\ell} V_k \, dx^\ell. \tag{13.93}$$

On the curve $x^\ell(u)$, we approximate $\Gamma^k_{i\ell}(x(u))$ and $V_k(u)$ as

$$\begin{aligned} \Gamma^k{}_{i\ell}(x) &= \Gamma^k{}_{i\ell}(x_0) + \Gamma^k{}_{i\ell,n}(x_0)\,(x-x_0)^n \\ V_k(u) &= V_k(0) + \Gamma^m{}_{kn}(x_0)\,V_m(0)\,(x-x_0)^n. \end{aligned} \tag{13.94}$$

So keeping only terms linear in $(x-x_0)^n$, we have

$$\begin{aligned} \Delta V_i = & \oint \Gamma^k{}_{i\ell}\,V_k\,dx^\ell \qquad (13.95) \\ &= \left[\Gamma^k{}_{i\ell,n}(x_0)\,V_k(0) + \Gamma^k{}_{i\ell}(x_0)\,\Gamma^m{}_{kn}(x_0)\,V_m(0)\right] \oint (x-x_0)^n\,dx^\ell \\ &= \left[\Gamma^k{}_{i\ell,n}(x_0)\,V_k(0) + \Gamma^m{}_{i\ell}(x_0)\,\Gamma^k{}_{mn}(x_0)\,V_k(0)\right] \oint (x-x_0)^n\,dx^\ell \end{aligned}$$

after interchanging the dummy indices k and m in the second term within the square brackets. The integral around the square is antisymmetric in n and ℓ and equal in absolute value to the area a^2 of the tiny square

$$\oint (x-x_0)^n\,dx^\ell = \pm\,a^2\,\epsilon_{n\ell}. \tag{13.96}$$

The overall sign depends upon whether the integral is clockwise or counterclockwise, what n and ℓ are, and what we mean by positive area. The integral picks out the part of the term between the brackets in the formula (13.95) that is antisymmetric in n and ℓ. We choose minus signs in (13.96) so that the change in the vector is

$$\Delta V_i = a^2\left[\Gamma^k{}_{in,\ell} - \Gamma^k{}_{i\ell,n} + \Gamma^k{}_{\ell m}\,\Gamma^m{}_{in} - \Gamma^k{}_{nm}\,\Gamma^m{}_{i\ell}\right]\,V_k. \tag{13.97}$$

The quantity between the brackets is **Riemann's curvature tensor**

$$R^k{}_{i\ell n} = \Gamma^k{}_{ni,\ell} - \Gamma^k{}_{\ell i,n} + \Gamma^k{}_{\ell m}\,\Gamma^m{}_{ni} - \Gamma^k{}_{nm}\,\Gamma^m{}_{\ell i}. \tag{13.98}$$

The sign convention is that of (Zee, 2013; Misner et al., 1973; Carroll, 2003; Schutz, 2009; Hartle, 2003; Cheng, 2010; Padmanabhan, 2010). Weinberg (Weinberg, 1972) uses the opposite sign. The covariant form $R_{ijk\ell}$ of Riemann's tensor is related to $R^k{}_{i\ell n}$ by

$$R_{ijk\ell} = g_{in}R^n{}_{jk\ell} \quad \text{and} \quad R^i{}_{jk\ell} = g^{in}R_{njk\ell}. \tag{13.99}$$

The Riemann curvature tensor is the commutator of two covariant derivatives. To see why, we first use the formula (13.78) for the covariant derivative

$D_n D_\ell V_i$ of the second-rank covariant tensor $D_\ell V_i$

$$\begin{aligned} D_n D_\ell V_i &= D_n \left(V_{i,\ell} - \Gamma^k{}_{\ell i} \, V_k \right) \\ &= V_{i,\ell n} - \Gamma^k{}_{\ell i,n} \, V_k - \Gamma^k{}_{\ell i} \, V_{k,n} \\ &\quad - \Gamma^j{}_{ni} \left(V_{j,\ell} - \Gamma^m{}_{\ell j} V_m \right) - \Gamma^m{}_{\ell n} \left(V_{i,m} - \Gamma^q{}_{im} V_q \right). \end{aligned} \tag{13.100}$$

Subtracting $D_\ell D_n V_i$, we find the commutator $[D_n, D_\ell] V_i$ to be the contraction of the curvature tensor $R^k{}_{i\ell n}$ (13.98) with the covariant vector V_k

$$[D_n, D_\ell] V_i = \left(\Gamma^k{}_{ni,\ell} - \Gamma^k{}_{\ell i,n} + \Gamma^k{}_{\ell j} \, \Gamma^j{}_{ni} - \Gamma^k{}_{nj} \, \Gamma^j{}_{\ell i} \right) V_k = R^k{}_{i\ell n} \, V_k. \tag{13.101}$$

Since $[D_n, D_\ell] V_i$ is a rank-3 covariant tensor and V_k is an arbitrary covariant vector, the quotient theorem (section 13.8) implies that the curvature tensor is a rank-4 tensor with one contravariant index.

If we define the matrix Γ_ℓ with row index k and column index i as $\Gamma^k_{i\ell}$

$$\Gamma_\ell = \begin{pmatrix} \Gamma^0{}_{\ell 0} & \Gamma^0{}_{\ell 1} & \Gamma^0{}_{\ell 2} & \Gamma^0{}_{\ell 3} \\ \Gamma^1{}_{\ell 0} & \Gamma^1{}_{\ell 1} & \Gamma^1{}_{\ell 2} & \Gamma^1{}_{\ell 3} \\ \Gamma^2{}_{\ell 0} & \Gamma^2{}_{\ell 1} & \Gamma^2{}_{\ell 2} & \Gamma^2{}_{\ell 3} \\ \Gamma^3{}_{\ell 0} & \Gamma^3{}_{\ell 1} & \Gamma^3{}_{\ell 2} & \Gamma^3{}_{\ell 3} \end{pmatrix}, \tag{13.102}$$

then we may write the covariant derivatives appearing in the curvature tensor $R^k{}_{i\ell n}$ as $D_\ell = \partial_\ell + \Gamma_\ell$ and $D_n = \partial_n + \Gamma_n$. In these terms, the curvature tensor is the i, k matrix element of their commutator

$$R^k{}_{i\ell n} = [\partial_\ell + \Gamma_\ell, \partial_n + \Gamma_n]^k{}_i = [D_\ell, D_n]^k{}_i \,. \tag{13.103}$$

The curvature tensor is therefore antisymmetric in its last two indexes

$$R^k{}_{i\ell n} = \, - R^k{}_{in\ell}. \tag{13.104}$$

The curvature tensor with all lower indices shares this symmetry

$$R_{ji\ell n} = g_{jk} R^k{}_{i\ell n} = \, - g_{jk} R^k{}_{in\ell} = \, - R_{jin\ell} \tag{13.105}$$

and has three others. In Riemann normal coordinates the derivatives of the metric vanish at any particular point x_*. In these coordinates, the Γ's all vanish, and the curvature tensor in terms of the Γ's with all lower indices (13.88) is after a cancellation

$$R_{ki\ell n} = \Gamma_{kni,\ell} - \Gamma_{k\ell i,n} = \tfrac{1}{2} \left(g_{kn,i\ell} - g_{ni,k\ell} - g_{k\ell,in} + g_{\ell i,kn} \right). \tag{13.106}$$

In these coordinates and therefore in all coordinates, $R_{ki\ell n}$ is antisymmetric in its first two indexes and symmetric under the interchange of its first and second pairs of indexes

$$R_{ijk\ell} = -R_{jik\ell} \quad \text{and} \quad R_{ijk\ell} = R_{k\ell ij}. \tag{13.107}$$

Cartan's equations of structure (13.328 & 13.330) imply (13.342) that the curvature tensor is antisymmetric in its last three indexes

$$0 = R^j{}_{[ik\ell]} = \frac{1}{3!}\left(R^j{}_{ik\ell} + R^j{}_{\ell ik} + R^j{}_{k\ell i} - R^j{}_{ki\ell} - R^j{}_{i\ell k} - R^j{}_{\ell ki}\right) \tag{13.108}$$

and obeys the cyclic identity

$$0 = R^j{}_{ik\ell} + R^j{}_{\ell ik} + R^j{}_{k\ell i}. \tag{13.109}$$

The vanishing (13.108) of $R_{i[jk\ell]}$ implies that the completely antisymmetric part of the Riemann tensor also vanishes

$$0 = R_{[ijk\ell]} = \frac{1}{4!}\left(R_{ijk\ell} - R_{jik\ell} - R_{ikj\ell} - R_{ij\ell k} + R_{jki\ell} \cdots\right). \tag{13.110}$$

The Riemann tensor also satisfies a Bianchi identity

$$0 = R^i_{j[k\ell;m]}. \tag{13.111}$$

These symmetries reduce 256 different functions $R_{ijk\ell}(x)$ to 20.

The **Ricci tensor** is the contraction

$$R_{in} = R^k{}_{ikn}. \tag{13.112}$$

The **curvature scalar** is the further contraction

$$R = g^{ni}\, R_{in}. \tag{13.113}$$

Example 13.14 (Curvature of the sphere S^2) While in four-dimensional spacetime indices run from 0 to 3, on the everyday sphere S^2 (example 13.4) they are just θ and ϕ. There are only eight possible affine connections, and because of the symmetry (13.65) in their lower indices $\Gamma^i_{\theta\phi} = \Gamma^i_{\phi\theta}$, only six are independent.

In the euclidian embedding space $\mathbb{E}^3$, the point $\boldsymbol{p}$ on a sphere of radius R has cartesian coordinates $\boldsymbol{p} = R\,(\sin\theta\cos\phi,\, \sin\theta\sin\phi,\, \cos\theta)$, so the two tangent 3-vectors are (13.37)

$$\begin{aligned} \boldsymbol{e}_\theta &= \boldsymbol{p}_{,\theta} = R\,(\cos\theta\,\cos\phi,\, \cos\theta\,\sin\phi,\, -\sin\theta) = R\,\hat{\boldsymbol{\theta}} \\ \boldsymbol{e}_\phi &= \boldsymbol{p}_{,\phi} = R\sin\theta\,(-\sin\phi,\, \cos\phi,\, 0) = R\sin\theta\,\hat{\boldsymbol{\phi}}. \end{aligned} \tag{13.114}$$

Their dot products form the metric (13.38)

$$g_{ik} = \begin{pmatrix} g_{\theta\theta} & g_{\theta\phi} \\ g_{\phi\theta} & g_{\phi\phi} \end{pmatrix} = \begin{pmatrix} \boldsymbol{e}_\theta \cdot \boldsymbol{e}_\theta & \boldsymbol{e}_\theta \cdot \boldsymbol{e}_\phi \\ \boldsymbol{e}_\phi \cdot \boldsymbol{e}_\theta & \boldsymbol{e}_\phi \cdot \boldsymbol{e}_\phi \end{pmatrix} = \begin{pmatrix} R^2 & 0 \\ 0 & R^2 \sin^2\theta \end{pmatrix} \tag{13.115}$$

which is diagonal with $g_{\theta\theta} = R^2$ and $g_{\phi\phi} = R^2 \sin^2\theta$. Differentiating the vectors $\boldsymbol{e}_\theta$ and $\boldsymbol{e}_\phi$, we find

$$\begin{aligned} e_{\theta,\theta} =& - R\,(\sin\theta\cos\phi,\, \sin\theta\sin\phi,\, \cos\theta) = -R\,\hat{\boldsymbol{r}} \\ e_{\theta,\phi} =& R\cos\theta\,(-\sin\phi,\, \cos\phi,\, 0) = R\cos\theta\,\hat{\boldsymbol{\phi}} \\ e_{\phi,\theta} =& e_{\theta,\phi} \\ e_{\phi,\phi} =& - R\sin\theta\,(\cos\phi,\, \sin\phi,\, 0)\,. \end{aligned} \tag{13.116}$$

The metric with upper indices g^{ij} is the inverse of the metric g_{ij}

$$(g^{ij}) = \begin{pmatrix} R^{-2} & 0 \\ 0 & R^{-2}\sin^{-2}\theta \end{pmatrix}, \tag{13.117}$$

so the dual vectors $\boldsymbol{e}^i = g^{ik} e_k$ are

$$\begin{aligned} \boldsymbol{e}^{\,\theta} &= R^{-1}\,(\cos\theta\,\cos\phi,\, \cos\theta\,\sin\phi,\, -\sin\theta) = R^{-1}\hat{\boldsymbol{\theta}} \\ \boldsymbol{e}^{\,\phi} &= \frac{1}{R\sin\theta}\,(-\sin\phi,\, \cos\phi,\, 0) = \frac{1}{R\sin\theta}\,\hat{\boldsymbol{\phi}}. \end{aligned} \tag{13.118}$$

The affine connections are given by (13.59) as

$$\Gamma^i{}_{jk} = \Gamma^i{}_{kj} = \boldsymbol{e}^{\,i} \cdot \boldsymbol{e}_{j,k}. \tag{13.119}$$

Since both $\boldsymbol{e}^\theta$ and $\boldsymbol{e}^{\,\phi}$ are perpendicular to $\hat{r}$, the affine connections $\Gamma^\theta{}_{\theta\theta}$ and $\Gamma^\phi{}_{\theta\theta}$ both vanish. Also, $e_{\phi,\phi}$ is orthogonal to $\hat{\phi}$, so $\Gamma^\phi{}_{\phi\phi} = 0$ as well. Similarly, $e_{\theta,\phi}$ is perpendicular to $\hat{\theta}$, so $\Gamma^\theta{}_{\theta\phi} = \Gamma^\theta{}_{\phi\theta}$ also vanishes.

The two nonzero affine connections are

$$\Gamma^\phi{}_{\theta\phi} = \boldsymbol{e}^{\,\phi} \cdot \boldsymbol{e}_{\theta,\phi} = R^{-1}\sin^{-1}\theta\,\hat{\boldsymbol{\phi}} \cdot R\cos\theta\,\hat{\boldsymbol{\phi}} = \cot\theta \tag{13.120}$$

and

$$\begin{aligned} \Gamma^\theta{}_{\phi\phi} = \boldsymbol{e}^{\,\theta} \cdot \boldsymbol{e}_{\phi,\phi} &= -\sin\theta\,(\cos\theta\,\cos\phi,\, \cos\theta\,\sin\phi,\, -\sin\theta) \cdot (\cos\phi,\, \sin\phi,\, 0) \\ &= -\sin\theta\cos\theta. \end{aligned} \tag{13.121}$$

The nonzero connections are $\Gamma^\phi_{\theta\phi} = \cot\theta$ and $\Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta$. So the matrices Γ_θ and Γ_ϕ, the derivative $\Gamma_{\phi,\theta}$, and the commutator $[\Gamma_\theta, \Gamma_\phi]$ are

$$\Gamma_\theta = \begin{pmatrix} 0 & 0 \\ 0 & \cot\theta \end{pmatrix} \quad \text{and} \quad \Gamma_\phi = \begin{pmatrix} 0 & -\sin\theta\cos\theta \\ \cot\theta & 0 \end{pmatrix} \tag{13.122}$$

$$\Gamma_{\phi,\theta} = \begin{pmatrix} 0 & \sin^2\theta - \cos^2\theta \\ -\csc^2\theta & 0 \end{pmatrix} \quad \text{and} \quad [\Gamma_\theta, \Gamma_\phi] = \begin{pmatrix} 0 & \cos^2\theta \\ \cot^2\theta & 0 \end{pmatrix}.$$

Both $[\Gamma_\theta, \Gamma_\theta]$ and $[\Gamma_\phi, \Gamma_\phi]$ vanish. So the commutator formula (13.103) gives for Riemann's curvature tensor

$$\begin{aligned} R^\theta{}_{\theta\theta\theta} &= [\partial_\theta + \Gamma_\theta, \partial_\theta + \Gamma_\theta]^\theta{}_\theta = 0 \\ R^\phi{}_{\theta\phi\theta} &= [\partial_\phi + \Gamma_\phi, \partial_\theta + \Gamma_\theta]^\phi{}_\theta = \left(\Gamma_{\theta,\phi}\right)^\phi{}_\theta + [\Gamma_\phi, \Gamma_\theta]^\phi{}_\theta = 1 \\ R^\theta{}_{\phi\theta\phi} &= [\partial_\theta + \Gamma_\theta, \partial_\phi + \Gamma_\phi]^\theta{}_\phi = -\left(\Gamma_{\theta,\phi}\right)^\theta{}_\phi + [\Gamma_\theta, \Gamma_\phi]^\theta{}_\phi = \sin^2\theta \\ R^\phi{}_{\phi\phi\phi} &= [\partial_\phi + \Gamma_\phi, \partial_\phi + \Gamma_\phi]^\phi{}_\phi = 0. \end{aligned} \tag{13.123}$$

The Ricci tensor (13.112) is the contraction $R_{mk} = R^n_{mnk}$, and so

$$\begin{aligned} R_{\theta\theta} &= R^\theta{}_{\theta\theta\theta} + R^\phi{}_{\theta\phi\theta} = 1 \\ R_{\phi\phi} &= R^\theta{}_{\phi\theta\phi} + R^\phi{}_{\phi\phi\phi} = \sin^2\theta. \end{aligned} \tag{13.124}$$

The curvature scalar (13.113) is the contraction $R = g^{km} R_{mk}$, and so since $g^{\theta\theta} = R^{-2}$ and $g^{\phi\phi} = R^{-2}\sin^{-2}\theta$, it is

$$R = g^{\theta\theta} R_{\theta\theta} + g^{\phi\phi} R_{\phi\phi} = R^{-2} + R^{-2} = \frac{2}{R^2} \tag{13.125}$$

for a 2-sphere of radius R. The scalar curvature is a constant because the sphere is a maximally symmetric space (section 13.24).

Gauss invented a formula for the curvature K of a surface; for all two-dimensional surfaces, his $K = R/2$. □

Example 13.15 (Curvature of a cylindrical hyperboloid) The points of a cylindrical hyperboloid in 3-space satisfy $R^2 = -x^2 - y^2 + z^2$ and may be parameterized as $p = r(\sinh\theta\cos\phi, \sinh\theta\sin\phi, \cosh\theta)$. The (orthogonal) coordinate basis vectors are

$$\begin{aligned} \boldsymbol{e}_\theta &= p_{,\theta} = r(\cosh\theta\cos\phi, \cosh\theta\sin\phi, \sinh\theta) \\ \boldsymbol{e}_\phi &= p_{,\phi} = r(-\sinh\theta\sin\phi, \sinh\theta\cos\phi, 0). \end{aligned} \tag{13.126}$$

The squared distance ds^2 between nearby points is

$$ds^2 = \boldsymbol{e}_\theta \cdot \boldsymbol{e}_\theta \, d\theta^2 + \boldsymbol{e}_\phi \cdot \boldsymbol{e}_\phi \, d\phi^2. \tag{13.127}$$

If the embedding metric is $m = \text{diag}(1, 1, -1)$, then ds^2 is

$$ds^2 = R^2\, d\theta^2 + R^2\, \sinh^2\theta\, d\phi^2 \tag{13.128}$$

and

$$(g_{ij}) = r^2 \begin{pmatrix} 1 & 0 \\ 0 & \sinh^2\theta \end{pmatrix}. \tag{13.129}$$

The Mathematica scripts GREAT.m and cylindrical_hyperboloid.nb compute the scalar curvature as $R = -2/r^2$. The surface is maximally symmetric with constant negative curvature. This chapter's programs and scripts are in Tensors_and_general_relativity at `github.com/kevinecahill`. □

Example 13.16 (Curvature of the sphere S^3) The three-dimensional sphere S^3 may be embedded isometrically in four-dimensional flat euclidian space $\mathbb{E}^4$ as the set of points $p = (x, y, z, w)$ that satisfy $L^2 = x^2 + y^2 + z^2 + w^2$. If we label its points as

$$p(\chi, \theta, \phi) = L(\sin\chi \sin\theta \cos\phi, \sin\chi \sin\theta \sin\phi, \sin\chi \cos\theta, \cos\chi), \tag{13.130}$$

then its coordinate basis vectors are

$$\begin{aligned} e_\chi &= p_{,\chi} = L(\cos\chi \sin\theta \cos\phi, \cos\chi \sin\theta \sin\phi, \cos\chi \cos\theta, -\sin\chi) \\ e_\theta &= p_{,\theta} = L(\sin\chi \cos\theta \cos\phi, \sin\chi \cos\theta \sin\phi, -\sin\chi \sin\theta, 0) \\ e_\phi &= p_{,\phi} = L(-\sin\chi \sin\theta \sin\phi, \sin\chi \sin\theta \cos\phi, 0, 0). \end{aligned} \tag{13.131}$$

The inner product of $\mathbb{E}^4$ is the four-dimensional dot-product. The basis vectors are orthogonal. In terms of the radial variable $r = L\sin\chi$, the squared distance ds^2 between two nearby points is

$$\begin{aligned} ds^2 &= e_\chi \cdot e_\chi d\chi^2 + e_\theta \cdot e_\theta d\theta^2 + e_\phi \cdot e_\phi d\phi^2 \\ &= L^2 \left(d\chi^2 + \sin^2\chi\, d\theta^2 + \sin^2\chi \sin^2\theta\, d\phi^2 \right) \\ &= \frac{dr^2}{1 - \sin^2\chi} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 = \frac{dr^2}{1 - (r/L)^2} + r^2 d\Omega^2 \end{aligned} \tag{13.132}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta\, d\phi^2$. In these coordinates, r, θ, ϕ, the metric is

$$g_{ik} = \begin{pmatrix} 1/(1 - (r/L)^2) & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2\theta \end{pmatrix}. \tag{13.133}$$

The Mathematica scripts GREAT.m and sphere_S3.nb compute the scalar curvature as

$$R = \frac{6}{L^2} \tag{13.134}$$

which is a constant because S^3 is maximally symmetric (section 13.24). □

Example 13.17 (Curvature of the hyperboloid H^3) The hyperboloid H^3 is a three-dimensional surface that can be isometrically embedded in the semi-euclidian spacetime $\mathbb{E}^{(1,3)}$ in which distances are $ds^2 = dx^2 + dy^2 + dz^2 - dw^2$, and w is a time coordinate. The points of H^3 satisfy $L^2 = -x^2 - y^2 - z^2 + w^2$. If we label them as

$$p(\chi, \theta, \phi) = L\left(\sinh\chi \sin\theta \cos\phi, \sinh\chi \sin\theta \sin\phi, \sinh\chi \cos\theta, \cosh\chi\right) \tag{13.135}$$

then the coordinate basis vectors or tangent vectors of H^3 are

$$\begin{aligned} e_\chi &= p_{,\chi} = L(\cosh\chi \sin\theta \cos\phi, \cosh\chi \sin\theta \sin\phi, \cosh\chi \cos\theta, \sinh\chi) \\ e_\theta &= p_{,\theta} = L(\sinh\chi \cos\theta \cos\phi, \sinh\chi \cos\theta \sin\phi, -\sinh\chi \sin\theta, 0) \\ e_\phi &= p_{,\phi} = L(-\sinh\chi \sin\theta \sin\phi, \sinh\chi \sin\theta \cos\phi, 0, 0). \end{aligned} \tag{13.136}$$

The basis vectors are orthogonal. In terms of the radial variable $r = L \sinh \chi / a$, the squared distance ds^2 between two nearby points is

$$\begin{aligned} ds^2 &= e_\chi \cdot e_\chi d\chi^2 + e_\theta \cdot e_\theta d\theta^2 + e_\phi \cdot e_\phi d\phi^2 \\ &= L^2\left(d\chi^2 + \sinh^2\chi \, d\theta^2 + \sinh^2\chi \sin^2\theta \, d\phi^2\right) \\ &= \frac{dr^2}{1 + \sinh^2\chi} + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2 = \frac{dr^2}{1 + (r/L)^2} + r^2 d\Omega^2. \end{aligned} \tag{13.137}$$

The Mathematica scripts GREAT.m and hyperboloid_H3.nb compute the scalar curvature of H^3 as

$$R = -\frac{6}{L^2}. \tag{13.138}$$

Its curvature is a constant because H^3 is maximally symmetric (section 13.24). The only maximally symmetric 3-dimensional manifolds are S^3, H^3, and euclidian space $\mathbb{E}^3$ whose line element is $ds^2 = dr^2 + r^2 d\Omega^2$.

They are the spatial parts of Friedmann-Lemaître-Robinson-Walker cosmologies (section 13.42). □

13.24 Maximally symmetric spaces

The spheres S^2 and S^3 (examples 13.4 & 13.16) and the hyperboloids H^2 and H^3 (examples 13.6 & 13.17) are maximally symmetric spaces. A space described by a metric $g_{ik}(x)$ is symmetric under a transformation $x \to x'$ if the distances $g_{ik}(x')dx'^i dx'^k$ and $g_{ik}(x)dx^i dx^k$ are the same. To see what this symmetry condition means, we consider the infinitesimal transformation $x'^\ell = x^\ell + \epsilon y^\ell(x)$ under which to lowest order $g_{ik}(x') = g_{ik}(x) + g_{ik,\ell}\epsilon y^\ell$ and $dx'^i = dx^i + \epsilon y^i_{,j} dx^j$. The symmetry condition requires

$$g_{ik}(x)dx^i dx^k = (g_{ik}(x) + g_{ik,\ell}\epsilon y^\ell)(dx^i + \epsilon y^i_{,j} dx^j)(dx^k + \epsilon y^k_{,m} dx^m) \quad (13.139)$$

or

$$0 = g_{ik,\ell}\, y^\ell + g_{im}\, y^m_{,k} + g_{jk}\, y^j_{,i}. \quad (13.140)$$

The vector field $y^i(x)$ must satisfy this condition if $x'^i = x^i + \epsilon y^i(x)$ is to be a symmetry of the metric $g_{ik}(x)$. By using the vanishing (13.83) of the covariant derivative of the metric tensor, we may write the condition on the symmetry vector $y^\ell(x)$ as (exercise 13.9)

$$0 = y_{i;k} + y_{k;i}. \quad (13.141)$$

The symmetry vector y^ℓ is a **Killing** vector (Wilhelm Killing, 1847–1923). We may use symmetry condition (13.140) or (13.141) either to find the symmetries of a space with a known metric or to find the metric with given symmetries.

Example 13.18 (Killing vectors of the sphere S^2) The first Killing vector is $(y_1^\theta, y_1^\phi) = (0, 1)$. Since the components of y_1 are constants, the symmetry condition (13.140) says $g_{ik,\phi} = 0$ which tells us that the metric is independent of ϕ. The other two Killing vectors are $(y_2^\theta, y_2^\phi) = (\sin\phi, \cot\theta\cos\phi)$ and $(y_3^\theta, y_3^\phi) = (\cos\phi, -\cot\theta\sin\phi)$. The symmetry condition (13.140) for $i = k = \theta$ and Killing vectors y_2 and y_3 tell us that $g_{\theta\phi} = 0$ and that $g_{\theta\theta,\theta} = 0$. So $g_{\theta\theta}$ is a constant, which we set equal to unity. Finally, the symmetry condition (13.140) for $i = k = \phi$ and the Killing vectors y_2 and y_3 tell us that $g_{\phi\phi,\theta} = 2\cot\theta g_{\phi\phi}$ which we integrate to $g_{\phi\phi} = \sin^2\theta$. The 2-dimensional space with Killing vectors y_1, y_2, y_3 therefore has the metric (13.115) of the sphere S^2. □

Example 13.19 (Killing vectors of the hyperboloid H^2) The metric (13.46) of the hyperboloid H^2 is diagonal with $g_{\theta\theta} = R^2$ and $g_{\phi\phi} = R^2\sinh^2\theta$. The Killing vector $(y_1^\theta, y_1^\phi) = (0, 1)$ satisfies the symmetry condition (13.140). Since $g_{\theta\theta}$ is independent of θ and ϕ, the $\theta\theta$ component of (13.140) implies that $y^\theta_{,\theta} = 0$. Since $g_{\phi\phi} = R^2\sinh^2\theta$, the $\phi\phi$ component of (13.140) says that $y^\phi_{,\phi} = -\coth\theta\, y^\theta$. The $\theta\phi$ and $\phi\theta$ components of (13.140) give

$y^\theta_{,\phi} = -\sinh^2\theta\, y^\phi_{,\theta}$. The vectors $y_2 = (y_2^\theta, y_2^\phi) = (\sin\phi, \coth\theta \sin\phi)$ and $y_3 = (y_3^\theta, y_3^\phi) = (\cos\phi, -\coth\theta\sin\phi)$ satisfy both of these equations. □

The **Lie derivative** $\mathcal{L}_y$ of a scalar field A is defined in terms of a vector field $y^\ell(x)$ as $\mathcal{L}_y A = y^\ell A_{,\ell}$. The Lie derivative $\mathcal{L}_y$ of a contravariant vector F^i is

$$\mathcal{L}_y F^i = y^\ell F^i_{,\ell} - F^\ell y^i_{,\ell} = y^\ell F^i_{;\ell} - F^\ell y^i_{;\ell} \tag{13.142}$$

in which the second equality follows from $y^\ell \Gamma^i_{\ell k} F^k = F^\ell \Gamma^i_{\ell k} y^k$. The Lie derivative $\mathcal{L}_y$ of a covariant vector V_i is

$$\mathcal{L}_y V_i = y^\ell V_{i,\ell} + V_\ell y^\ell_{,i} = y^\ell V_{i;\ell} + V_\ell y^\ell_{;i}. \tag{13.143}$$

Similarly, the Lie derivative $\mathcal{L}_y$ of a rank-2 covariant tensor T_{ik} is

$$\mathcal{L}_y T_{ik} = y^\ell T_{ik,\ell} + T_{\ell k} y^\ell_{,i} + T_{i\ell} y^\ell_{,k}. \tag{13.144}$$

We see now that the condition (13.140) that a vector field y^ℓ be a symmetry of a metric g_{jm} is that its Lie derivative

$$\mathcal{L}_y g_{ik} = g_{ik,\ell}\, y^\ell + g_{im}\, y^m_{,k} + g_{jk}\, y^j_{,i} = 0 \tag{13.145}$$

must vanish.

A maximally symmetric space (or spacetime) in d dimensions has d translation symmetries and $d(d-1)/2$ rotational symmetries which gives a total of $d(d+1)/2$ symmetries associated with $d(d+1)/2$ Killing vectors. Thus for $d = 2$, there is one rotation and two translations. For $d = 3$, there are three rotations and three translations. For $d = 4$, there are six rotations and four translations.

A maximally symmetric space has a curvature tensor (13.99) that is simply related to its metric tensor

$$R_{ijk\ell} = c\,(g_{ik} g_{j\ell} - g_{i\ell} g_{jk}) \tag{13.146}$$

where c is a constant (Zee, 2013, IX.6). Since $g^{ki} g_{ik} = g^k_k = d$ is the number of dimensions of the space(time), the Ricci tensor (13.112) and the curvature scalar (13.113) of a maximally symmetric space are

$$R_{j\ell} = g^{ki} R_{ijk\ell} = c\,(d-1)\, g_{j\ell} \quad \text{and} \quad R = g^{\ell j} R_{j\ell} = c\, d(d-1). \tag{13.147}$$

13.25 Principle of equivalence

Since the metric tensor $g_{ij}(x)$ is real and symmetric, it can be diagonalized at any point $p(x)$ by a 4×4 orthogonal matrix $O(x)$

$$O^{\mathsf{T}}{}_i{}^k \, g_{k\ell} \, O^\ell{}_j = \begin{pmatrix} e_0 & 0 & 0 & 0 \\ 0 & e_1 & 0 & 0 \\ 0 & 0 & e_2 & 0 \\ 0 & 0 & 0 & e_3 \end{pmatrix} \tag{13.148}$$

which arranges the four real eigenvalues e_i of the matrix $g_{ij}(x)$ in the order $e_0 \le e_1 \le e_2 \le e_3$. Thus the coordinate transformation

$$\frac{\partial x^k}{\partial x'^i} = \frac{O^{\mathsf{T}}{}_i{}^k}{\sqrt{|e_i|}} \tag{13.149}$$

takes any spacetime metric $g_{k\ell}(x)$ with one negative and three positive eigenvalues into the Minkowski metric η_{ij} of flat spacetime

$$g_{k\ell}(x) \, \frac{\partial x^k}{\partial x'^i} \, \frac{\partial x^\ell}{\partial x'^j} = g'_{ij}(x') = \eta_{ij} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{13.150}$$

at the point $p(x) = p(x')$.

The **principle of equivalence** says that in these free-fall coordinates x', the physical laws of gravity-free special relativity apply in a suitably small region about the point $p(x) = p(x')$. It follows from this principle that the metric g_{ij} of spacetime accounts for all the effects of gravity.

In the x' coordinates, the invariant squared separation dp^2 is

$$\begin{aligned} dp^2 = g'_{ij} \, dx'^i dx'^j &= e'_i(x') \cdot e'_j(x') \, dx'^i dx'^j \\ &= e'^a_i(x') \eta_{ab} e'^b_j(x') \, dx'^i dx'^j = \delta^a_i \eta_{ab} \delta^b_j \, dx'^i dx'^j \\ &= \eta_{ij} \, dx'^i dx'^j = (\boldsymbol{dx'})^2 - (dx'^0)^2 = ds^2. \end{aligned} \tag{13.151}$$

If $\boldsymbol{dx'} = 0$, then $dt' = \sqrt{-ds^2}/c$ is the **proper time** elapsed between events p and $p + dp$. If $dt' = 0$, then ds is the **proper distance** between the events.

The x' coordinates are not unique because every Lorentz transformation (section 12.1) leaves the metric η invariant. Coordinate systems in which $g_{ij}(x') = \eta_{ij}$ are called **Lorentz**, **inertial**, or **free-fall** coordinate systems.

The congruency transformation (1.351 & 13.148–13.150) preserves the signs of the eigenvalues e_i which make up the **signature** $(-1, 1, 1, 1)$ of the metric tensor.

13.26 Tetrads

We defined the metric tensor as the dot product (13.34) or (13.42) of tangent vectors, $g_{k\ell}(x) = e_k(x) \cdot e_\ell(x)$. If instead we invert the equation (13.150) that relates the metric tensor to the flat metric

$$g_{k\ell}(x) = \frac{\partial x'^a}{\partial x^k}\, \eta_{ab}\, \frac{\partial x'^b}{\partial x^\ell} \tag{13.152}$$

then we can express the metric in terms of four 4-vectors

$$c^a_k(x) = \frac{\partial x'^a}{\partial x^k} \quad \text{as} \quad g_{k\ell}(x) = c^a_k(x)\, \eta_{ab}\, c^b_\ell(x) \tag{13.153}$$

in which η_{ij} is the 4×4 metric (13.150) of flat Minkowski space. Cartan's four 4-vectors $c^a_i(x)$ are called a **moving frame**, a **tetrad**, and a **vierbein**.

Because $L^a{}_c(x)\, \eta_{ab}\, L^b{}_d(x) = \eta_{cd}$, every spacetime-dependent Lorentz transformation $L(x)$ maps one set of tetrads $c^c_k(x)$ to another set of tetrads $c'^a_k(x) = L^a{}_c(x)\, c^c_k(x)$ that represent the same metric

$$\begin{aligned} c'^a_k(x)\, \eta_{ab}\, c'^b_\ell(x) &= L^a{}_c(x)\, c^c_k(x)\, \eta_{ab}\, L^b{}_d(x)\, c^d_\ell(x) \\ &= c^c_k(x)\, \eta_{cd}\, c^d_\ell(x) = g_{k\ell}(x). \end{aligned} \tag{13.154}$$

Cartan's tetrad is four 4-vectors c_i that give the metric tensor as $g_{ik} = c_i \cdot c_k = \vec{c}_i \cdot \vec{c}_k - c^0_i c^0_k$. The dual tetrads $c^i_a = g^{ik} \eta_{ab} c^b_k$ satisfy

$$c^i_a c^a_k = \delta^i_k \quad \text{and} \quad c^i_a c^b_i = \delta^b_a. \tag{13.155}$$

The metric $g_{k\ell}(x)$ is symmetric, $g_{k\ell}(x) = g_{\ell k}(x)$, so it has 10 independent components at each spacetime point x. The four 4-vectors c^a_k have 16 components, but a Lorentz transformation $L(x)$ has 6 components. So the tetrads have $16 - 6 = 10$ independent components at each spacetime point.

The distinction between tetrads and tangent basis vectors is that each tetrad $c^a{}_k$ has 4 components, $a = 0, 1, 2, 3$, while each basis vector $e^\alpha{}_k(x)$ has as many components $\alpha = 0, 1, 2, 3, \ldots$ as there are dimensions in the minimal semi-euclidian embedding space $\mathbb{E}^{1,n}$ where $n \le 19$ (Aké et al., 2018). Both represent the metric

$$g_{k\ell}(x) = \sum_{a,b=0}^{3} c^a{}_k(x)\, \eta_{ab}\, c^b{}_\ell(x) = \sum_{\alpha,\beta=0}^{n} e^\alpha{}_k(x)\, \eta'_{\alpha\beta}\, e^\beta{}_\ell(x) \tag{13.156}$$

in which η' is like η but with n diagonal elements that are unity. (Élie Cartan, 1869–1951)

13.27 Scalar densities and $g = |\det(g_{ik})|$

Let g be the absolute value of the determinant of the metric tensor g_{ik}

$$g = g(x) = |\det(g_{ik}(x))|. \tag{13.157}$$

Under a coordinate transformation, $\sqrt{g}$ becomes

$$\sqrt{g'} = \sqrt{g'(x')} = \sqrt{|\det(g'_{ik}(x'))|} = \sqrt{\left|\det\left(\frac{\partial x^j}{\partial x'^i}\frac{\partial x^\ell}{\partial x'^k}\, g_{j\ell}(x)\right)\right|}. \tag{13.158}$$

The definition (1.204) of a determinant and the product rule (1.225) for determinants tell us that

$$\sqrt{g'(x')} = \sqrt{\left|\det\left(\frac{\partial x^j}{\partial x'^i}\right)\det\left(\frac{\partial x^\ell}{\partial x'^k}\right)\det(g_{j\ell})\right|} = |J(x/x')|\sqrt{g(x)} \tag{13.159}$$

where $J(x/x')$ is the jacobian (section 1.21) of the coordinate transformation

$$J(x/x') = \det\left(\frac{\partial x^j}{\partial x'^i}\right). \tag{13.160}$$

A quantity $s(x)$ is a **scalar density** of weight w if it transforms as

$$s'(x') = [J(x'/x)]^w s(x). \tag{13.161}$$

Thus the transformation rule (13.159) says that the determinant $\det(g_{ik})$ is a scalar density of weight minus two

$$\det(g'_{ik}(x')) = \, = [J(x/x')]^2 g(x) = [J(x'/x)]^{-2}\det(g_{j\ell}(x)). \tag{13.162}$$

We saw in section 1.21 that under a coordinate transformation $x \to x'$ the d-dimensional element of volume in the new coordinates d^dx' is related to that in the old coordinates d^dx by a jacobian

$$d^dx' = J(x'/x)\, d^dx = \det\left(\frac{\partial x'^i}{\partial x^j}\right) d^dx. \tag{13.163}$$

Thus the product $\sqrt{g}\; d^dx$ changes at most by the sign of the jacobian $J(x'/x)$ when $x \to x'$

$$\sqrt{g'}\, d^dx' = |J(x/x')|\, J(x'/x)\,\sqrt{g(x)}\, d^dx = \pm\sqrt{g(x)}\, d^dx. \tag{13.164}$$

The quantity $\sqrt{g}\ d^4x$ is the invariant scalar $\sqrt{g}\ |d^4x|$ so that if $L(x)$ is a scalar, then the integral over spacetime

$$\int L(x) \sqrt{g}\ d^4x \tag{13.165}$$

is invariant under general coordinate transformations. The Levi-Civita tensor provides a fancier definition.

13.28 Levi-Civita's symbol and tensor

In 3 dimensions, Levi-Civita's **symbol** $\epsilon_{ijk} \equiv \epsilon^{ijk}$ is totally antisymmetric with $\epsilon_{123} = 1$ in all coordinate systems. In 4 space or spacetime dimensions, Levi-Civita's **symbol** $\epsilon_{ijk\ell} \equiv \epsilon^{ijk\ell}$ is totally antisymmetric with $\epsilon_{1234} = 1$ or equivalently with $\epsilon_{0123} = 1$ in all coordinate systems. In n dimensions, Levi-Civita's symbol $\epsilon_{i_1 i_2 \dots i_n}$ is totally antisymmetric with $\epsilon_{123\dots n} = 1$ or $\epsilon_{012\dots n-1} = 1$.

We can turn his symbol into a pseudotensor by multiplying it by the square root of the absolute value of the determinant of a rank-2 covariant tensor. A natural choice is the metric tensor. In a right-handed coordinate system in which the tangent vector e_0 points (orthochronously) toward the future, the Levi-Civita **tensor** $\eta_{ijk\ell}$ is the totally antisymmetric rank-4 covariant tensor

$$\eta_{ijk\ell}(x) = \sqrt{g(x)}\ \epsilon_{ijk\ell} \tag{13.166}$$

in which $g(x) = |\det g_{mn}(x)|$ is (13.157) the absolute value of the determinant of the metric tensor g_{mn}. In a different system of coordinates x', the Levi-Civita tensor $\eta_{ijk\ell}(x')$ differs from (13.166) by the sign s of the jacobian $J(x'/x)$ of any coordinate transformation to x' from a right-handed, orthochronous coordinate system x

$$\eta_{ijk\ell}(x') = s(x')\ \sqrt{g(x')}\ \epsilon_{ijk\ell}. \tag{13.167}$$

The transformation rule (13.159) and the definition (1.204) and product rule (1.225) of determinants show that $\eta_{ijk\ell}$ transforms as a rank-4 covariant tensor

$$\begin{aligned} \eta'_{ijk\ell}(x') &= s(x')\ \sqrt{g'(x')}\, \epsilon_{ijk\ell} = s(x')\ |J(x/x')| \sqrt{g(x)}\, \epsilon_{ijk\ell} \\ &= J(x/x') \sqrt{g(x)}\, \epsilon_{ijk\ell} = \det\left(\frac{\partial x}{\partial x'}\right) \sqrt{g}\, \epsilon_{ijk\ell} \\ &= \frac{\partial x^t}{\partial x'^i} \frac{\partial x^u}{\partial x'^j} \frac{\partial x^v}{\partial x'^k} \frac{\partial x^w}{\partial x'^\ell} \sqrt{g}\, \epsilon_{tuvw} = \frac{\partial x^t}{\partial x'^i} \frac{\partial x^u}{\partial x'^j} \frac{\partial x^v}{\partial x'^k} \frac{\partial x^w}{\partial x'^\ell}\, \eta_{tuvw}. \end{aligned} \tag{13.168}$$

Raising the indices of η, we have using σ as the sign of $\det(g_{ik})$

$$\begin{aligned}\eta^{ijk\ell} &= g^{it}\, g^{ju}\, g^{kv}\, g^{\ell w}\, \eta_{tuvw} = g^{it}\, g^{ju}\, g^{kv}\, g^{\ell w}\, \sqrt{g}\, \epsilon_{tuvw} = \sqrt{g}\, \epsilon_{ijk\ell} \det(g^{mn}) \\ &= \sqrt{g}\, \epsilon_{ijk\ell}/\det(g_{mn}) = \sigma\, \epsilon_{ijk\ell}/\sqrt{g} \equiv \sigma\, \epsilon^{ijk\ell}/\sqrt{g}. \end{aligned} \tag{13.169}$$

In terms of the Hodge star (14.151), the invariant volume element is

$$\sqrt{g}\, |d^4x| = \,*\,1 = \frac{1}{4!}\, \eta_{ijk\ell}\; dx^i \wedge dx^j \wedge dx^k \wedge dx^\ell. \tag{13.170}$$

13.29 Divergence of a contravariant vector

The contracted covariant derivative of a contravariant vector is a **scalar** known as the **divergence**,

$$\nabla \cdot V = V^i_{;i} = V^i_{,i} + V^k\, \Gamma^i_{ki}. \tag{13.171}$$

Because $g_{ik} = g_{ki}$, in the sum over i of the connection (13.59)

$$\Gamma^i{}_{ki} = \tfrac{1}{2} g^{i\ell} \left(g_{i\ell,k} + g_{\ell k,i} - g_{ki,\ell} \right) \tag{13.172}$$

the last two terms cancel because they differ only by the interchange of the dummy indices i and ℓ

$$g^{i\ell} g_{\ell k,i} = g^{\ell i} g_{ik,\ell} = g^{i\ell} g_{ki,\ell}. \tag{13.173}$$

So the contracted connection collapses to

$$\Gamma^i{}_{ki} = \tfrac{1}{2} g^{i\ell} g_{i\ell,k}. \tag{13.174}$$

There is a nice formula for this last expression. To derive it, let $\underline{g} \equiv g_{i\ell}$ be the 4×4 matrix whose elements are those of the covariant metric tensor $g_{i\ell}$. Its determinant, like that of any matrix, is the cofactor sum (1.213) along any row or column, that is, over ℓ for fixed i or over i for fixed ℓ

$$\det(\underline{g}) = \sum_{i \text{ or } \ell} g_{i\ell}\, C_{i\ell} \tag{13.175}$$

in which the cofactor $C_{i\ell}$ is $(-1)^{i+\ell}$ times the determinant of the reduced matrix consisting of the matrix $\underline{g}$ with row i and column ℓ omitted. Thus the partial derivative of $\det \underline{g}$ with respect to the $i\ell$th element $g_{i\ell}$ is

$$\frac{\partial \det(\underline{g})}{\partial g_{i\ell}} = C_{i\ell} \tag{13.176}$$

in which we allow $g_{i\ell}$ and $g_{\ell i}$ to be independent variables for the purposes of this differentiation. The inverse $g^{i\ell}$ of the metric tensor $\underline{g}$, like the inverse (1.215) of any matrix, is the transpose of the cofactor matrix divided by its determinant $\det(\underline{g})$

$$g^{i\ell} = \frac{C_{\ell i}}{\det(\underline{g})} = \frac{1}{\det(\underline{g})} \frac{\partial \det(\underline{g})}{\partial g_{\ell i}}. \tag{13.177}$$

Using this formula and the chain rule, we may write the derivative of the determinant $\det(\underline{g})$ as

$$\det(\underline{g})_{,k} = \frac{\partial \det(\underline{g})}{\partial g_{i\ell}} \, g_{i\ell,k} = \det(\underline{g}) \, g^{\ell i} \, g_{i\ell,k} \tag{13.178}$$

and so since $g_{i\ell} = g_{\ell i}$, the contracted connection (13.174) is

$$\Gamma^i{}_{ki} = \tfrac{1}{2} g^{i\ell} g_{i\ell,k} = \frac{\det(\underline{g})_{,k}}{2\det(\underline{g})} = \frac{|\det(\underline{g})|_{,k}}{2|\det(\underline{g})|} = \frac{g_{,k}}{2g} = \frac{(\sqrt{g})_{,k}}{\sqrt{g}} \tag{13.179}$$

in which $g \equiv \left|\det(\underline{g})\right|$ is the absolute value of the determinant of the metric tensor.

Thus from (13.171 & 13.179), we arrive at our formula for the covariant divergence of a contravariant vector:

$$\nabla \cdot V = V^i_{;i} = V^i_{,i} + \Gamma^i_{ki} V^k = V^k_{,k} + \frac{(\sqrt{g})_{,k}}{\sqrt{g}} V^k = \frac{(\sqrt{g}\, V^k)_{,k}}{\sqrt{g}}. \tag{13.180}$$

Example 13.20 (Maxwell's inhomogeneous equations) An important application of this divergence formula (13.180) is the generally covariant form (14.157) of Maxwell's inhomogeneous equations

$$\frac{1}{\sqrt{g}} \left(\sqrt{g} F^{k\ell} \right)_{,\ell} = \mu_0 j^k. \tag{13.181}$$

□

Example 13.21 (Energy-momentum tensor) Another application is to the divergence of the symmetric energy-momentum tensor $T^{ij} = T^{ji}$

$$\begin{aligned} T^{ij}_{;i} &= T^{ij}_{,i} + \Gamma^i{}_{ki} T^{kj} + \Gamma^j{}_{mi} T^{im} \\ &= \frac{(\sqrt{g} T^{kj})_k}{\sqrt{g}} + \Gamma^j{}_{mi} T^{im}. \end{aligned} \tag{13.182}$$

□

13.30 Covariant laplacian

In flat 3-space, we write the laplacian as $\nabla \cdot \nabla = \nabla^2$ or as $\triangle$. In euclidian coordinates, both mean $\partial_x^2 + \partial_y^2 + \partial_z^2$. In flat minkowski space, one often turns the triangle into a square and writes the 4-laplacian as $\Box = \triangle - \partial_0^2$.

The gradient $f_{,k}$ of a scalar field f is a covariant vector, and $f^{,i} = g^{ik} f_{,k}$ is its contravariant form. The **invariant laplacian** $\Box f$ of a scalar field f is the covariant divergence $f^{,i}_{;i}$. We may use our formula (13.180) for the divergence of a contravariant vector to write it in these equivalent ways

$$\Box f = f^{,i}_{;i} = (g^{ik} f_{,k})_{;i} = \frac{(\sqrt{g}\, f^{,i})_{,i}}{\sqrt{g}} = \frac{(\sqrt{g}\, g^{ik} f_{,k})_{,i}}{\sqrt{g}}. \tag{13.183}$$

13.31 Principle of stationary action in general relativity

The invariant proper time for a particle to move along a path $x^i(t)$

$$T = \int_{\tau_1}^{\tau_2} d\tau = \frac{1}{c} \int \Big(- g_{i\ell} dx^i dx^\ell \Big)^{\frac{1}{2}} \tag{13.184}$$

is extremal and stationary on free-fall paths called **geodesics**. We can identify a geodesic by computing the variation $\delta d\tau$

$$\begin{aligned} c\delta d\tau = \delta \sqrt{-g_{i\ell} dx^i dx^\ell} &= \frac{-\delta(g_{i\ell}) dx^i dx^\ell - 2 g_{i\ell} dx^i \delta dx^\ell}{2\sqrt{-g_{i\ell} dx^i dx^\ell}} \\ &= -\frac{g_{i\ell,k}}{2c} \delta x^k u^i u^\ell d\tau - \frac{g_{i\ell}}{c} u^i \delta dx^\ell = -\frac{g_{i\ell,k}}{2c} \delta x^k u^i u^\ell d\tau - \frac{g_{i\ell}}{c} u^i d\delta x^\ell \end{aligned} \tag{13.185}$$

in which $u^\ell = dx^\ell/d\tau$ is the 4-velocity (12.20). The path is extremal if

$$0 = c\delta T = c \int_{\tau_1}^{\tau_2} \delta d\tau = -\frac{1}{c} \int_{\tau_1}^{\tau_2} \left(\tfrac{1}{2} g_{i\ell,k} \delta x^k u^i u^\ell + g_{i\ell} u^i \frac{d\delta x^\ell}{d\tau} \right) d\tau \tag{13.186}$$

which we integrate by parts keeping in mind that $\delta x^\ell(\tau_2) = \delta x^\ell(\tau_1) = 0$

$$\begin{aligned} 0 &= -\int_{\tau_1}^{\tau_2} \left(\tfrac{1}{2} g_{i\ell,k} \delta x^k u^i u^\ell - \frac{d(g_{i\ell} u^i)}{d\tau} \delta x^\ell \right) d\tau \\ &= -\int_{\tau_1}^{\tau_2} \left(\tfrac{1}{2} g_{i\ell,k} \delta x^k u^i u^\ell - g_{i\ell,k} u^i u^k \delta x^\ell - g_{i\ell} \frac{du^i}{d\tau} \delta x^\ell \right) d\tau. \end{aligned} \tag{13.187}$$

Now interchanging the dummy indices ℓ and k on the second and third terms, we have

$$0 = -\int_{\tau_1}^{\tau_2} \left(\tfrac{1}{2} g_{i\ell,k} u^i u^\ell - g_{ik,\ell} u^i u^\ell - g_{ik} \frac{du^i}{d\tau} \right) \delta x^k d\tau \tag{13.188}$$

or since δx^k is arbitrary

$$0 = \tfrac{1}{2} g_{i\ell,k} u^i u^\ell - g_{ik,\ell} u^i u^\ell - g_{ik} \frac{du^i}{d\tau}. \tag{13.189}$$

If we multiply this equation of motion by g^{rk} and note that $g_{ik,\ell} u^i u^\ell = g_{\ell k,i} u^i u^\ell$, then we find

$$0 = \frac{du^r}{d\tau} + \tfrac{1}{2} g^{rk} \left(g_{ik,\ell} + g_{\ell k,i} - g_{i\ell,k} \right) u^i u^\ell. \tag{13.190}$$

So using the symmetry $g_{i\ell} = g_{\ell i}$ and the formula (13.87) for $\Gamma^r{}_{i\ell}$, we get

$$0 = \frac{du^r}{d\tau} + \Gamma^r{}_{i\ell}\, u^i u^\ell \qquad \text{or} \qquad 0 = \frac{d^2 x^r}{d\tau^2} + \Gamma^r{}_{i\ell} \frac{dx^i}{d\tau} \frac{dx^\ell}{d\tau} \tag{13.191}$$

which is the geodesic equation. In empty space, particles fall along **geodesics independently of their masses**.

One gets the same geodesic equation from the simpler action principle

$$0 = \delta \int_{\lambda_1}^{\lambda_2} g_{i\ell}(x) \frac{dx^i}{d\lambda} \frac{dx^\ell}{d\lambda} \, d\lambda \implies 0 = \frac{d^2 x^r}{d\lambda^2} + \Gamma^r{}_{i\ell} \frac{dx^i}{d\lambda} \frac{dx^\ell}{d\lambda}. \tag{13.192}$$

The right-hand side of the geodesic equation (13.191) is a contravariant vector because (Weinberg, 1972) under general coordinate transformations, the inhomogeneous terms arising from $\ddot{x}^r$ cancel those from $\Gamma^r{}_{i\ell} \dot{x}^i \dot{x}^\ell$. Here and often in what follows we'll use dots to mean proper-time derivatives.

The action for a particle of mass m and charge q in a gravitational field $\Gamma^r{}_{i\ell}$ and an electromagnetic field A_i is

$$S = -mc \int \Big(- g_{i\ell} dx^i dx^\ell \Big)^{\frac{1}{2}} + \frac{q}{c} \int_{\tau_1}^{\tau_2} A_i(x)\, dx^i \tag{13.193}$$

in which the interaction $q \int A_i dx^i$ is invariant under general coordinate transformations. By (12.59 & 13.188), the first-order change in S is

$$\delta S = m \int_{\tau_1}^{\tau_2} \left[\tfrac{1}{2} g_{i\ell,k} u^i u^\ell - g_{ik,\ell} u^i u^\ell - g_{ik} \frac{du^i}{d\tau} + \frac{q}{mc} \left(A_{i,k} - A_{k,i} \right) u^i \right] \delta x^k d\tau \tag{13.194}$$

and so by combining the Lorentz force law (12.60) and the geodesic equation (13.191) and by writing $F^{ri} \dot{x}_i$ as $F^r{}_i \, \dot{x}^i$, we have

$$0 = \frac{d^2 x^r}{d\tau^2} + \Gamma^r{}_{i\ell} \frac{dx^i}{d\tau} \frac{dx^\ell}{d\tau} - \frac{q}{m} F^r{}_i \frac{dx^i}{d\tau} \tag{13.195}$$

as the equation of motion of a particle of mass m and charge q. It is striking how nearly perfect the electromagnetism of Faraday and Maxwell is.

The action of the electromagnetic field interacting with an electric current j^k in a gravitational field is

$$S = \int \left[-\tfrac{1}{4} F_{k\ell} F^{k\ell} + \mu_0 A_k j^k \right] \sqrt{g}\, d^4x \tag{13.196}$$

in which $\sqrt{g}\, d^4x$ is the invariant volume element. After an integration by parts, the first-order change in the action is

$$\delta S = \int \left[-\frac{\partial}{\partial x^\ell} \left(F^{k\ell} \sqrt{g} \right) + \mu_0 j^k \sqrt{g} \right] \delta A_k \, d^4x, \tag{13.197}$$

and so the inhomogeneous Maxwell equations in a gravitational field are

$$\frac{\partial}{\partial x^\ell} \left(\sqrt{g}\, F^{k\ell} \right) = \mu_0 \sqrt{g}\, j^k. \tag{13.198}$$

The action of a scalar field ϕ of mass m in a gravitational field is

$$S = \frac{1}{2} \int \left(-\phi_{,i}\, g^{ik} \phi_{,k} - m^2 \phi^2 \right) \sqrt{g}\, d^4x. \tag{13.199}$$

After an integration by parts, the first-order change in the action is

$$\delta S = \int \delta\phi \left[\left(\sqrt{g}\, g^{ik} \phi_{,k} \right)_{,i} - m^2 \sqrt{g}\, \phi \right] d^4x \tag{13.200}$$

which yields the equation of motion

$$\left(\sqrt{g}\, g^{ik} \phi_{,k} \right)_{,i} - m^2 \sqrt{g}\, \phi = 0. \tag{13.201}$$

The action of the gravitational field itself is a spacetime integral of the Riemann scalar (13.113) divided by Newton's constant

$$S = \frac{c^3}{16\pi G} \int R \sqrt{g}\; d^4x. \tag{13.202}$$

Its variation leads to Einstein's equations (section 13.35).

13.32 Equivalence principle and geodesic equation

The **principle of equivalence** (section 13.25) says that in any gravitational field, one may choose **free-fall coordinates** in which all physical laws take the same form as in special relativity without acceleration or gravitation—at least over a suitably small volume of spacetime. Within this volume and in these coordinates, things behave as they would at rest deep in empty space far from any matter or energy. The volume must be small enough so that the gravitational field is constant throughout it. Such free-fall coordinate systems are called **local Lorentz frames** and **local inertial frames**.

Example 13.22 (Elevators) When a modern elevator starts going down from a high floor, it accelerates downward at something less than the local acceleration of gravity. One feels less pressure on one's feet; one feels lighter. After accelerating downward for a few seconds, the elevator assumes a constant downward speed, and then one feels the normal pressure of one's weight on one's feet. The elevator seems to be slowing down for a stop, but actually it has just stopped accelerating downward.

What if the cable snapped, and a frightened passenger dropped his laptop? He could catch it very easily as it would not seem to fall because the elevator, the passenger, and the laptop would all fall at the same rate. The physics in the falling elevator would be the same as if the elevator were at rest in empty space far from any gravitational field. The laptop's clock would tick as fast as it would at rest in the absence of gravity, but to an observer on the ground it would appear slower.

What if a passenger held an electric charge? Observers in the falling elevator would see a static electric field around the charge, but observers on the ground could detect radiation from the accelerating charge. □

Example 13.23 (Proper time) If the events are the ticks of a clock, then the proper time between ticks $d\tau/c$ is the time between the ticks of the clock at rest or at speed zero if the clock is accelerating. The proper lifetime $d\tau_\ell/c$ of an unstable particle is the average time it takes to decay at speed zero. In arbitrary coordinates, this proper lifetime is

$$c^2 d\tau_\ell^2 = -ds^2 = -g_{ik}(x)\, dx^i dx^k. \tag{13.203}$$

□

Example 13.24 (Clock hypothesis) The apparent lifetime of an unstable particle is independent of the acceleration of the particle even when the particle is subjected to centripetal accelerations of 10^{19} m/s^2(Bailey et al., 1977) and to longitudinal accelerations of 10^{16}m/s^2(Roos et al., 1980). □

The transformation from arbitrary coordinates x^k to free-fall coordinates y^i changes the metric $g_{j\ell}$ to the diagonal metric η_{ik} of flat spacetime $\eta =$

diag$(-1,1,1,1)$, which has two indices and is not a Levi-Civita tensor. Algebraically, this transformation is a congruence (1.353)

$$\eta_{ik} = \frac{\partial x^j}{\partial y^i}\, g_{j\ell}\, \frac{\partial x^\ell}{\partial y^k}. \tag{13.204}$$

The geodesic equation (13.191) follows from the **principle of equivalence** (Weinberg, 1972; Hobson et al., 2006). Suppose a particle is moving under the influence of gravitation alone. Then one may choose free-fall coordinates $y(x)$ so that the particle obeys the force-free equation of motion

$$\frac{d^2 y^i}{d\tau^2} = 0 \tag{13.205}$$

with $d\tau$ the proper time $d\tau^2 = -\eta_{ik}\, dy^i dy^k$. The chain rule applied to $y^i(x)$ in (13.205) gives

$$\begin{aligned} 0 &= \frac{d}{d\tau}\left(\frac{\partial y^i}{\partial x^k}\frac{dx^k}{d\tau}\right) \\ &= \frac{\partial y^i}{\partial x^k}\frac{d^2 x^k}{d\tau^2} + \frac{\partial^2 y^i}{\partial x^k \partial x^\ell}\frac{dx^k}{d\tau}\frac{dx^\ell}{d\tau}. \end{aligned} \tag{13.206}$$

We multiply by $\partial x^m/\partial y^i$ and use the identity

$$\frac{\partial x^m}{\partial y^i}\frac{\partial y^i}{\partial x^k} = \delta^m_k \tag{13.207}$$

to write the equation of motion (13.205) in the x-coordinates

$$\frac{d^2 x^m}{d\tau^2} + \Gamma^m{}_{k\ell}\,\frac{dx^k}{d\tau}\frac{dx^\ell}{d\tau} = 0. \tag{13.208}$$

This is the geodesic equation (13.191) in which the affine connection is

$$\Gamma^m{}_{k\ell} = \frac{\partial x^m}{\partial y^i}\frac{\partial^2 y^i}{\partial x^k \partial x^\ell}. \tag{13.209}$$

13.33 Weak static gravitational fields

Newton's equations describe slow motion in a weak static gravitational field. Because the motion is slow, we neglect u^i compared to u^0 and simplify the geodesic equation (13.191) to

$$0 = \frac{du^r}{d\tau} + \Gamma^r{}_{00}\,(u^0)^2. \tag{13.210}$$

Because the gravitational field is static, we neglect the time derivatives $g_{k0,0}$ and $g_{0k,0}$ in the connection formula (13.87) and find for $\Gamma^r{}_{00}$

$$\Gamma^r{}_{00} = \tfrac{1}{2}\, g^{rk}\, \left(g_{0k,0} + g_{0k,0} - g_{00,k}\right) = -\tfrac{1}{2}\, g^{rk}\, g_{00,k} \tag{13.211}$$

with $\Gamma^0{}_{00} = 0$. Because the field is weak, the metric can differ from η_{ij} by only a tiny tensor $g_{ij} = \eta_{ij} + h_{ij}$ so that to first order in $|h_{ij}| \ll 1$ we have $\Gamma^r{}_{00} = -\frac{1}{2}\, h_{00,r}$ for $r = 1, 2, 3$. With these simplifications, the geodesic equation (13.191) reduces to

$$\frac{d^2x^r}{d\tau^2} = \tfrac{1}{2}\,(u^0)^2\, h_{00,r} \qquad \text{or} \qquad \frac{d^2x^r}{d\tau^2} = \frac{1}{2}\left(\frac{dx^0}{d\tau}\right)^2 h_{00,r}. \tag{13.212}$$

So for slow motion, the ordinary acceleration is described by Newton's law

$$\frac{d^2\boldsymbol{x}}{dt^2} = \frac{c^2}{2}\,\nabla h_{00}. \tag{13.213}$$

If ϕ is his potential, then for slow motion in weak static fields

$$g_{00} = -1 + h_{00} = -1 - 2\phi/c^2 \qquad \text{and so} \qquad h_{00} = -\,2\phi/c^2. \tag{13.214}$$

Thus, if the particle is at a distance r from a mass M, then $\phi = -\,GM/r$ and $h_{00} = -2\phi/c^2 = 2GM/rc^2$ and so

$$\frac{d^2\boldsymbol{x}}{dt^2} = -\,\nabla\phi = \nabla\frac{GM}{r} = -\,GM\frac{\boldsymbol{r}}{r^3}. \tag{13.215}$$

How weak are the static gravitational fields we know about? The dimensionless ratio ϕ/c^2 is 10^{-39} on the surface of a proton, 10^{-9} on the Earth, 10^{-6} on the surface of the sun, and 10^{-4} on the surface of a white dwarf.

13.34 Gravitational time dilation

The proper time (example 13.23) interval $d\tau$ of a clock at rest in the weak, static gravitational field (13.210–13.215) satisfies equation (13.203)

$$c^2 d\tau^2 = -\,ds^2 = -g_{ik}(x)\, dx^i dx^k = -g_{00}\, c^2 dt^2 = (1 + 2\phi/c^2)c^2 dt^2. \tag{13.216}$$

So if two clocks are at rest at distances r and $r + h$ from the center of the Earth, then the times dt_r and dt_{r+h} between ticks of the clock at r and the one at $r + h$ are related by the proper time $d\tau$ of the ticks of the clock

$$(c^2 + 2\phi(r))dt_r^2 = c^2 d\tau^2 = (c^2 + 2\phi(r + h))dt_{r+h}^2. \tag{13.217}$$

Since $\phi(r) = -GM/r$, the potential at $r+h$ is $\phi(r+h) \approx \phi(r) + gh$, and so the ratio of the tick time dt_r of the lower clock at r to the tick time of the upper clock at $r+h$ is

$$\frac{dt_r}{dt_{r+h}} = \frac{\sqrt{c^2 + 2\phi(r+h)}}{\sqrt{c^2 + 2\phi(r)}} = \frac{\sqrt{c^2 + 2\phi(r) + 2gh}}{\sqrt{c^2 + 2\phi(r)}} \approx 1 + \frac{gh}{c^2}. \tag{13.218}$$

The lower clock is slower.

Example 13.25 (Pound and Rebka) Pound and Rebka in 1960 used the Mössbauer effect to measure the blue shift of light falling down a 22.6 m shaft. They found $(\nu_\ell - \nu_u)/\nu = gh/c^2 = 2.46 \times 10^{-15}$. (Robert Pound 1919–2010, Glen Rebka 1931–2015) `media.physics.harvard.edu/video/?id=LOEB_POUND_092591.flv`

□

Example 13.26 (Redshift of the sun) A photon emitted with frequency ν_0 at a distance r from a mass M would be observed at spatial infinity to have frequency $\nu = \nu_0\sqrt{-g_{00}} = \nu_0\sqrt{1 - 2MG/c^2 r}$ for a redshift of $\Delta\nu = \nu_0 - \nu$. Since the Sun's dimensionless potential $\phi_\odot/c^2$ is $-MG/c^2 r = -2.12 \times 10^{-6}$ at its surface, sunlight is shifted to the red by 2 parts per million. □

13.35 Einstein's equations

If we make an action that is a scalar, invariant under general coordinate transformations, and then apply to it the principle of stationary action, we will get tensor field equations that are invariant under general coordinate transformations. If the metric of spacetime is among the fields of the action, then the resulting theory will be a possible theory of gravity. If we make the action as simple as possible, it will be Einstein's theory.

To make the action of the gravitational field, we need a scalar. Apart from the scalar $\sqrt{g}\, d^4x = \sqrt{g}\, c\, dt\, d^3x$, where $g = |\det(g_{ik})|$, the simplest scalar we can form from the metric tensor and its first and second derivatives is the scalar curvature R which gives us the **Einstein-Hilbert action**

$$S_{EH} = \frac{c^3}{16\pi G}\int R\sqrt{g}\, d^4x = \frac{c^3}{16\pi G}\int g^{ik}\, R_{ik}\sqrt{g}\, d^4x \tag{13.219}$$

in which $G = 6.7087 \times 10^{-39}\, \hbar c\, (\mathrm{GeV}/c^2)^{-2} = 6.6742 \times 10^{-11}\, \mathrm{m^3\, kg^{-1}\, s^{-2}}$ is Newton's constant.

If $\delta g^{ik}(x)$ is a tiny local change in the inverse metric, then the rule $\delta \det A = \det A \, \mathrm{Tr}(A^{-1}\delta A)$ (1.228), valid for any nonsingular, nondefective matrix A) together with the identity $0 = \delta(g^{ik} g_{k\ell}) = \delta g^{ik}\, g_{k\ell} + g^{ik}\, \delta g_{k\ell}$ and the notation $\underline{g}$ for the metric tensor $g_{j\ell}$ considered as a matrix imply that

$$\delta\sqrt{g} = \frac{\det \underline{g}}{2g\sqrt{g}}\, \delta \det \underline{g} = \frac{(\det \underline{g})^2 g^{ik}\, \delta g_{ik}}{2g\sqrt{g}} \; = \; -\frac{1}{2}\sqrt{g}\; g_{ik}\, \delta g^{ik}. \qquad (13.220)$$

So the first-order change in the action density is

$$\begin{aligned} \delta\left(g^{ik}\, R_{ik}\, \sqrt{g}\right) &= R_{ik}\, \sqrt{g}\; \delta g^{ik} + g^{ik}\, R_{ik}\; \delta\sqrt{g} + g^{ik}\, \sqrt{g}\; \delta R_{ik} \\ &= \left(R_{ik} - \frac{1}{2}\, R\, g_{ik}\right) \sqrt{g}\; \delta g^{ik} + g^{ik}\, \sqrt{g}\; \delta R_{ik}. \end{aligned} \qquad (13.221)$$

The product $g^{ik}\delta R_{ik}$ is a scalar, so we can evaluate it in any coordinate system. In a local inertial frame, where $\Gamma^a{}_{bc} = 0$ and g_{de} is constant, this invariant variation of the Ricci tensor (13.112) is

$$\begin{aligned} g^{ik}\delta R_{ik} &= g^{ik}\, \delta\left(\Gamma^n{}_{in,k} - \Gamma^n{}_{ik,n}\right) = g^{ik}\left(\partial_k\, \delta\Gamma^n{}_{in} - \partial_n\, \delta\Gamma^n{}_{ik}\right) \\ &= g^{ik}\, \partial_k\, \delta\Gamma^n{}_{in} - g^{in}\, \partial_k\, \delta\Gamma^k{}_{in} = \partial_k \left(g^{ik}\, \delta\Gamma^n{}_{in} - g^{in}\, \delta\Gamma^k{}_{in}\right). \end{aligned} \qquad (13.222)$$

The transformation law (13.66) for the affine connection shows that the variations $\delta\Gamma^n{}_{in}$ and $\delta\Gamma^k{}_{in}$ are tensors although the connections themselves aren't. Thus, we can evaluate this invariant variation of the Ricci tensor in any coordinate system by replacing the derivatives with covariant ones getting

$$g^{ik}\delta R_{ik} = \left(g^{ik}\, \delta\Gamma^n{}_{in} - g^{in}\, \delta\Gamma^k{}_{in}\right)_{;k} \qquad (13.223)$$

which we recognize as the covariant divergence (13.180) of a contravariant vector. The last term in the first-order change (13.221) in the action density is therefore a surface term whose variation vanishes for tiny local changes δg^{ik} of the metric

$$\sqrt{g}\, g^{ik}\delta R_{ik} = \left[\sqrt{g}\left(g^{ik}\, \delta\Gamma^n{}_{in} - g^{in}\, \delta\Gamma^k{}_{in}\right)\right]_{,k}. \qquad (13.224)$$

Hence the variation of S_{EH} is simply

$$\delta S_{EH} = \frac{c^3}{16\pi G}\int \left(R_{ik} - \frac{1}{2}\, g_{ik} R\right)\sqrt{g}\; \delta g^{ik}\, d^4x. \qquad (13.225)$$

The principle of least action $\delta S_{EH} = 0$ now gives us **Einstein's equations**

for empty space:

$$R_{ik} - \frac{1}{2}\, g_{ik}\, R = 0. \tag{13.226}$$

The tensor $G_{ik} = R_{ik} - \frac{1}{2} g_{ik}\, R$ is Einstein's tensor.

Taking the trace of Einstein's equations (13.226), we find that the scalar curvature R and the Ricci tensor R_{ik} are zero in empty space:

$$R = 0 \quad \text{and} \quad R_{ik} = 0. \tag{13.227}$$

The **energy-momentum tensor** T_{ik} is the source of the gravitational field. It is defined so that the change in the action of the matter fields due to a tiny local change $\delta g^{ik}(x)$ in the metric is

$$\delta S_m = -\frac{1}{2c}\int T_{ik}\,\sqrt{g}\;\delta g^{ik}\, d^4x = \frac{1}{2c}\int T^{ik}\,\sqrt{g}\;\delta g_{ik}\, d^4x \tag{13.228}$$

in which the identity $\delta g^{ik} = -\, g^{ij} g^{\ell k} \delta g_{j\ell}$ explains the sign change. Now the principle of least action $\delta S = \delta S_{EH} + \delta S_m = 0$ yields Einstein's equations in the presence of matter and energy

$$R_{ik} - \frac{1}{2}\, g_{ik}\, R = \frac{8\pi G}{c^4} T_{ik}. \tag{13.229}$$

Taking the trace of both sides, we get

$$R = -\frac{8\pi G}{c^4}\, T \quad \text{and} \quad R_{ik} = \frac{8\pi G}{c^4}\left(T_{ik} - \frac{T}{2} g_{ik}\right). \tag{13.230}$$

13.36 Energy-momentum tensor

The action S_m of the matter fields is a scalar that is invariant under general coordinate transformations. In particular, a tiny local general coordinate transformation $x'^a = x^a + \epsilon^a(x)$ leaves S_m invariant

$$0 = \delta S_m = \int \delta\left(L(\phi_i(x))\,\sqrt{g(x)}\right)\, d^4x. \tag{13.231}$$

The vanishing change $\delta S_m = \delta S_{m\phi} + \delta S_{mg}$ has a part $\delta S_{m\phi}$ due to the changes in the fields $\delta\phi_i(x)$ and a part δS_{mg} due to the change in the metric δg^{ik}. The principle of stationary action tells us that the change $\delta S_{m\phi}$ is zero as long as the fields obey the classical equations of motion. The definition (13.228) of the energy-momentum tensor now tells us that

$$0 = \delta S_m = \delta S_{mg} = \frac{1}{2c}\int T^{ik}\,\sqrt{g}\;\delta g_{ik}\, d^4x. \tag{13.232}$$

We take the change in S_m to be

$$\begin{aligned} \delta S_m &= \int L'(\phi_i'(x'))\,\sqrt{g'(x')}\,d^4x' - \int L(\phi_i(x))\,\sqrt{g(x)}\,d^4x \\ &= \int L'(\phi_i'(x))\,\sqrt{g'(x)}\,d^4x - \int L(\phi_i(x))\,\sqrt{g(x)}\,d^4x. \end{aligned} \tag{13.233}$$

So using the identity $\delta g^{ik}\,g_{k\ell} = -g^{ik}\,\delta g_{k\ell}$, the definition (13.68) of the covariant derivative of a covariant vector, and the formula (13.87) for the connection in terms of the metric, we find to lowest order in the change $\epsilon^a(x)$ in x^a that the change in the metric is

$$\begin{aligned} \delta g_{ik} &= g'_{ik}(x) - g_{ik}(x) = g'_{ik}(x') - g_{ik}(x) - (g'_{ik}(x') - g'_{ik}(x)) \\ &= (\delta^a_i - \epsilon^a_{,i})(\delta^b_k - \epsilon^b_{,k})g_{ab} - \epsilon^c g_{ik,c} \\ &= -\, g_{ib}\,\epsilon^b_{,k} - g_{ak}\,\epsilon^a_{,i} - \epsilon^c\, g_{ik,c} \\ &= -\, g_{ib}(g^{bc}\,\epsilon_c)_{,k} - g_{ak}(g^{ac}\,\epsilon_c)_{,i} - \epsilon^c\, g_{ik,c} \\ &= -\, \epsilon_{i,k} - \epsilon_{k,i} - \epsilon_c\, g_{ib}\, g^{bc}_{,k} - \epsilon_c\, g_{ak}\, g^{ac}_{,i} - \epsilon^c\, g_{ik,c} \\ &= -\, \epsilon_{i,k} - \epsilon_{k,i} + \epsilon_c\, g^{bc}\, g_{ib,k} + \epsilon_c\, g^{ac}\, g_{ak,i} - \epsilon^c\, g_{ik,c} \\ &= -\, \epsilon_{i,k} - \epsilon_{k,i} + \epsilon_c\, g^{ac}\left(g_{ia,k} + g_{ak,i} - g_{ik,a}\right) \\ &= -\, \epsilon_{i,k} - \epsilon_{k,i} + \epsilon_c\, \Gamma^c{}_{ik} + \epsilon_c\, \Gamma^c{}_{ki} = -\, \epsilon_{i;k} - \epsilon_{k;i}. \end{aligned} \tag{13.234}$$

Combining this result (13.234) with the vanishing (13.232) of the change δS_{mg}, we have

$$0 = \int T^{ik}\,\sqrt{g}\,\left(\epsilon_{i;k} + \epsilon_{k;i}\right) d^4x. \tag{13.235}$$

Since the energy-momentum tensor is symmetric, we may combine the two terms, integrate by parts, divide by $\sqrt{g}$, and so find that the covariant divergence of the energy-momentum tensor is zero

$$0 = T^{ik}_{;k} = T^{ik}_{,k} + \Gamma^k{}_{ak}\,T^{ia} + \Gamma^i{}_{ak}\,T^{ak} = \frac{1}{\sqrt{g}}(\sqrt{g}T^{ik})_{,k} + \Gamma^i{}_{ak}\,T^{ak} \tag{13.236}$$

when the fields obey their equations of motion. In a given inertial frame, only the total energy, momentum, and angular momentum of both the matter and the gravitational field are conserved.

13.37 Perfect fluids

In many cosmological models, the energy-momentum tensor is assumed to be that of a **perfect fluid**, which is isotropic in its rest frame, does not conduct heat, and has zero viscosity. The **energy-momentum** tensor T_{ij} of a perfect fluid moving with 4-velocity u^i (12.20) is

$$T_{ij} = p\, g_{ij} + (\frac{p}{c^2} + \rho)\, u_i\, u_j \tag{13.237}$$

in which p and ρ are the pressure and mass density of the fluid in its *rest frame* and g_{ij} is the spacetime metric. Einstein's equations (13.229) then are

$$R_{ik} - \frac{1}{2}\, g_{ik}\, R = \frac{8\pi G}{c^4} T_{ik} = \frac{8\pi G}{c^4} \left[p\, g_{ij} + (\frac{p}{c^2} + \rho)\, u_i\, u_j \right]. \tag{13.238}$$

An important special case is the energy-momentum tensor due to a nonzero value of the energy density of the vacuum. In this case $p = -c^2\rho$ and the energy-momentum tensor is

$$T_{ij} = p\, g_{ij} = -c^2 \rho\, g_{ij} \tag{13.239}$$

in which $T_{00} = c^2\rho$ is the (presumably constant) value of the energy density of the ground state of the theory. This energy density ρ is a plausible candidate for the **dark-energy** density. It is equivalent to a **cosmological constant** $\Lambda = 8\pi G\rho$.

On small scales, such as that of our solar system, one may neglect matter and dark energy. So in empty space and on small scales, the energy-momentum tensor vanishes $T_{ij} = 0$ along with its trace and the scalar curvature $T = 0 = R$, and Einstein's equations (13.230) are

$$R_{ij} = 0. \tag{13.240}$$

13.38 Gravitational waves

The nonlinear properties of Einstein's equations (13.229– 13.230) are important on large scales of distance (sections 13.42 & 13.43) and near great masses (sections 13.39 & 13.40). But throughout most of the mature universe, it is helpful to linearize them by writing the metric as the metric η_{ik} of empty, flat spacetime (12.3) plus a tiny deviation h_{ik}

$$g_{ik} = \eta_{ik} + h_{ik}. \tag{13.241}$$

To first order in h_{ik}, the affine connection (13.87) is

$$\Gamma^k{}_{i\ell} = \tfrac{1}{2}\, g^{kj} \left(g_{ji,\ell} + g_{j\ell,i} - g_{i\ell,j} \right) = \tfrac{1}{2}\, \eta^{kj} \left(h_{ji,\ell} + h_{j\ell,i} - h_{i\ell,j} \right) \tag{13.242}$$

and the Ricci tensor (13.112) is the contraction

$$R_{i\ell} = R^k{}_{ik\ell} = [\partial_k + \Gamma_k, \partial_\ell + \Gamma_\ell]^k{}_i = \Gamma^k{}_{\ell i,k} - \Gamma^k{}_{ki,\ell}. \tag{13.243}$$

Since $\Gamma^k{}_{i\ell} = \Gamma^k{}_{\ell i}$ and $h_{ik} = h_{ki}$, the linearized Ricci tensor is

$$\begin{aligned} R_{i\ell} &= \tfrac{1}{2}\,\eta^{kj}\left(h_{ji,\ell} + h_{j\ell,i} - h_{i\ell,j}\right)_{,k} - \tfrac{1}{2}\,\eta^{kj}\left(h_{ji,k} + h_{jk,i} - h_{ik,j}\right)_{,\ell} \\ &= \tfrac{1}{2}\left(h^k{}_{\ell,ik} + h_{ik,\ell}{}^{,k} - h_{i\ell,k}{}^{,k} - h^k{}_{k,i\ell}\right). \end{aligned} \tag{13.244}$$

We can simplify Einstein's equations (13.230) in empty space $R_{i\ell} = 0$ by using coordinates in which h_{ik} obeys (exercise 13.17) de Donder's harmonic gauge condition $h^i{}_{k,i} = \frac{1}{2}(\eta^{j\ell} h_{j\ell})_{,k} \equiv \frac{1}{2} h_{,k}$. In this gauge, the linearized Einstein equations in empty space are

$$R_{i\ell} = -\tfrac{1}{2}\,h_{i\ell,k}{}^{,k} = 0 \quad \text{or} \quad \left(c^2\nabla^2 - \partial_0^2\right) h_{i\ell} = 0. \tag{13.245}$$

On 14 September 2015, the LIGO collaboration detected the merger of two black holes of 29 and 36 solar masses which liberated $3M_\odot c^2$ of energy. By 2017, LIGO and Virgo had detected gravitational waves from the mergers of six pairs of black holes and one pair of neutron stars and had set an upper limit of $c^2 m_g < 2 \times 10^{-25}$ eV on the mass of the graviton.

13.39 Schwarzschild's solution

In 1916, Schwarzschild solved Einstein's field equations (13.240) in empty space $R_{ij} = 0$ outside a static mass M and found as the metric

$$ds^2 = -\left(1 - \frac{2MG}{c^2 r}\right) c^2 dt^2 + \left(1 - \frac{2MG}{c^2 r}\right)^{-1} dr^2 + r^2\, d\Omega^2 \tag{13.246}$$

in which $d\Omega^2 = d\theta^2 + \sin^2\theta\, d\phi^2$.

The Mathematica scripts GREAT.m and Schwarzschild.nb give for the Ricci tensor and the scalar curvature $R_{ik} = 0$ and $R = 0$, which show that the metric obeys Einstein's equations and that the singularity in

$$g_{rr} = \left(1 - \frac{2MG}{c^2 r}\right)^{-1} \tag{13.247}$$

at the Schwarzschild radius $r_s = 2MG/c^2$ is an artifact of the coordinates. Schwarzschild.nb also computes the affine connections.

The Schwarzschild radius of the Sun $r_s = 2M_\odot G/c^2 = 2.95$ km is far less than the Sun's radius $r_\odot = 6.955 \times 10^5$ km, beyond which his metric applies. (Karl Schwarzschild, 1873–1916)

13.40 Black holes

Suppose an uncharged, spherically symmetric star of mass M has collapsed within a sphere of radius r_b less than its Schwarzschild radius or horizon $r_h = r_s = 2MG/c^2$. Then for $r > r_b$, the Schwarzschild metric (13.246) is correct. The time dt measured on a clock outside the gravitational field is related to the proper time $d\tau$ on a clock fixed at $r \geq 2MG/c^2$ by (13.216)

$$dt = d\tau/\sqrt{-g_{00}} = d\tau/\sqrt{1 - \frac{2MG}{c^2 r}}. \tag{13.248}$$

The time dt measured away from the star becomes infinite as r approaches the horizon $r_h = r_s = 2MG/c^2$. To outside observers, a clock at the horizon r_h seems frozen in time.

Due to the gravitational redshift (13.248), light of frequency ν_p emitted at $r \geq 2MG/c^2$ will have frequency ν

$$\nu = \nu_p \sqrt{-g_{00}} = \nu_p \sqrt{1 - \frac{2MG}{c^2 r}} \tag{13.249}$$

when observed at great distances. Light coming from the surface at $r_s = 2MG/c^2$ is redshifted to zero frequency $\nu = 0$. The star is black. It is a black hole with a horizon at its Schwarzschild radius $r_h = r_s = 2MG/c^2$, although there is no singularity there. If the radius of the Sun were less than its Schwarzschild radius of 2.95 km, then the Sun would be a black hole. The radius of the Sun is 6.955×10^5 km.

Black holes are not black. Stephen Hawking (1942–2018) showed (Hawking, 1975) that the intense gravitational field of a black hole of mass M radiates at a temperature

$$T = \frac{\hbar\, c^3}{8\pi\, k\, G\, M} = \frac{\hbar c}{4\pi k r_h} = \frac{\hbar g}{2\pi k c} \tag{13.250}$$

in which $k = 8.617 \times 10^{-5}\,\text{eV K}^{-1}$ is Boltzmann's constant, $\hbar$ is Planck's constant $h = 6.626 \times 10^{-34}\,\text{J s}$ divided by 2π, $\hbar = h/(2\pi)$, and $g = GM/r_h^2$ is the gravitational acceleration at $r = r_h$. More generally, a detector in vacuum subject to a uniform acceleration a (in its instantaneous rest frame) sees a temperature $T = \hbar a/(2\pi k c)$ (Alsing and Milonni, 2004).

In a region of empty space where the pressure p and the chemical potentials μ_j all vanish, the change (7.111) in the internal energy $U = c^2 M$ of a black hole of mass M is $c^2 dM = TdS$ where S is its entropy. So the change dS in the entropy of a black hole of mass $M = c^2 r_h/(2G)$ and temperature $T = \hbar c/(4\pi k r_h)$ (13.250) is

$$dS = \frac{c^2}{T} dM = \frac{4\pi c^2 k r_h}{(\hbar c)} dM = \frac{4\pi c k r_h}{\hbar} \frac{c^2}{2G} dr_h = \frac{\pi c^3 k}{G\hbar} 2r_h dr_h. \tag{13.251}$$

Integrating, we get a formula for the entropy of a black hole in terms of its area (Bekenstein, 1973; Hawking, 1975)

$$S = \frac{\pi c^3 k}{G\hbar} r_h^2 = \frac{c^3 k}{\hbar G} \frac{A}{4} \tag{13.252}$$

where $A = 4\pi r_h^2$ is the area of the horizon of the black hole.

A black hole is entirely converted into radiation after a time

$$t = \frac{5120\,\pi\, G^2}{\hbar\, c^4} M^3 \tag{13.253}$$

proportional to the cube of its mass M.

13.41 Rotating black holes

A half-century after Einstein invented general relativity, Roy Kerr invented the metric for a mass M rotating with angular momentum $J = GMa/c$. Two years later, Newman and others generalized the Kerr metric to one of charge Q. In Boyer-Lindquist coordinates, its line element is

$$\begin{aligned} ds^2 = &- \frac{\Delta}{\rho^2} \left(dt - a \sin^2\theta d\phi\right)^2 \\ &+ \frac{\sin^2\theta}{\rho^2} \left((r^2 + a^2)d\phi - a\, dt\right)^2 + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2 \end{aligned} \tag{13.254}$$

where $\rho^2 = r^2 + a^2 \cos^2\theta$ and $\Delta = r^2 + a^2 - 2GMr/c^2 + Q^2$. The Mathematica script Kerr_black_hole.nb shows that the Kerr-Newman metric for the uncharged case, $Q = 0$, has $R_{ik} = 0$ and $R = 0$ and so is a solution of Einstein's equations in empty space (13.230) with zero scalar curvature.

A rotating mass drags nearby masses along with it. The daily rotation of the Earth moves satellites to the East by tens of meters per year. The **frame dragging** of extremal black holes with $J \lesssim GM^2/c$ approaches the speed of light (Ghosh et al., 2018). (Roy Kerr, 1934–)

13.42 Spatially symmetric spacetimes

Einstein's equations (13.230) are second-order, nonlinear partial differential equations for 10 unknown functions $g_{ik}(x)$ in terms of the energy-momentum tensor $T_{ik}(x)$ throughout the universe, which of course we don't know. The problem is not quite hopeless, however. The ability to choose arbitrary coordinates, the appeal to symmetry, and the choice of a reasonable form for T_{ik} all help.

Astrophysical observations tell us that the universe extends at least 46 billion light years in all directions; that it is **flat** or very nearly flat; and that the cosmic microwave background (CMB) radiation is isotropic to one part in 10^5 apart from a Doppler shift due the motion of the Sun at 370 km/s towards the constellation Leo. These microwave photons have been moving freely since the universe became cool enough for hydrogen atoms to be stable. Observations of clusters of galaxies reveal an expanding universe that is homogeneous on suitably large scales of distance. Thus as far as we know, the universe is **homogeneous** and **isotropic** in space, but not in time.

There are only three **maximally symmetric** 3-dimensional spaces: euclidian space $\mathbb{E}^3$, the sphere S^3 (example 13.16), and the hyperboloid H^3 (example 13.17). Their line elements may be written in terms of a distance L as

$$ds^2 = \frac{dr^2}{1 - k\,r^2/L^2} + r^2 d\Omega^2 \tag{13.255}$$

in which $k = 1$ for the sphere, $k = 0$ for euclidian space, and $k = -1$ for the hyperboloid. The **Friedmann-Lemaître-Robinson-Walker** (FLRW) cosmologies add to these spatially symmetric line elements a **dimensionless scale factor** $a(t)$ that describes the expansion (or contraction) of space

$$ds^2 = -c^2 dt^2 + a^2(t)\left(\frac{dr^2}{1 - k\,r^2/L^2} + r^2\,d\theta^2 + r^2\,\sin^2\theta\,d\phi^2\right). \tag{13.256}$$

The FLRW metric is

$$g_{ik}(t,r,\theta,\phi) = \begin{pmatrix} -c^2 & 0 & 0 & 0 \\ 0 & a^2/(1 - k\,r^2/L^2) & 0 & 0 \\ 0 & 0 & a^2\,r^2 & 0 \\ 0 & 0 & 0 & a^2\,r^2\,\sin^2\theta \end{pmatrix}. \tag{13.257}$$

The constant k determines whether the spatial universe is **open** $k = -1$, **flat** $k = 0$, or **closed** $k = 1$. The coordinates $x^0, x^1, x^2, x^3 \equiv t, r, \theta, \phi$ are **comoving** in that a detector at rest at r, θ, ϕ records the CMB as isotropic with no Doppler shift.

The metric (13.257) is diagonal; its inverse g^{ij} also is diagonal

$$g^{ik} = \begin{pmatrix} -c^{-2} & 0 & 0 & 0 \\ 0 & (1-k\,r^2/L^2)/a^2 & 0 & 0 \\ 0 & 0 & (a\,r)^{-2} & 0 \\ 0 & 0 & 0 & (a\,r\,\sin\theta)^{-2} \end{pmatrix}. \tag{13.258}$$

One may use the formula (13.87) to compute the affine connection in terms of the metric and its inverse as ${\Gamma^k}_{i\ell} = \frac{1}{2} g^{kj}(g_{ji,\ell} + g_{j\ell,i} - g_{\ell i,j})$. It usually is easier, however, to use the action principle (13.192) to derive the geodesic equation directly and then to read its expressions for the ${\Gamma^i}_{jk}$'s. So we require that the integral

$$0 = \delta \int \left(-c^2 t'^2 + \frac{a^2\,r'^2}{1-kr^2/L^2} + a^2\,r^2\,\theta'^2 + a^2\,r^2\,\sin^2\theta\,\phi'^2 \right) d\lambda, \tag{13.259}$$

in which a prime means derivative with respect to λ, be stationary with respect to the tiny variations $\delta t(\lambda), \delta r(\lambda), \delta\theta(\lambda)$, and $\delta\phi(\lambda)$. By varying $t(\lambda)$, we get the equation

$$0 = t'' + \frac{a\dot{a}}{c^2}\left(\frac{r'^2}{1-kr^2/L^2} + r^2\,\theta'^2 + r^2\,\sin^2\theta\,\phi'^2 \right) = t'' + {\Gamma^t}_{i\ell} x'^i x'^\ell \tag{13.260}$$

which tells us that the nonzero ${\Gamma^t}_{jk}$'s are

$${\Gamma^t}_{rr} = \frac{a\dot{a}}{c^2(1-kr^2/L^2)}, \quad {\Gamma^t}_{\theta\theta} = \frac{a\dot{a}\,r^2}{c^2}, \quad \text{and} \quad {\Gamma^t}_{\phi\phi} = \frac{a\dot{a}\,r^2\sin^2\theta}{c^2}. \tag{13.261}$$

By varying $r(\lambda)$ we get (with more effort)

$$0 = r'' + \frac{rr'^2 k/L^2}{(1-kr^2/L^2)} + 2\frac{\dot{a}\,t'r'}{a} - r\,(1-\frac{kr^2}{L^2})(\theta'^2 + \sin^2\theta\,\phi'^2). \tag{13.262}$$

So we find that ${\Gamma^r}_{tr} = \dot{a}/a$,

$${\Gamma^r}_{rr} = \frac{kr}{L^2-kr^2}, \quad {\Gamma^r}_{\theta\theta} = -r + \frac{kr^3}{L^2}, \quad \text{and} \quad {\Gamma^r}_{\phi\phi} = \sin^2\theta\,{\Gamma^r}_{\theta\theta}. \tag{13.263}$$

Varying $\theta(\lambda)$ gives

$$\begin{gathered} 0 = \theta'' + 2\frac{\dot{a}}{a}t'\theta' + \frac{2}{r}\theta' r' - \sin\theta\cos\theta\phi'^2 \quad \text{and} \\ {\Gamma^\theta}_{t\theta} = \frac{\dot{a}}{a}, \quad {\Gamma^\theta}_{r\theta} = \frac{1}{r}, \quad \text{and} \quad {\Gamma^\theta}_{\phi\phi} = -\sin\theta\cos\theta. \end{gathered} \tag{13.264}$$

Finally, varying $\phi(\lambda)$ gives

$$0 = \phi'' + 2\frac{\dot a}{a}t'\phi' + 2\frac{r'\phi'}{r} + 2\cot\theta\,\theta'\phi' \quad \text{and}$$
$$\Gamma^{\phi}{}_{t\phi} = \frac{\dot a}{a}, \quad \Gamma^{\phi}{}_{r\phi} = \frac{1}{r}, \quad \text{and} \quad \Gamma^{\phi}{}_{\theta\phi} = \cot\theta. \tag{13.265}$$

Other Γ's are either zero or related by the symmetry $\Gamma^k{}_{i\ell} = \Gamma^k{}_{\ell i}$.

Our formulas for the Ricci (13.112) and curvature (13.103) tensors give

$$R_{00} = R^k{}_{0k0} = [D_k, D_0]^k{}_0 = [\partial_k + \Gamma_k, \partial_0 + \Gamma_0]^k{}_0. \tag{13.266}$$

Because $[D_0, D_0] = 0$, we need only compute $[D_1, D_0]^1{}_0$, $[D_2, D_0]^2{}_0$, and $[D_3, D_0]^3{}_0$. Using the formulas (13.261–13.265) for the Γ's and keeping in mind (13.102) that the element of row r and column c of the ℓth gamma matrix is $\Gamma^r{}_{\ell c}$, we find

$$[D_1, D_0]^1{}_0 = \Gamma^1{}_{00,1} - \Gamma^1{}_{10,0} + \Gamma^1{}_{1j}\Gamma^j{}_{00} - \Gamma^1{}_{0j}\Gamma^j{}_{10} = -(\dot a/a)_{,0} - (\dot a/a)^2$$
$$[D_2, D_0]^2{}_0 = \Gamma^2{}_{00,2} - \Gamma^2{}_{20,0} + \Gamma^2{}_{2j}\Gamma^j{}_{00} - \Gamma^2{}_{0j}\Gamma^j{}_{20} = -(\dot a/a)_{,0} - (\dot a/a)^2$$
$$[D_3, D_0]^3{}_0 = \Gamma^3{}_{00,3} - \Gamma^3{}_{30,0} + \Gamma^3{}_{3j}\Gamma^j{}_{00} - \Gamma^3{}_{0j}\Gamma^j{}_{30} = -(\dot a/a)_{,0} - (\dot a/a)^2$$
$$R_{tt} = R_{00} = [D_k, D_0]^k{}_0 = -3(\dot a/a)_{,0} - 3(\dot a/a)^3 = -3\ddot a/a. \tag{13.267}$$

Thus for $R_{rr} = R_{11} = R^k{}_{1k1} = [D_k, D_1]^k{}_1 = [\partial_k + \Gamma_k, \partial_1 + \Gamma_1]^k{}_1$, we get

$$R_{rr} = [D_k, D_1]^k{}_1 = \frac{a\ddot a + 2\dot a^2 + 2kc^2/L^2}{c^2(1 - kr^2/L^2)} \tag{13.268}$$

(exercise 13.21), and for $R_{22} = R_{\theta\theta}$ and $R_{33} = R_{\phi\phi}$ we find

$$R_{\theta\theta} = [(a\ddot a + 2\dot a^2 + 2kc^2/L^2)r^2]/c^2 \quad \text{and} \quad R_{\phi\phi} = \sin^2\theta R_{\theta\theta} \tag{13.269}$$

(exercises 13.22 & 13.23). And so the scalar curvature $R = g^{ab}R_{ba}$ is

$$R = g^{ab}R_{ba} = -\frac{R_{00}}{c^2} + \frac{(1 - kr^2/L^2)R_{11}}{a^2} + \frac{R_{22}}{a^2r^2} + \frac{R_{33}}{a^2r^2\sin^2\theta}$$
$$= 6\,\frac{a\ddot a + \dot a^2 + kc^2/L^2}{c^2a^2}. \tag{13.270}$$

It is, of course, quicker to use the Mathematica script FLRW.nb.

13.43 Friedmann-Lemaître-Robinson-Walker cosmologies

The energy-momentum tensor (13.237) of a perfect fluid moving at 4-velocity u_i is $T_{ik} = pg_{ik} + (p/c^2 + \rho)u_iu_k$ where p and ρ are the pressure and mass density of the fluid in its rest frame. In the comoving coordinates of the

FLRW metric (13.257), the 4-velocity (12.20) is $u^i = (1, 0, 0, 0)$, and the energy-momentum tensor (13.237) is

$$T_{ij} = \begin{pmatrix} -c^2\rho\, g_{00} & 0 & 0 & 0 \\ 0 & p\, g_{11} & 0 & 0 \\ 0 & 0 & p\, g_{22} & 0 \\ 0 & 0 & 0 & p\, g_{33} \end{pmatrix}. \tag{13.271}$$

Its trace is

$$T = g^{ij}\, T_{ij} = -c^2\rho + 3p. \tag{13.272}$$

Thus successively using our formulas (13.257) for $g_{00} = -c^2$, (13.267) for $R_{00} = -3\ddot{a}/a$, (13.271) for T_{ij}, and (13.272) for T, we can write the 00 Einstein equation (13.230) as the second-order equation

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right). \tag{13.273}$$

It is nonlinear because ρ and $3p$ depend upon a. The sum $c^2\rho + 3p$ determines the acceleration $\ddot{a}$ of the scale factor $a(t)$; when it is negative, it accelerates the expansion. If we combine Einstein's formula for the scalar curvature $R = -8\pi G\, T/c^4$ (13.230) with the FLRW formulas for R (13.270) and for the trace T (13.272) of the energy-momentum tensor, we get

$$\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{c^2 k}{L^2 a^2} = \frac{4\pi G}{3}\left(\rho - \frac{3p}{c^2}\right). \tag{13.274}$$

Using the 00-equation (13.273) to eliminate the second derivative $\ddot{a}$, we find

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\,\rho - \frac{c^2 k}{L^2 a^2} \tag{13.275}$$

which is a first-order nonlinear equation. It and the second-order equation (13.273) are known as the **Friedmann equations**.

The left-hand side of the first-order Friedmann equation (13.275) is the square of the **Hubble rate**

$$H = \frac{\dot{a}}{a} \tag{13.276}$$

which is an inverse time or a frequency. Its present value H_0 is the **Hubble constant**.

In terms of H, the first-order Friedmann equation (13.275) is

$$H^2 = \frac{8\pi G}{3}\,\rho - \frac{c^2 k}{L^2 a^2}. \tag{13.277}$$

An absolutely flat universe has $k = 0$, and therefore its density must be

$$\rho_c = \frac{3H^2}{8\pi G} \tag{13.278}$$

which is the **critical mass density**.

13.44 Density and pressure

The 0-th energy-momentum conservation law (13.236) is

$$0 = T^{0a}_{\;;a} = \partial_a T^{0a} + \Gamma^a{}_{ca} T^{0c} + \Gamma^0{}_{ca} T^{ca}. \tag{13.279}$$

For a perfect fluid of 4-velocity u^a, the energy-momentum tensor (13.271) is $T^{ik} = (\rho + p/c^2)u^i u^k + p g^{ik}$ in which ρ and p are the mass density and pressure of the fluid in its rest frame. The comoving frame of the Friedmann-Lemaître-Robinson-Walker metric (13.257) is the rest frame of the fluid. In these coordinates, the 4-velocity u^a is $(1, 0, 0, 0)$, and the energy-momentum tensor is diagonal with $T^{00} = \rho$ and $T^{jj} = p g^{jj}$ for $j = 1, 2, 3$. Our connection formulas (13.261) tell us that $\Gamma^0{}_{00} = 0$, that $\Gamma^0{}_{jj} = \dot{a} g_{jj}/(c^2 a)$, and that $\Gamma^j{}_{0j} = 3\dot{a}/a$. Thus the conservation law (13.279) becomes for spatial j

$$\begin{aligned} 0 &= \partial_0 T^{00} + \Gamma^j{}_{0j} T^{00} + \Gamma^0{}_{jj} T^{jj} \\ &= \dot{\rho} + 3\,\frac{\dot{a}}{a}\,\rho + \sum_{j=1}^{3} \frac{\dot{a}\, g_{jj}}{c^2 a}\, p\, g^{jj} = \dot{\rho} + 3\frac{\dot{a}}{a}\left(\rho + \frac{p}{c^2}\right). \end{aligned} \tag{13.280}$$

Thus

$$\dot{\rho} = -\,\frac{3\dot{a}}{a}(\rho + \frac{p}{c^2}), \quad \text{and so} \quad \frac{d\rho}{da} = -\frac{3}{a}\left(\rho + \frac{p}{c^2}\right). \tag{13.281}$$

The energy density ρ is composed of fractions ρ_i each contributing its own partial pressure p_i according to its own **equation of state**

$$p_i = c^2 w_i \rho_i \tag{13.282}$$

in which w_i is a constant. The rate of change (13.282) of the density ρ_i is then

$$\frac{d\rho_i}{da} = -\,\frac{3}{a}\,(1 + w_i)\,\rho_i. \tag{13.283}$$

In terms of the present density ρ_{i0} and scale factor a_0, the solution is

$$\rho_i = \rho_{i0}\left(\frac{a_0}{a}\right)^{3(1+w_i)}. \tag{13.284}$$

There are three important kinds of density. The dark-energy density ρ_Λ is assumed to be like a cosmological constant Λ or like the energy density of the vacuum, so it is independent of the scale factor a and has $w_\Lambda = -1$.

A universe composed only of **dust** or **non-relativistic collisionless matter** has no pressure. Thus $p = w\rho = 0$ with $\rho \neq 0$, and so $w = 0$. So the matter density falls inversely with the volume

$$\rho_m = \rho_{m0} \left(\frac{a_0}{a}\right)^3. \tag{13.285}$$

The density of radiation ρ_r has $w_r = 1/3$ because wavelengths scale with the scale factor, and so there's an extra factor of a

$$\rho_r = \rho_{r0} \left(\frac{a_0}{a}\right)^4. \tag{13.286}$$

The total density ρ varies with a as

$$\rho = \rho_\Lambda + \rho_{m0} \left(\frac{a_0}{a}\right)^3 + \rho_{r0} \left(\frac{a_0}{a}\right)^4. \tag{13.287}$$

This mass density ρ, the Friedmann equations (13.273 & 13.275), and the physics of the standard model have caused our universe to evolve as in Fig. 13.1 over the past 14 billion years.

13.45 How the scale factor evolves with time

The first-order Friedmann equation (13.275) expresses the square of the instantaneous Hubble rate $H = \dot{a}/a$ in terms of the density ρ and the scale factor $a(t)$

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\,\rho - \frac{c^2 k}{L^2 a^2} \tag{13.288}$$

in which $k = \pm 1$ or 0. The critical density $\rho_c = 3H^2/(8\pi G)$ (13.278) is the one that satisfies this equation for a flat ($k = 0$) universe. Its present value is $\rho_{c0} = 3H_0^2/(8\pi G) = 8.599 \times 10^{-27}$ kg m^{-3}. Dividing Friedmann's equation by the square of the present Hubble rate H_0^2, we get

$$\frac{H^2}{H_0^2} = \frac{1}{H_0^2}\left(\frac{\dot{a}}{a}\right)^2 = \frac{1}{H_0^2}\left(\frac{8\pi G}{3}\,\rho - \frac{c^2 k}{a^2 L^2}\right) = \frac{\rho}{\rho_{0c}} - \frac{c^2 k}{a^2 H_0^2 L^2} \tag{13.289}$$

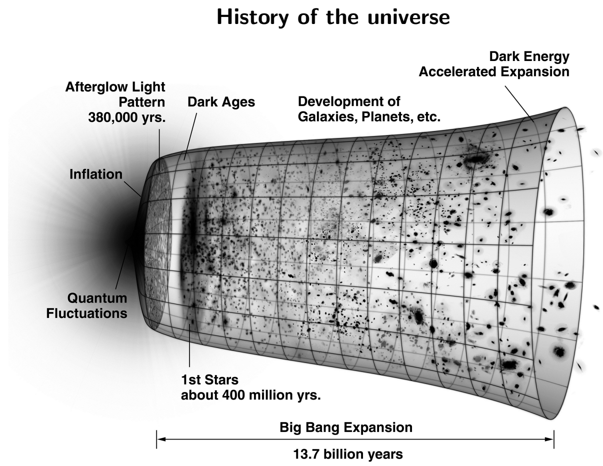

Figure 13.1 NASA/WMAP Science Team's timeline of the known universe.

in which ρ is the total density (13.287)

$$\frac{H^2}{H_0^2} = \frac{\rho_\Lambda}{\rho_{c0}} + \frac{\rho_m}{\rho_{c0}} + \frac{\rho_r}{\rho_{c0}} - \frac{c^2 k}{a^2 H_0^2 L^2}$$
$$= \frac{\rho_\Lambda}{\rho_{c0}} + \frac{\rho_{m0}}{\rho_{c0}}\frac{a_0^3}{a^3} + \frac{\rho_{r0}}{\rho_{c0}}\frac{a_0^4}{a^4} - \frac{c^2 k}{a_0^2 H_0^2 L^2}\frac{a_0^2}{a^2}. \tag{13.290}$$

The Planck collaboration use a model in which the energy density of the universe is due to radiation, matter, and a cosmological constant Λ. Only about 18.79% of the matter in their model is composed of baryons, $\Omega_b = 0.05845 \pm 0.0003$. Most of the matter is transparent and is called **dark matter**. They assume the dark matter is composed of particles that have masses in excess of a keV so that they are heavy enough to have been nonrelativistic or "cold" when the universe was about a year old (Peebles, 1982). The energy density of the cosmological constant Λ is known as **dark energy**. The Planck collaboration use this Λ-cold-dark-matter (ΛCDM) model and their CMB data to estimate the Hubble constant as $H_0 = 67.66\text{km}/(\text{s}\,\text{Mpc}) = 2.1927 \times 10^{-18}\,\text{s}^{-1}$ and the density ratios $\Omega_\Lambda = \rho_\Lambda/\rho_{c0}$, $\Omega_m = \rho_{m0}/\rho_{c0}$, and $\Omega_k \equiv -c^2 k/(a_0 H_0 L)^2$ as listed in the table (13.1) (Aghanim *et al.*, 2018). The Riess group use the Gaia observatory to calibrate Cepheid stars and type Ia supernovas as standard candles for measuring distances to remote galaxies. The distances and redshifts of these galaxies give the Hubble constant as $H_0 = 73.48 \pm 1.66$ (Riess et al., 2018). As this book goes to press, the 9% discrepancy between the Planck and Riess H_0's is unexplained.

Table 13.1 *Cosmological parameters of the Planck collaboration*

H_0 (km/(s Mpc)	Ω_Λ	Ω_m	Ω_k
67.66 ± 0.42	0.6889 ± 0.0056	0.3111 ± 0.0056	0.0007 ± 0.0037

To estimate the ratio $\Omega_r = \rho_{r0}/\rho_{c0}$ of radiation densities, one may use the present temperature $T_0 = 2.7255 \pm 0.0006$ K (Fixsen, 2009) of the CMB radiation and the formula (5.110) for the energy density of photons

$$\rho_\gamma = \frac{8\pi^5\,(k_B T_0)^4}{15h^3c^5} = 4.6451 \times 10^{-31} \text{ kg m}^{-3}. \tag{13.291}$$

Adding in three kinds of neutrinos and antineutrinos at $T_{0\nu} = (4/11)^{1/3}\,T_0$, we get for the present density of massless and nearly massless particles (Weinberg, 2010, section 2.1)

$$\rho_{r0} = \left[1 + 3\left(\frac{7}{8}\right)\left(\frac{4}{11}\right)^{4/3}\right]\,\rho_\gamma = 7.8099 \times 10^{-31} \text{ kg m}^{-3}. \tag{13.292}$$

The fraction Ω_r the present energy density that is due to radiation is then

$$\Omega_r = \frac{\rho_{r0}}{\rho_{c0}} = 9.0824 \times 10^{-5}. \tag{13.293}$$

In terms of Ω_r and of the Ω's in the table (13.1), the formula (13.290) for H^2/H_0^2 is

$$\frac{H^2}{H_0^2} = \Omega_\Lambda + \Omega_k\,\frac{a_0^2}{a^2} + \Omega_m\,\frac{a_0^3}{a^3} + \Omega_r\,\frac{a_0^4}{a^4}. \tag{13.294}$$

Since $H = \dot{a}/a$, one has $dt = da/(aH) = H_0^{-1}(da/a)(H_0/H)$, and so with $x = a/a_0$, the time interval dt is

$$dt = \frac{1}{H_0}\,\frac{dx}{x}\,\frac{1}{\sqrt{\Omega_\Lambda + \Omega_k\,x^{-2} + \Omega_m\,x^{-3} + \Omega_r\,x^{-4}}}. \tag{13.295}$$

Integrating and setting the origin of time $t(0) = 0$ and the scale factor at the present time equal to unity $a_0 = 1$, we find that the time $t(a)$ that $a(t)$ took to grow from 0 to $a(t)$ is

$$t(a) = \frac{1}{H_0}\int_0^a \frac{dx}{\sqrt{\Omega_\Lambda\,x^2 + \Omega_k + \Omega_m\,x^{-1} + \Omega_r\,x^{-2}}}. \tag{13.296}$$

This integral gives the age of the universe as $t(1) = 13.789$ Gyr; the Planck-collaboration value is 13.787 ± 0.020 Gyr (Aghanim *et al.*, 2018). Figure 13.2 plots the scale factor $a(t)$ and the redshift $z(t) = 1/a - 1$ as functions of the time t (13.296) for the first 14 billion years after the time $t = 0$ of infinite redshift. A photon emitted with wavelength λ at time $t(a)$ now has wavelength $\lambda_0 = \lambda/a(t)$. The change in its wavelength is $\Delta\lambda = \lambda\, z(t) = \lambda\,(1/a - 1) = \lambda_0 - \lambda$.

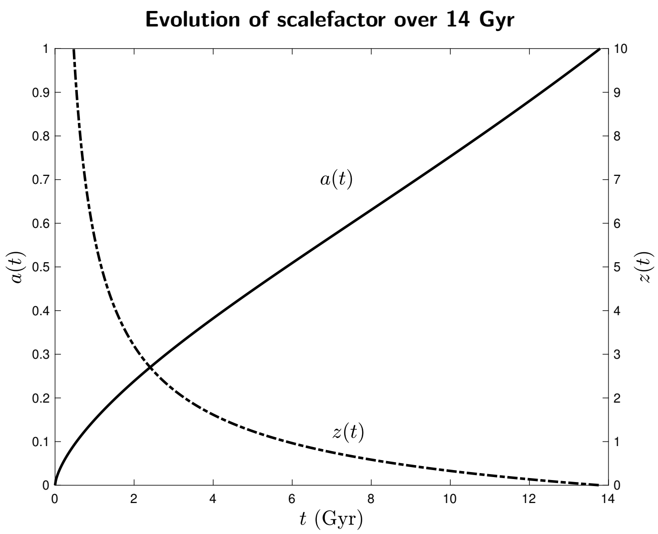

Figure 13.2 The scale factor $a(t)$ (solid, left axis) and redshift $z(t)$ (dotdash, right axis) are plotted against the time (13.296) in Gyr. This chapter's Fortran, Matlab, and Mathematica scripts are in Tensors_and_general_relativity at `github.com/kevinecahill`.

13.46 The first hundred thousand years

Figure 13.3 plots the scale factor $a(t)$ as given by the integral (13.296) and the densities of radiation $\rho_r(t)$ and matter $\rho_m(t)$ for the first 100,000 years after the time of infinite redshift. Because wavelengths grow with the scale factor, the radiation density (13.286) is proportional to the inverse fourth power of the scale factor $\rho_r(t) = \rho_{r0}/a^4(t)$. The density of radiation therefore was dominant at early times when the scale factor was small. Keeping only

$\Omega_r = 0.6889$ in the integral (13.296), we get

$$t = \frac{a^2}{2H_0\sqrt{\Omega_r}} \quad \text{and} \quad a(t) = \Omega_r^{1/4}\sqrt{2\,H_0\,t}. \tag{13.297}$$

Since the radiation density $\rho_r(t) = \rho_{r0}/a^4(t)$ also is proportional to the fourth power of the temperature $\rho_r(t) \sim T^4$, the temperature varied as the inverse of the scale factor $T \sim 1/a(t) \sim t^{-1/2}$ during the era of radiation.

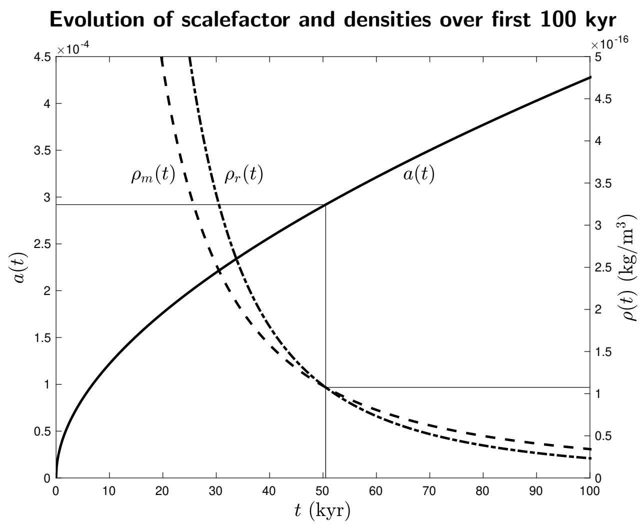

Figure 13.3 The Planck-collaboration scale factor a (solid), radiation density ρ_r (dotdash), and matter density ρ_m (dashed) are plotted as functions of the time (13.296) in kyr. The era of radiation ends at $t = 50,506$ years when the two densities are equal to 1.0751×10^{-16} kg/m^3, $a = 2.919 \times 10^{-4}$, and $z = 3425$.

In cold-dark-matter models, when the temperature was in the range $10^{12} > T > 10^{10}\,\text{K}$ or $m_\mu c^2 > kT > m_e c^2$, where m_μ is the mass of the muon and m_e that of the electron, the radiation was mostly electrons, positrons, photons, and neutrinos, and the relation between the time t and the temperature T was $t \sim 0.994 \;\text{ sec} \times (10^{10}\,\text{K}/T)^2$ (Weinberg, 2010, ch. 3). By 10^9 K, the positrons had annihilated with electrons, and the neutrinos fallen out of equilibrium. Between 10^9 K and 10^6 K, when the energy density of nonrelativistic particles became relevant, the time-temperature relation was $t \sim 1.78 \;\text{ sec} \times (10^{10}\,\text{K}/T)^2$ (Weinberg, 2010, ch. 3). During the first three minutes (Weinberg, 1988) of the era of radiation, quarks and gluons formed hadrons, which decayed into protons and neutrons. As the neutrons decayed ($\tau = 877.7$ s), they and the protons formed the light elements—principally hydrogen, deuterium, and helium in a process called **big-bang nucleosynthesis**.

The density of nonrelativistic matter (13.285) falls as the third power of the scale factor $\rho_m(t) = \rho_{m0}/a^3(t)$. The more rapidly falling density of radiation $\rho_r(t)$ crosses it 50,506 years after the Big Bang as indicated by the vertical line in the figure (13.3). This time $t = 50,506$ yr and redshift $z = 3425$ mark the end of the **era of radiation**.

13.47 The next ten billion years

The **era of matter** began about 50,506 years after the time of infinite redshift when the matter density ρ_m first exceeded the radiation density ρ_r. Some 330,000 years later at $t \sim 380,000$ yr, the universe had cooled to about $T = 3000$ K or $kT = 0.26\,\text{eV}$—a temperature at which less than 1% of the hydrogen was ionized. At this redshift of $z = 1090$, the plasma of ions and electrons became a **transparent** gas of neutral hydrogen and helium with trace amounts of deuterium, helium-3, and lithium-7. The photons emitted or scattered at that time as 0.26 eV or 3000 K photons have redshifted down to become the 2.7255 K photons of the cosmic microwave background (CMB) radiation. This time of last scattering and first transparency often and inexplicably is called **recombination**.

If we approximate time periods $t - t_m$ during the era of matter by keeping only Ω_m in the integral (13.296), then we get

$$t - t_m = \frac{2\,a^{2/3}}{3H_0\sqrt{\Omega_m}} \quad \text{and} \quad a(t) = \left(\frac{3H_0\sqrt{\Omega_m}\,(t - t_m)}{2}\right)^{2/3} \tag{13.298}$$

in which t_m is a time well inside the era of matter.

Between 10 and 17 million years after the Big Bang, the temperature of the known universe fell from 373 to 273 K. If by then the supernovas of very early, very heavy stars had produced carbon, nitrogen, and oxygen, biochemistry may have started during this period of 7 million years. Stars did form at least as early as 180 million years after the Big Bang (Bowman et al., 2018).

The era of matter lasted until the energy density of matter $\rho_m(t)$, falling as $\rho_m(t) = \rho_{m0}/a^3(t)$ had dropped to the energy density of dark energy $\rho_\Lambda = 5.9238 \times 10^{-27}\text{kg/m}^3$. This happened at $t = 10.228$ Gyr as indicated by the first vertical line in the figure (13.4).

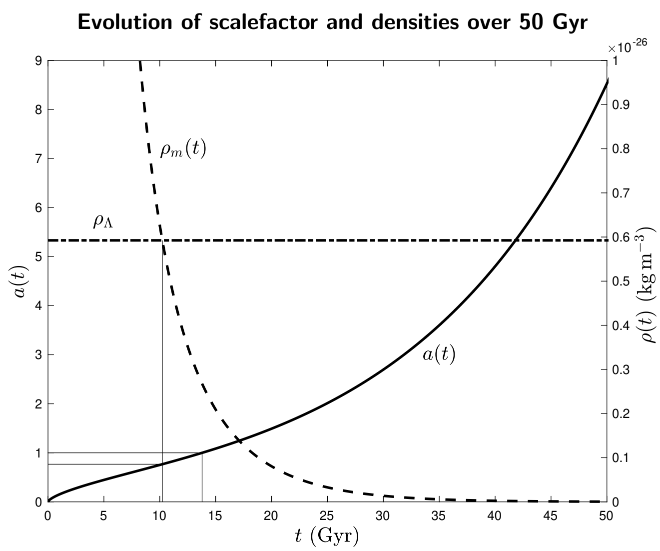

Figure 13.4 The scale factor a (solid), the vacuum density ρ_Λ (dotdash), and the matter density ρ_m (dashed) are plotted as functions of the time (13.296) in Gyr. The era of matter ends at $t = 10.228$ Gyr (first vertical line) when the two densities are equal to 5.9238×10^{-27}kg m^{-3} and $a = 0.7672$. The present time t_0 is 13.787 Gyr (second vertical line) at which $a(t) = 1$.

13.48 Era of dark energy

The era of dark energy began 3.6 billion years ago at a redshift of $z = 0.3034$ when the energy density of the universe $\rho_m + \rho_\Lambda$ was twice that of empty space, $\rho = 2\,\rho_\Lambda = 1.185 \times 10^{-26}$ kg/m^3. The energy density of matter now is only 31.11% of the energy density of the universe, and it is falling as the cube of the scale factor $\rho_m(t) = \rho_{m0}/a^3(t)$. In another 20 billion years, the energy density of the universe will have declined almost all the way to the dark-energy density $\rho_\Lambda = 5.9238 \times 10^{-27}$ kg/m^3 or $(1.5864 \text{ meV})^4/(\hbar^3 c^5)$. At that time t_Λ and in the indefinite future, the only significant term in the integral (13.296) will be the vacuum energy. Neglecting the others and replacing $a_0 = 1$ with $a_\Lambda = a(t_\Lambda)$, we find

$$t(a/a_\Lambda) - t_\Lambda = \frac{\log(a/a_\Lambda)}{H_0\sqrt{\Omega_\Lambda}} \quad \text{or} \quad a(t) = e^{H_0\sqrt{\Omega_\Lambda}(t-t_\Lambda)}\, a(t_\Lambda) \qquad (13.299)$$

in which $t_\Lambda \gtrsim 35$ Gyr.

13.49 Before the Big Bang

The ΛCDM model is remarkably successful (Aghanim *et al.*, 2018). But it does not explain why the CMB is so isotropic, apart from a Doppler shift, and why the universe is so flat (Guth, 1981). A brief period of rapid exponential growth like that of the era of dark energy may explain the isotropy and the flatness.

Inflation occurs when the potential energy ρ dwarfs the energy of matter and radiation. The internal energy of the universe then is proportional to its volume $U = c^2 \rho V$, and its pressure p as given by the thermodynamic relation

$$p = -\frac{\partial U}{\partial V} = -c^2 \rho \tag{13.300}$$

is *negative*. The second-order Friedmann equation (13.273)

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right) = \frac{8\pi G \rho}{3} \tag{13.301}$$

then implies exponential growth like that of the era of dark energy (13.299)

$$a(t) = e^{\sqrt{8\pi G\rho/3}\, t}\, a(0). \tag{13.302}$$

The origin of the potential-energy density ρ is unknown.

In **chaotic inflation** (Linde, 1983), a scalar field ϕ fluctuates to a mean value $\langle \phi \rangle_i$ that makes its potential-energy density ρ huge. The field remains at or close to the value $\langle \phi \rangle_i$, and the scale factor inflates rapidly and exponentially (13.302) until time t at which the potential energy of the universe is

$$E = c^2 \rho\, e^{\sqrt{24\pi G\rho}\, t}\, V(0) \tag{13.303}$$

where $V(0)$ is the spatial volume in which the field ϕ held the value $\langle \phi \rangle_i$. After time t, the field returns to its mean value $\langle 0|\phi|0\rangle$ in the ground state $|0\rangle$ of the theory, and the huge energy E is released as radiation in a Big Bang. The energy E_g of the gravitational field caused by inflation is negative, $E_g = -E$, and energy is conserved. Chaotic inflation plausibly explains why there is a universe: a quantum fluctuation made it. If the universe did arise from a quantum fluctuation, other quantum fluctuations would occur elsewhere inflating new universes and making a **multiverse**.

If a quantum fluctuation gives a field ϕ a spatially constant mean value $\langle\phi\rangle_i \equiv \phi$ in an initial volume $V(0)$, then the equations for the scale factor (13.301) and for the scalar field (13.201) simplify to

$$H = \left(\frac{8\pi G\rho}{3}\right)^{1/2} \quad \text{and} \quad \ddot{\phi} = -3\,H\,\dot{\phi} - \frac{m^2c^4}{\hbar^2}\phi \tag{13.304}$$

in which ρ is the mass density of the potential energy of the field ϕ. The term $-3H\dot{\phi}$ is a kind of gravitational friction. It may explain why a field ϕ sticks at the value $\langle\phi\rangle_i$ long enough to resolve the isotropy and flatness puzzles.

The anti-de Sitter ($k = -1$) spacetime $a(t) = c/(L\,\lambda)\,\sin(\lambda\,t)$ of example 7.63 and bouncing cosmologies (Steinhardt et al., 2002; Ijjas and Steinhardt, 2018) explain the flatness and isotropy of the universe as due to repeated collapses and rebirths. Experiments will tell us whether inflation or bouncing or something else actually occurred (Akrami *et al.*, 2018).

13.50 Yang-Mills theory

The gauge transformation of an **abelian** gauge theory like electrodynamics multiplies a *single* charged field by a spacetime-dependent *phase factor* $\phi'(x) = \exp(iq\theta(x))\,\phi(x)$. Yang and Mills generalized this gauge transformation to one that multiplies a *vector* ϕ of matter fields by a spacetime dependent *unitary matrix* $U(x)$

$$\phi'_a(x) = \sum_{b=1}^{n} U_{ab}(x)\,\phi_b(x) \quad \text{or} \quad \phi'(x) = U(x)\,\phi(x) \tag{13.305}$$

and showed how to make the action of the theory invariant under such **non-abelian** gauge transformations. (The fields ϕ are scalars for simplicity.)

Since the matrix U is unitary, inner products like $\phi^\dagger(x)\,\phi(x)$ are automatically invariant

$$\left(\phi^\dagger(x)\,\phi(x)\right)' = \phi^\dagger(x)U^\dagger(x)U(x)\phi(x) = \phi^\dagger(x)\phi(x). \tag{13.306}$$

But inner products of derivatives $\partial^i\phi^\dagger\,\partial_i\phi$ are not invariant because the derivative acts on the matrix $U(x)$ as well as on the field $\phi(x)$.

Yang and Mills made derivatives $D_i\phi$ that transform like the fields ϕ

$$(D_i\phi)' = U\,D_i\phi. \tag{13.307}$$

To do so, they introduced **gauge-field matrices** A_i that play the role of the connections Γ_i in general relativity and set

$$D_i = \partial_i + A_i \tag{13.308}$$

in which A_i like ∂_i is antihermitian. They required that under the gauge transformation (13.305), the gauge-field matrix A_i transform to A_i' in such a way as to make the derivatives transform as in (13.307)

$$(D_i\phi)' = \left(\partial_i + A_i'\right)\phi' = \left(\partial_i + A_i'\right)U\phi = U\,D_i\phi = U\left(\partial_i + A_i\right)\phi. \tag{13.309}$$

So they set

$$\left(\partial_i + A_i'\right)U\phi = U\left(\partial_i + A_i\right)\phi \quad \text{or} \quad \left(\partial_i U\right)\phi + A_i' U\phi = U A_i\,\phi \tag{13.310}$$

and made the gauge-field matrix A_i transform as

$$A_i' = U A_i U^{-1} - \left(\partial_i U\right)U^{-1}. \tag{13.311}$$

Thus under the gauge transformation (13.305), the derivative $D_i\phi$ transforms as in (13.307), like the vector ϕ in (13.305), and the inner product of covariant derivatives

$$\left[\left(D^i\phi\right)^\dagger D_i\phi\right]' = \left(D^i\phi\right)^\dagger U^\dagger U D_i\phi = \left(D^i\phi\right)^\dagger D_i\phi \tag{13.312}$$

remains invariant.

To make an invariant action density for the gauge-field matrices A_i, they used the transformation law (13.309) which implies that $D_i' U\phi = U D_i\,\phi$ or $D_i' = U D_i\,U^{-1}$. So they defined their generalized Faraday tensor as

$$F_{ik} = [D_i, D_k] = \partial_i A_k - \partial_k A_i + [A_i, A_k] \tag{13.313}$$

so that it transforms covariantly

$$F_{ik}' = U F_{ik} U^{-1}. \tag{13.314}$$

They then generalized the action density $F_{ik}F^{ik}$ of electrodynamics to the trace $\mathrm{Tr}\left(F_{ik}F^{ik}\right)$ of the square of the Faraday matrices which is invariant under gauge transformations since

$$\mathrm{Tr}\left(U F_{ik} U^{-1} U F^{ik} U^{-1}\right) = \mathrm{Tr}\left(U F_{ik} F^{ik} U^{-1}\right) = \mathrm{Tr}\left(F_{ik}F^{ik}\right). \tag{13.315}$$

As an action density for fermionic matter fields, they replaced the ordinary derivative in Dirac's formula $\overline{\psi}(\gamma^i\partial_i + m)\psi$ by the covariant derivative (13.308) to get $\overline{\psi}(\gamma^i D_i + m)\psi$ (Chen-Ning Yang 1922–, Robert L. Mills 1927–1999).

In an abelian gauge theory, the square of the 1-form $A = A_i\,dx^i$ vanishes

$A^2 = A_i\, A_k\, dx^i \wedge dx^k = 0$, but in a nonabelian gauge theory the gauge fields are matrices, and $A^2 \neq 0$. The sum $dA + A^2$ is the Faraday 2-form

$$F = dA + A^2 = (\partial_i\, A_k + A_i\, A_k)\; dx^i \wedge dx^k \qquad (13.316)$$
$$= \tfrac{1}{2}\,(\partial_i\, A_k - \partial_k\, A_i + [A_i,\, A_k])\; dx^i \wedge dx^k = \tfrac{1}{2} F_{ik}\, dx^i \wedge dx^k.$$

The scalar matter fields ϕ may have self-interactions described by a potential $V(\phi)$ such as $V(\phi) = \lambda(\phi^\dagger\phi - m^2/\lambda)^2$ which is positive unless $\phi^\dagger\phi = m^2/\lambda$. The kinetic action of these fields is $(D^i\phi)^\dagger D_i\phi$. At low temperatures, these scalar fields assume mean values $\langle 0|\phi|0\rangle = \phi_0$ in the vacuum with $\phi_0^\dagger\phi_0 = m^2/\lambda$ so as to minimize their potential energy density $V(\phi)$ and their kinetic action $(D^i\phi)^\dagger D_i\phi = (\partial^i\phi + A^i\phi)^\dagger(\partial_i\phi + A_i\phi)$ is approximately $\phi_0^\dagger\, A^i\, A_i\, \phi_0$. The gauge-field matrix $A^i_{ab} = i\, t^\alpha_{ab} A^i_\alpha$ is a linear combination of the generators t^α of the gauge group. So the action of the scalar fields contains the term $\phi_0^\dagger\, A^i\, A_i\, \phi_0 = -\, M^2_{\alpha\beta}\, A^i_\alpha\, A_{i\beta}$ in which the mass-squared matrix for the gauge fields is $M^2_{\alpha\beta} = \phi_0^{*a}\, t^\alpha_{ab}\, t^\beta_{bc}\, \phi_0^c$. This **Higgs mechanism** gives masses to those linear combinations $b_{\beta i}\, A_\beta$ of the gauge fields for which $M^2_{\alpha\beta}\, b_{\beta i} = m_i^2 b_{\alpha i} \neq 0$.

The Higgs mechanism also gives masses to the fermions. The mass term m in the Yang-Mills-Dirac action is replaced by something like $c\,\phi$ in which c is a constant, different for each fermion. In the vacuum and at low temperatures, each fermion acquires as its mass $c\,\phi_0$. On 4 July 2012, physicists at CERN's Large Hadron Collider announced the discovery of a Higgs-like particle with a mass near 12 5 GeV/c^2 (Peter Higgs 1929 –).

13.51 Cartan's spin connection and structure equations

Cartan's tetrads (13.153) $c^a_{\ k}(x)$ are the rows and columns of the orthogonal matrix that turns the flat-space metric η_{ab} into the curved-space metric $g_{ik} = c^a_i\, \eta_{ab} c^b_k$. Early-alphabet letters $a, b, c, d, \cdots = 0, 1, 2, 3$ are Lorentz indexes, and middle-to-late letters $i, j, k, \ell, \cdots = 0, 1, 2, 3$ are spacetime indexes. Under a combined local Lorentz (13.154) and general coordinate transformation the tetrads transform as

$$c'^a_{\ k}(x') = L^a_{\ b}(x')\, \frac{\partial x^\ell}{\partial x'^k}\, c^b_{\ \ell}(x). \qquad (13.317)$$

The covariant derivative of a tetrad $D_\ell c$ must transform as

$$(D_\ell\, c^a_{\ k})'(x') = L^a_{\ b}(x')\, \frac{\partial x^i}{\partial x'^\ell}\, \frac{\partial x^j}{\partial x'^k}\, D_i\, c^b_{\ j}(x). \qquad (13.318)$$

We can use the affine connection $\Gamma^j{}_{k\ell}$ and the formula (13.68) for the covariant derivative of a covariant vector to cope with the index j. And we can treat the Lorentz index b like an index of a nonabelian group as in section 13.50 by introducing a gauge field $\omega^a{}_{\ell b}$

$$D_\ell \, c^a{}_k = c^a{}_{,\ell} - \Gamma^j{}_{k\ell} \, c^a{}_j + \omega^a{}_{\ell b} \, c^b{}_k. \tag{13.319}$$

The affine connection is defined so as to make the covariant derivative of the tangent basis vectors vanish

$$D_\ell \, e^\alpha_k = e^\alpha_{k,\ell} - \Gamma^j{}_{k\ell} e^\alpha_j = 0 \tag{13.320}$$

in which the Greek letter α labels the coordinates $0, 1, 2, \ldots, n$ of the embedding space. We may verify this relation by taking the inner product in the embedding space with the dual tangent vector e^i_α

$$e^i_\alpha \Gamma^j{}_{k\ell} e^\alpha_j = \delta^i_j \Gamma^j{}_{k\ell} = \Gamma^i{}_{k\ell} = e^i_\alpha e^\alpha_{k,\ell} = e^i \cdot e_{k,\ell} \tag{13.321}$$

which is the definition (13.59) of the affine connection, $\Gamma^i{}_{k\ell} = e^i \cdot e_{k,\ell}$. So too the **spin connection** $\omega^a{}_{b\,\ell}$ is defined so as to make the covariant derivative of the tetrad vanish

$$D_\ell \, c^a{}_k = c^a{}_{k;\ell} = c^a{}_{k,\ell} - \Gamma^j{}_{k\ell} \, c^a_j + \omega^a{}_{d\,\ell} \, c^d_k = 0. \tag{13.322}$$

The dual tetrads c^k_b are doubly orthonormal:

$$c^k_b \, c^b_i = \delta^k_i \quad \text{and} \quad c^k_a \, c^b_k = \delta^b_a. \tag{13.323}$$

Thus using their orthonormality, we have $\omega^a{}_{d\,\ell} \, c^d_k \, c^k_b = \omega^a{}_{d\,\ell} \, \delta^d_b = \omega^a{}_{b\,\ell}$, and so the spin connection is

$$\omega^a{}_{b\,\ell} = \, - \, c^k_b \big(c^a_{k,\ell} - \Gamma^j_{k\ell} \, c^a_j \big) = c^a_j \, c^k_b \, \Gamma^j{}_{k\ell} - c^a_{k,\ell} \, c^k_b. \tag{13.324}$$

In terms of the differential forms (section 12.6)

$$c^a = c^a_k \, dx^k \quad \text{and} \quad \omega^a{}_b = \omega^a{}_{b\,\ell} \, dx^\ell \tag{13.325}$$

we may use the exterior derivative to express the vanishing (13.322) of the covariant derivative $c^a{}_{k;\ell}$ as

$$dc^a = c^a_{k,\ell} \, dx^\ell \wedge dx^k = \Big(\Gamma^j{}_{k\ell} \, c^a_j - \omega^a{}_{b\,\ell} \, c^b_k \Big) \, dx^\ell \wedge dx^k. \tag{13.326}$$

But the affine connection $\Gamma^j{}_{k\ell}$ is symmetric in k and ℓ while the wedge product $dx^\ell \wedge dx^k$ is antisymmetric in k and ℓ. Thus we have

$$dc^a = c^a_{k,\ell} \, dx^\ell \wedge dx^k = \, - \, \omega^a{}_{b\,\ell} \, dx^\ell \wedge c^b_k \, dx^k \tag{13.327}$$

or with $c \equiv c^a_k dx^k$ and $\omega \equiv \omega^a{}_{b\,\ell} dx^\ell$

$$dc = -\omega \wedge c \tag{13.328}$$

which is **Cartan's first equation of structure**. Cartan's curvature 2-form is

$$\begin{aligned} R^a{}_b &= \tfrac{1}{2}\, c^a_j\, c^i_b\, R^j{}_{ik\ell}\, dx^k \wedge dx^\ell \\ &= \tfrac{1}{2}\, c^a_j\, c^i_b \Big[\Gamma^j{}_{\ell i,k} - \Gamma^j{}_{ki,\ell} + \Gamma^j{}_{kn}\, \Gamma^n{}_{\ell i} - \Gamma^j{}_{\ell n}\, \Gamma^n{}_{ki} \Big] dx^k \wedge dx^\ell . \end{aligned} \tag{13.329}$$

His **second equation of structure** expresses $R^a{}_b$ as

$$R^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b \tag{13.330}$$

or more simply as

$$R = d\omega + \omega \wedge \omega. \tag{13.331}$$

A more compact notation, similar to that of Yang-Mills theory, uses Cartan's covariant exterior derivative

$$D \equiv d + \omega \wedge \tag{13.332}$$

to express his two structure equations as

$$D\,c = 0 \quad \text{and} \quad R = D\,\omega. \tag{13.333}$$

To derive Cartan's second structure equation (13.330), we let the exterior derivative act on the 1-form $\omega^a{}_b$

$$d\omega^a{}_b = d(\omega^a{}_{b\,\ell}\, dx^\ell) = \omega^a{}_{b\,\ell,k}\, dx^k \wedge dx^\ell \tag{13.334}$$

and add the 2-form $\omega^a{}_c \wedge \omega^c{}_b$

$$\omega^a{}_c \wedge \omega^c{}_b = \omega^a{}_{c\,k}\, \omega^c{}_{b\,\ell}\, dx^k \wedge dx^\ell \tag{13.335}$$

to get

$$S^a{}_b = (\omega^a{}_{b\,\ell,k} + \omega^a{}_{c\,k}\, \omega^c{}_{b\,\ell})\, dx^k \wedge dx^\ell \tag{13.336}$$

which we want to show is Cartan's curvature 2-form $R^a{}_b$ (13.329). First we replace $\omega^a{}_{\ell b}$ with its equivalent (13.324) $c^a_j\, c^i_b\, \Gamma^j{}_{i\ell} - c^a_{i,\ell}\, c^i_b$

$$\begin{aligned} S^a{}_b = \Big[&(c^a_j\, c^i_b\, \Gamma^j{}_{i\ell} - c^a_{i,\ell}\, c^i_b)_{,k} + (c^a_j\, c^i_c\, \Gamma^j{}_{ik} - c^a_{i,k}\, c^i_c)(c^c_n\, c^p_b\, \Gamma^n{}_{p\ell} - c^c_{p,\ell}\, c^p_b) \Big] \\ &\times\, dx^k \wedge dx^\ell . \end{aligned} \tag{13.337}$$

The terms proportional to $\Gamma^j{}_{i\ell,k}$ are equal to those in the definition (13.329) of Cartan's curvature 2-form. Among the remaining terms in ${S^a}_b$, those independent of Γ are after explicit antisymmetrization

$$S_0 = c^a_{i,k}\, c^i_{b,\ell} - c^a_{i,\ell}\, c^i_{b,k} + c^a_{i,k}\, c^i_c\, c^c_{p,\ell}\, c^p_b - c^a_{i,\ell}\, c^i_c\, c^c_{p,k}\, c^p_b \tag{13.338}$$

which vanishes (exercise 13.36) because $c^i_{b,k} = -c^i_c\, c^c_{p,k}\, c^p_b$. The terms in ${S^a}_b$ that are linear in Γ's also vanish (exercise 13.37). Finally, the terms in ${S^a}_b$ that are quadratic in Γ's are

$$\begin{aligned} c^a_j\, c^i_c\, c^c_n\, c^p_b\, {\Gamma^j}_{ik}\, {\Gamma^n}_{p\ell}\, dx^k \wedge dx^\ell &= c^a_j\, \delta^i_n\, c^p_b\, {\Gamma^j}_{ik}\, {\Gamma^n}_{p\ell}\, dx^k \wedge dx^\ell \\ &= c^a_j\, c^p_b\, {\Gamma^j}_{nk}\, {\Gamma^n}_{p\ell}\, dx^k \wedge dx^\ell \end{aligned} \tag{13.339}$$

and these match those of Cartan's curvature 2-form ${R^a}_b$ (13.329). Since ${S^a}_b = {R^a}_b$, Cartan's second equation of structure (13.330) follows.

Example 13.27 (Cyclic identity for the curvature tensor) We can use Cartan's structure equations to derive the cyclic identity (13.109) of the curvature tensor. We apply the exterior derivative (whose square $dd = 0$) to Cartan's first equation of structure (13.328) and then use it and his second equation of structure (13.330) to write the result as

$$0 = d(dc + \omega \wedge c) = (d\omega) \wedge c - \omega \wedge dc = (d\omega + \omega \wedge \omega) \wedge c = R \wedge c. \tag{13.340}$$

The definition (13.329) of Cartan's curvature 2-form R and of his 1-form (13.325) now give

$$\begin{aligned} 0 = R \wedge c =& \tfrac{1}{2}\, c^a_j\, c^i_b\, {R^j}_{ik\ell}\, dx^k \wedge dx^\ell \wedge c^b_m dx^m \\ =& \tfrac{1}{2}\, c^a_j\, {R^j}_{ik\ell}\, dx^k \wedge dx^\ell \wedge dx^i \end{aligned} \tag{13.341}$$

which implies that

$$0 = {R^j}_{[ik\ell]} = \frac{1}{3!}\left({R^j}_{ik\ell} + {R^j}_{\ell ik} + {R^j}_{k\ell i} - {R^j}_{ki\ell} - {R^j}_{i\ell k} - {R^j}_{\ell ki}\right). \tag{13.342}$$

But since Riemann's tensor is antisymmetric in its last two indices (13.104), we can write this result more simply as the cyclic identity (13.109) for the curvature tensor

$$0 = {R^j}_{ik\ell} + {R^j}_{\ell ik} + {R^j}_{k\ell i}. \tag{13.343}$$

□

The vanishing of the covariant derivative of the flat-space metric

$$0 = \eta_{ab;k} = \eta_{ab,k} - {\omega^c}_{ak}\eta_{cb} - {\omega^c}_{bk}\eta_{ac} = -\omega_{bak} - \omega_{abk} \tag{13.344}$$

shows that the spin connection is antisymmetric in its Lorentz indexes

$$\omega_{abk} = -\,\omega_{bak} \quad \text{and} \quad {\omega^{ab}}_k = -\,{\omega^{ba}}_k. \tag{13.345}$$

Under a general coordinate transformation and a local Lorentz transformation, the spin connection (13.324) transforms as

$$\omega'^{a}{}_{b\,\ell} = \frac{\partial x^i}{\partial x'^\ell}\left[L^a{}_d\,\omega^d{}_{e\,i} - (\partial_i L^a{}_e)\right] L^{-1e}{}_b. \tag{13.346}$$

13.52 Spin-one-half fields in general relativity

The action density (11.329) of a free Dirac field is $L = -\bar\psi\,(\gamma^a\partial_a + m)\,\psi$ in which $a = 0,1,2,3$; ψ is a 4-component Dirac field; $\bar\psi = \psi^\dagger\beta = i\psi^\dagger\gamma^0$; and m is a mass. Tetrads $c^a_k(x)$ turn the flat-space indices a into curved-space indices i, so one first replaces $\gamma^a\partial_a$ by $\gamma^a\,c^\ell_a\partial_\ell$. The next step is to use the spin connection (13.324) to correct for the effect of the derivative ∂_ℓ on the field ψ. The fully covariant derivative is $D_\ell = \partial_\ell - \frac{1}{8}\,\omega^{ab}{}_\ell\,[\gamma_a,\gamma_b]$ where $\omega^{ab}{}_\ell = \omega^a{}_{c\,\ell}\,\eta^{bc}$, and the action density is $L = -\,\bar\psi(\gamma^a c^\ell_a\,D_\ell + m)\psi$.

13.53 Gauge theory and vectors

This section is optional on a first reading.

We can formulate Yang-Mills theory in terms of vectors as we did relativity. To accomodate noncompact groups, we generalize the unitary matrices $U(x)$ of the Yang-Mills gauge group to nonsingular matrices $V(x)$ that act on n matter fields $\psi^a(x)$ as $\psi'^a(x) = V^a_b(x)\,\psi^b(x)$. The field $\Psi(x) = e_a(x)\,\psi^a(x)$ will be gauge invariant $\Psi'(x) = \Psi(x)$ if the vectors $e_a(x)$ transform as $e'_a(x) = e_b(x)\,V^{-1b}{}_a(x)$. We are summing over repeated indices from 1 to n and often will suppress explicit mention of the spacetime coordinates. In this compressed notation, the field Ψ is gauge invariant because

$$\Psi' = e'_a\,\psi'^a = e_b\,V^{-1b}{}_a\,V^a{}_c\,\psi^c = e_b\,\delta^b{}_c\,\psi^c = e_b\,\psi^b = \Psi \tag{13.347}$$

which is $e'^{\mathsf{T}}\psi' = e^{\mathsf{T}}V^{-1}V\psi = e^{\mathsf{T}}\psi$ in matrix notation.

The inner product of two basis vectors is an internal "metric tensor"

$$e^*_a\cdot e_b = \sum_{\alpha=1}^{N}\sum_{\beta=1}^{N} e^{\alpha*}_a\,\eta_{\alpha\beta}e^\beta_b = \sum_{\alpha=1}^{N} e^{\alpha*}_a\,e^\alpha_b = g_{ab} \tag{13.348}$$

in which for simplicity I used the the N-dimensional identity matrix for the metric η. As in relativity, we'll assume the matrix g_{ab} to be nonsingular. We then can use its inverse to construct dual vectors $e^a = g^{ab}e_b$ that satisfy $e^{a\dagger}\cdot e_b = \delta^a_b$.

The free Dirac action density of the invariant field Ψ

$$\overline{\Psi}(\gamma^i \partial_i + m)\Psi = \overline{\psi}_a e^{a\dagger}(\gamma^i \partial_i + m) e_b \psi^b = \overline{\psi}_a \left[\gamma^i (\delta^a{}_b \partial_i + e^{a\dagger} \cdot e_{b,i}) + m\delta^a{}_b\right] \psi^b \tag{13.349}$$

is the full action of the component fields ψ^b

$$\overline{\Psi}(\gamma^i \partial_i + m)\Psi = \overline{\psi}_a(\gamma^i D^a_{i\,b} + m\,\delta^a{}_b)\psi^b = \overline{\psi}_a \left[\gamma^i(\delta^a{}_b \partial_i + A^a_{i\,b}) + m\,\delta^a{}_b\right] \psi^b \tag{13.350}$$

if we identify the gauge-field matrix as $A^a_{i\,b} = e^{a\dagger} \cdot e_{b,i}$ in harmony with the definition (13.59) of the affine connection $\Gamma^k_{i\ell} = e^k \cdot e_{\ell,i}$.

Under the gauge transformation $e'_a = e_b\, V^{-1b}{}_a$, the metric matrix transforms as

$$g'_{ab} = V^{-1c*}{}_a\, g_{cd}\, V^{-1d}{}_b \qquad \text{or as} \qquad g' = V^{-1\dagger}\, g\, V^{-1} \tag{13.351}$$

in matrix notation. Its inverse goes as $g'^{-1} = V\, g^{-1}\, V^\dagger$.

The gauge-field matrix $A^a_{i\,b} = e^{a\dagger} \cdot e_{b,i} = g^{ac} e^\dagger_c \cdot e_{b,i}$ transforms as

$$A'^a_{i\,b} = g'^{ac} e'^\dagger_a \cdot e'_{b,i} = V^a{}_c A^c_{id} V^{-1d}{}_b + V^a{}_c V^{-1c}{}_{b,i} \tag{13.352}$$

or as $A'_i = V A_i V^{-1} + V \partial_i V^{-1} = V A_i V^{-1} - (\partial_i V)\, V^{-1}$.

By using the identity $e^{a\dagger} \cdot e_{c,i} = \,- e^{a\dagger}_{,i} \cdot e_c$, we may write (exercise 13.39) the Faraday tensor as

$$F^a_{ijb} = [D_i, D_j]^a{}_b = e^{a\dagger}_{,i} \cdot e_{b,j} - e^{a\dagger}_{,i} \cdot e_c\; e^{c\dagger} \cdot e_{b,j} - e^{a\dagger}_{,j} \cdot e_{b,i} + e^{a\dagger}_{,j} \cdot e_c\; e^{c\dagger} \cdot e_{b,i}. \tag{13.353}$$

If $n = N$, then

$$\sum_{c=1}^{n} e^\alpha_c\; e^{\beta c*} = \delta^{\alpha\beta} \quad \text{and} \quad F^a_{ijb} = 0. \tag{13.354}$$

The Faraday tensor vanishes when $n = N$ because the dimension of the embedding space is too small to allow the tangent space to have different orientations at different points x of spacetime. The Faraday tensor, which represents internal curvature, therefore must vanish. One needs at least three dimensions in which to bend a sheet of paper. The embedding space must have $N > 2$ dimensions for $SU(2)$, $N > 3$ for $SU(3)$, and $N > 5$ for $SU(5)$.

The covariant derivative of the internal metric matrix

$$g_{;i} = g_{,i} - g A_i - A^\dagger_i g \tag{13.355}$$

does not vanish and transforms as $(g_{;i})' = V^{-1\dagger} g_{,i} V^{-1}$. A suitable action density for it is the trace $\mathrm{Tr}(g_{;i} g^{-1} g^{;i} g^{-1})$. If the metric matrix assumes a (constant, hermitian) mean value g_0 in the vacuum at low temperatures, then its action is

$$m^2 \mathrm{Tr}\left[(g_0 A_i + A_i^\dagger g_0) g_0^{-1} (g_0 A^i + A^{i\dagger} g_0) g_0^{-1}\right] \tag{13.356}$$

which is a mass term for the matrix of gauge bosons

$$W_i = g_0^{1/2}\, A_i\, g_0^{-1/2} + g_0^{-1/2}\, A_i^\dagger\, g_0^{1/2}. \tag{13.357}$$

This mass mechanism also gives masses to the fermions. To see how, we write the Dirac action density (13.350) as

$$\overline{\psi}_a \left[\gamma^i (\delta^a{}_b \partial_i + A^a_{i\,b}) + m\, \delta^a{}_b\right] \psi^b = \overline{\psi}^a \left[\gamma^i (g_{ab}\partial_i + g_{ac} A^c_{i\,b}) + m\, g_{ab}\right] \psi^b. \tag{13.358}$$

Each fermion now gets a mass $m\, c_i$ proportional to an eigenvalue c_i of the hermitian matrix g_0.

This mass mechanism does not leave behind scalar bosons. Whether nature ever uses it is unclear.

Further reading

Einstein Gravity in a Nutshell (Zee, 2013), *Gravitation* (Misner et al., 1973), *Gravitation and Cosmology* (Weinberg, 1972), *Cosmology* (Weinberg, 2010), *General Theory of Relativity* (Dirac, 1996), *Spacetime and Geometry* (Carroll, 2003), *Exact Space-Times in Einstein's General Relativity* (Griffiths and Podolsky, 2009), *Gravitation: Foundations and Frontiers* (Padmanabhan, 2010), *Modern Cosmology* (Dodelson, 2003), *The primordial density perturbation: Cosmology, inflation and the origin of structure* (Lyth and Liddle, 2009), *A First Course in General Relativity* (Schutz, 2009), *Gravity: An Introduction to Einstein's General Relativity* (Hartle, 2003), and *Relativity, Gravitation and Cosmology: A Basic Introduction* (Cheng, 2010).

Exercises

13.1 Use the flat-space formula $d\boldsymbol{p} = \hat{\boldsymbol{x}}\, dx + \hat{\boldsymbol{y}}\, dy + \hat{\boldsymbol{z}}\, dz$ to compute the change $\boldsymbol{dp}$ due to $d\rho$, $d\phi$, and dz, and so derive expressions for the orthonormal basis vectors $\hat{\boldsymbol{\rho}}$, $\hat{\boldsymbol{\phi}}$, and $\hat{\boldsymbol{z}}$ in terms of $\hat{\boldsymbol{x}}$, $\hat{\boldsymbol{y}}$, and $\hat{\boldsymbol{z}}$.

13.2 Similarly, compute the change $\boldsymbol{dp}$ due to dr, $d\theta$, and $d\phi$, and so derive expressions for the orthonormal basis vectors $\hat{\boldsymbol{r}}$, $\hat{\boldsymbol{\theta}}$, and $\hat{\boldsymbol{\phi}}$ in terms of $\hat{\boldsymbol{x}}$, $\hat{\boldsymbol{y}}$, and $\hat{\boldsymbol{z}}$.

13.3 (a) Using the formulas you found in exercise 13.2 for the basis vectors of spherical coordinates, compute the derivatives of the unit vectors $\hat{\boldsymbol{r}}$, $\hat{\boldsymbol{\theta}}$, and $\hat{\boldsymbol{\phi}}$ with respect to the variables r, θ, and ϕ and express them in terms of the basis vectors $\hat{\boldsymbol{r}}$, $\hat{\boldsymbol{\theta}}$, and $\hat{\boldsymbol{\phi}}$. (b) Using the formulas of (a) and our expression (2.16) for the gradient in spherical coordinates, derive the formula (2.33) for the laplacian $\boldsymbol{\nabla} \cdot \boldsymbol{\nabla}$.

13.4 Show that for any set of basis vectors $v_1, \dots, v_n$ and an inner product that is either positive definite (1.78–1.81) or indefinite (1.78–1.79 & 1.81 & 1.84), the inner products $g_{ik} = (v_i, v_k)$ define a matrix g_{ik} that is nonsingular and that therefore has an inverse. Hint: Show that the matrix g_{ik} cannot have a zero eigenvalue without violating either the condition (1.80) that it be positive definite or the condition (1.84) that it be nondegenerate.

13.5 Show that the inverse metric (13.48) transforms as a rank-2 contravariant tensor.

13.6 Show that if A_k is a covariant vector, then $g^{ik} A_k$ is a contravariant vector.

13.7 Show that in terms of the parameter $k = (a/R)^2$, the metric and line element (13.46) are given by (13.47).

13.8 Show that the connection $\Gamma^k_{i\ell}$ transforms as (13.66) and so is not a tensor.

13.9 Use the vanishing (13.83) of the covariant derivative of the metric tensor, to write the condition (13.140) in terms of the covariant derivatives of the symmetry vector (13.141).

13.10 Embed the points $\boldsymbol{p} = R(\cosh\theta, \sinh\theta\cos\phi, \sinh\theta\sin\phi)$ with tangent vectors (13.44) and line element (13.45) in the euclidian space $\mathbb{E}^3$. Show that the line element of this embedding is

$$\begin{aligned} ds^2 &= R^2 \left(\cosh^2\theta + \sinh^2\theta\right) d\theta^2 + R^2 \sinh^2\theta \, d\phi^2 \\ &= a^2 \left(\frac{(1+2kr^2)dr^2}{1+kr^2} + r^2 \, d\phi^2 \right) \end{aligned} \tag{13.359}$$

which describes a hyperboloid that is not maximally symmetric.

13.11 If you have Mathematica, imitate example 13.15 and find the scalar curvature R (13.113) of the line element (13.359) of the cylindrical hyperboloid embedded in euclidian 3-space $\mathbb{E}^3$.

13.12 Consider the torus with coordinates θ, ϕ labeling the arbitrary point

$$\boldsymbol{p} = (\cos\phi(R + r\sin\theta), \sin\phi(R + r\sin\theta), r\cos\theta) \tag{13.360}$$

in which $R > r$. Both θ and ϕ run from 0 to 2π. (a) Find the basis vectors e_θ and e_ϕ. (b) Find the metric tensor and its inverse.

13.13 For the same torus, (a) find the dual vectors e^θ and e^ϕ and (b) find the nonzero connections Γ^i_{jk} where i, j, k take the values θ and ϕ.

13.14 For the same torus, (a) find the two Christoffel matrices Γ_θ and Γ_ϕ, (b) find their commutator $[\Gamma_\theta, \Gamma_\phi]$, and (c) find the elements $R^\theta_{\theta\theta\theta}$, $R^\phi_{\theta\phi\theta}$, $R^\theta_{\phi\theta\phi}$, and $R^\phi_{\phi\phi\phi}$ of the curvature tensor.

13.15 Find the curvature scalar R of the torus with points (13.360). Hint: In these four problems, you may imitate the corresponding calculation for the sphere in Sec. 13.23.

13.16 Show that $\delta g^{ik} = -g^{is} g^{kt} \delta g_{st}$ or equivalently that $dg^{ik} = -g^{is} g^{kt} dg_{st}$ by differentiating the identity $g^{ik}\, g_{k\ell} = \delta^i_\ell$.

13.17 Let $g_{ik} = \eta_{ik} + h_{ik}$ and $x'^n = x^n + \epsilon^n$. To lowest order in ϵ and h, (a) show that in the x' coordinates $h'_{ik} = h_{ik} - \epsilon_{i,k} - \epsilon_{k,i}$ and (b) find an equation for ϵ that puts h' in de Donder's gauge ${h'}^i_{k,i} = \frac{1}{2}(\eta^{j\ell} h'_{j\ell})_{,k}$.

13.18 Just to get an idea of the sizes involved in black holes, imagine an isolated sphere of matter of uniform density ρ that as an initial condition is all at rest within a radius r_b. Its radius will be less than its Schwarzschild radius if

$$r_b < \frac{2MG}{c^2} = 2\left(\frac{4}{3}\pi r_b^3 \rho\right)\frac{G}{c^2}. \tag{13.361}$$

If the density ρ is that of water under standard conditions (1 gram per cc), for what range of radii r_b might the sphere be or become a black hole? Same question if ρ is the density of dark energy.

13.19 Embed the points

$$p = (ct, aL\sin\chi\sin\theta\cos\phi, aL\sin\chi\sin\theta\sin\phi, aL\sin\chi\cos\theta, aL\cos\chi) \tag{13.362}$$

in the flat semi-euclidian space $\mathbb{E}^{(1,4)}$ with metric $(-1, 1, 1, 1, 1)$ and derive the metric (13.257) with $k = 1$.

13.20 For the points $p = (ct, a\sin\theta\cos\phi, a\sin\theta\sin\phi, a\cos\theta)$, derive the metric (13.257) with $k = 0$.

13.21 Show that the 11 component of Ricci's tensor R_{11} is

$$R_{11} = [D_k, D_1]^k{}_1 = \frac{a\ddot{a} + 2\dot{a}^2 + 2kc^2/L^2}{c^2(1 - kr^2/L^2)}. \tag{13.363}$$

13.22 Show that the 22 component of Ricci's tensor R_{22} is

$$R_{22} = [D_k, D_2]^k{}_2 = \frac{(a\ddot{a} + 2\dot{a}^2 + 2kc^2/L^2)r^2}{c^2}. \tag{13.364}$$

13.23 Show that the 33 component of Ricci's tensor R_{33} is

$$R_{33} = [D_k, D_3]^k{}_3 = \frac{(a\ddot{a} + 2\dot{a}^2 + 2kc^2/L^2)r^2\sin^2\theta}{c^2}. \tag{13.365}$$

13.24 Embed the points

$$p = (ct, aL \sinh\chi \sin\theta \cos\phi, aL \sinh\chi \sin\theta \sin\phi, aL \sinh\chi \cos\theta, aL \cosh\chi) \quad (13.366)$$

in the flat semi-euclidian space $\mathbb{E}^{(2,3)}$ with metric $(-1, 1, 1, 1, -1)$ and derive the line element (13.256) and the metric (13.257) with $k = -1$.

13.25 Derive the second-order FLRW equation (13.273) from the formulas (13.257) for $g_{00} = -c^2$, (13.267) for $R_{00} = -3\ddot{a}/a$, (13.271) for T_{ij}, and (13.272) for T.

13.26 Derive the second-order FLRW equation (13.274) from Einstein's formula for the scalar curvature $R = -8\pi G\, T/c^4$ (13.230), the FLRW formulas for R (13.270) and for the trace T (13.272) of the energy-momentum tensor.

13.27 Assume there had been no inflation, no era of radiation, and no dark energy. In this case, the magnitude of the difference $|\Omega - 1|$ would have increased as $t^{2/3}$ over the past 13.8 billion years. Show explicitly how close to unity Ω would have had to have been at $t = 1\,\text{s}$ so as to satisfy the observational constraint $|\Omega_0 - 1| < 0.036$ on the present value of Ω.

13.28 Derive the relation (13.284) between the energy density ρ and the scale factor $a(t)$ from the conservation law (13.281) and the equation of state $p_i = w_i \rho_i$.

13.29 For constant $\rho = -p/c^2$ and $k = 1$, set $g^2 = 8\pi G\rho/3$ and use the Friedmann equations (13.273 & 13.275) and the boundary condition that the minimum of $a(t) > 0$ is at $t = 0$ to derive the formula $a(t) = c \cosh(gt)/(Lg)$.

13.30 Use the Friedmann equations (13.273 & 13.288) with $w = -1$, ρ constant, $k = -1$, and the boundary conditions $a(0) = 0$ and $\dot{a}(0) > 0$ to derive the formula $a(t) = c \sinh(gt)/(Lg)$ where again $g^2 = 8\pi G\rho/3$.

13.31 Use the Friedmann equations (13.273 & 13.288) with $w = -1$, ρ constant, and $k = 0$ to derive the formula $a(t) = a(0)\, e^{\pm gt}$.

13.32 Use the constancy of $8\pi G\rho\, a^4/3 = f^2$ for radiation ($w = 1/3$) and the Friedmann equations (13.273 & 13.288) to show that if $k = 0$, $a(0) = 0$, and $a(t) > 0$, then $a(t) = \sqrt{2ft}$ where $f > 0$.

13.33 Show that if the matrix $U(x)$ is nonsingular, then

$$(\partial_i U)\, U^{-1} = -U\, \partial_i U^{-1}. \quad (13.367)$$

13.34 The gauge-field matrix is a linear combination $A_k = -ig\, t^b\, A_k^b$ of the generators t^b of a representation of the gauge group. The generators obey the commutation relations

$$[t^a, t^b] = i f_{abc} t^c \quad (13.368)$$

in which the f_{abc} are the structure constants of the gauge group. Show that under a gauge transformation (13.311)

$$A'_i = U A_i U^{-1} - (\partial_i U)\, U^{-1} \tag{13.369}$$

by the unitary matrix $U = \exp(-ig\lambda^a t^a)$ in which λ^a is infinitesimal, the gauge-field matrix A_i transforms as

$$-igA'^a_i t^a = -igA^a_i t^a - ig^2 f_{abc}\lambda^a A^b_i t^c + ig\partial_i\lambda^a t^a. \tag{13.370}$$

Show further that the gauge field transforms as

$$A'^a_i = A^a_i - \partial_i\lambda^a - g f_{abc} A^b_i \lambda^c. \tag{13.371}$$

13.35 Show that if the vectors $e_a(x)$ are orthonormal, then $e^{a\dagger}\cdot e_{c,i} = -e^{a\dagger}_{,i}\cdot e_c$.

13.36 Use the equation $0 = \delta^a_{b,k} = (c^a_i\, c^i_b)_{,k}$ to show that $c^i_{b,k} = -c^i_c\, c^c_{p,k}\, c^p_b$. Then use this result to show that the Γ-free terms S_0 (13.338) vanish.

13.37 Show that terms in ${S^a}_b$ (13.337) linear in the Γ's vanish.

13.38 Derive the formula (13.346) for how the spin connection (13.324) changes under a Lorentz transformation and a general change of coordinates.

13.39 Use the identity of exercise 13.35 to derive the formula (13.353) for the nonabelian Faraday tensor.

13.40 Show that the dual tetrads $c^i_a = g^{ik}\eta_{ab}c^b_k$ are dual (13.155).

13.41 Write Dirac's action density in the explicitly hermitian form $L_D = -\frac{1}{2}\overline{\psi}\gamma^i\partial_i\psi - \frac{1}{2}\left[\overline{\psi}\gamma^i\partial_i\psi\right]^\dagger$ in which the field ψ has the invariant form $\psi = e_a\psi_a$ and $\overline{\psi} = i\psi^\dagger\gamma^0$. Use the identity $\left[\overline{\psi}_a\gamma^i\psi_b\right]^\dagger = -\overline{\psi}_b\gamma^i\psi_a$ to show that the gauge-field matrix A_i defined as the coefficient of $\overline{\psi}_a\gamma^i\psi_b$ as in $\overline{\psi}_a\gamma^i(\partial_i + iA_{iab})\psi_b$ is hermitian $A^*_{iab} = A_{iba}$.