Spinors

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1 Lorentz invariance and causality

In chapter 5 of *The Quantum Theory of Fields*, Weinberg shows that in order for fields to respond as

$$U(L,a)\psi_\ell(x)U^{-1}(L,a) = \sum_{\ell'} D_{\ell\ell'}(L^{-1})\psi_{\ell'}(Lx+a) \tag{1}$$

under a Lorentz transformation L followed by a translation a, and also to either commute or anticommute at spacelike separations (causality), the fields must use spinors or polarization vectors of specific forms. These forms are so specific, that Weinberg *derives* the Dirac equation from them. His treatment is the gold standard, but it is long and complicated. In my iron-standard treatment, I will use the Dirac equation to derive the spinors for arbitrary momentum from Weinberg's zero-momentum spinors.

In Schwartz's metric, if $\phi_a(x)$ is any four-component field that obeys the Klein-Gordon equation

$$\left(\partial_\mu \partial^\mu + m^2\right)\phi_a(x) = 0, \tag{2}$$

then the field (example (6.10) of *Physical Mathematics*)

$$\psi_a(x) = \left(i\partial_\mu \gamma^\mu + m\right)_{ab}\phi_b(x) \tag{3}$$

obeys Dirac's equation

$$\left(i\partial_\mu \gamma^\mu - m\right)\psi(x) = 0. \tag{4}$$

We expand a spin-one-half field as

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3} \sum_{s=-}^{+} \left[u(p,s)a(p,s)e^{-ipx} + v(p,s)b^\dagger(p,s)e^{ipx}\right]. \tag{5}$$

Since the Lorentz-invariant phase factor $\exp(-ipx)$ obeys the Klein-Gordon equation, any spinor of the form

$$u = (i\partial_\mu \gamma^\mu + m)\, u_0\, e^{-ipx} = (p_\mu \gamma^\mu + m)\, u_0\, e^{-ipx} \tag{6}$$

obeys Dirac's equation. So if the gamma matrices are

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \vec{\gamma} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix}, \qquad \gamma^5 = \gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3. \tag{7}$$

then the momentum-space spinors for particles are

$$u(p,s) = (p_\mu \gamma^\mu + m)\, u_0(s) = \begin{pmatrix} m & p\sigma \\ p\bar{\sigma} & m \end{pmatrix} u_0(s) \tag{8}$$

in which

$$p\sigma = p^0 - \vec{p}\cdot\vec{\sigma}, \qquad p\bar{\sigma} = p^0 + \vec{p}\cdot\vec{\sigma}. \tag{9}$$

For $\vec{p} = 0$, we have

$$u(0,s) = \left(m\gamma^0 + m\right) u_0(s) = \begin{pmatrix} m & m \\ m & m \end{pmatrix} u_0(s). \tag{10}$$

In general, $u_0(s)$ is

$$u_0(s) = \begin{pmatrix} \xi \\ \zeta \end{pmatrix}, \tag{11}$$

so

$$u(0,s) = \begin{pmatrix} m & m \\ m & m \end{pmatrix} \begin{pmatrix} \xi \\ \zeta \end{pmatrix} = m \begin{pmatrix} \xi + \zeta \\ \xi + \zeta \end{pmatrix}. \tag{12}$$

That is, only the sum $\xi + \zeta$ matters, so we put $\xi = \zeta$ and set

$$u(0,s) = \sqrt{m} \begin{pmatrix} \alpha(s) \\ \alpha(s) \end{pmatrix}. \tag{13}$$

The lower 2-spinor must be the same as the upper 2-spinor. Weinberg's choice is

$$\alpha(+) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \text{and} \qquad \alpha(-) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{14}$$

So the $\vec{p} = 0$ spinors for particles are in Schwartz's normalization

$$u(0,+) = \sqrt{m} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \qquad \text{and} \qquad u(0,-) = \sqrt{m} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} \tag{15}$$

For arbitrary $\vec{p}$, the spin-up spinor for particles is

$$\begin{aligned}
u(p,+) &= \frac{1}{\sqrt{2(E+m)}} \begin{pmatrix} m & p\sigma \\ p\bar{\sigma} & m \end{pmatrix} \begin{pmatrix} \alpha(+) \\ \alpha(+) \end{pmatrix} = \frac{1}{\sqrt{2(E+m)}} \begin{pmatrix} m & p\sigma \\ p\bar{\sigma} & m \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \\
&= \frac{1}{\sqrt{2(E+m)}} \begin{pmatrix} m & E - \vec{p}\cdot\vec{\sigma} \\ E + \vec{p}\cdot\vec{\sigma} & m \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \\
&= \frac{1}{\sqrt{2(E+m)}} \begin{pmatrix} m & 0 & E - p_3 & -p_1 + ip_2 \\ 0 & m & -p_1 - ip_2 & E + p_3 \\ E + p_3 & p_1 - ip_2 & m & 0 \\ p_1 + ip_2 & E - p_3 & 0 & m \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \\
&= \frac{1}{\sqrt{2(E+m)}} \begin{pmatrix} m + E - p_3 \\ -p_1 - ip_2 \\ m + E + p_3 \\ p_1 + ip_2 \end{pmatrix}.
\end{aligned} \tag{16}$$

In the massless limit, the spinor for a particle with spin up and momentum $p = (p, 0, 0, p)$ is

$$u(p,+) = \frac{1}{\sqrt{2E}} \begin{pmatrix} 0 \\ 0 \\ 2E \\ 0 \end{pmatrix} = \sqrt{2E} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \tag{17}$$

which shows that only the right-handed particle, the lower two components, can have spin and momentum in the $\hat{z}$ direction.

The spin-down spinor is

$$u(p,-) = \frac{1}{\sqrt{2(E+m)}} \begin{pmatrix} m & p\sigma \\ p\bar{\sigma} & m \end{pmatrix} \begin{pmatrix} \alpha(-) \\ \alpha(-) \end{pmatrix} = \frac{1}{\sqrt{2(E+m)}} \begin{pmatrix} m & p\sigma \\ p\bar{\sigma} & m \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}. \tag{18}$$

We usually don't need to know all four components of the spinors, but just in case the spin-down spinor is

$$
\begin{aligned}
u(p,-) &= \frac{1}{\sqrt{2(E+m)}} \begin{pmatrix} m & 0 & E-p_3 & -p_1+ip_2 \\ 0 & m & -p_1-ip_2 & E+p_3 \\ E+p_3 & p_1-ip_2 & m & 0 \\ p_1+ip_2 & E-p_3 & 0 & m \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} \\
&= \frac{1}{\sqrt{2(E+m)}} \begin{pmatrix} -p_1+ip_2 \\ m+E+p_3 \\ p_1-ip_2 \\ E-p_3+m \end{pmatrix}.
\end{aligned} \tag{19}
$$

In the massless limit, the spinor for a particle with spin down and momentum $p = (p, 0, 0, p)$ is

$$
u(p,-) = \frac{1}{\sqrt{2E}} \begin{pmatrix} 0 \\ 2E \\ 0 \\ 0 \end{pmatrix} = \sqrt{2E} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \tag{20}
$$

which shows that only the left-handed particle, the upper two components, can have spin in the $-\hat{z}$ direction and momentum in the $\hat{z}$ direction.

For antiparticles, any spinor like

$$
v = (i\partial_\mu \gamma^\mu + m)\, v_0\, e^{ipx} = (-p_\mu \gamma^\mu + m)\, v_0\, e^{ipx} \tag{21}
$$

obeys Dirac's equation. So the momentum-space spinors for antiparticles are

$$
v(p,s) = (-p_\mu \gamma^\mu + m)\, v_0(s) = \begin{pmatrix} m & -p\sigma \\ -p\bar{\sigma} & m \end{pmatrix} v_0(s). \tag{22}
$$

For $\vec{p} = 0$, we have

$$
v(0,s) = \left(-m\gamma^0 + m\right) v_0(s) = \begin{pmatrix} m & -m \\ -m & m \end{pmatrix} v_0(s). \tag{23}
$$

In general, $v_0(s)$ is

$$
v_0(s) = \begin{pmatrix} \xi \\ \zeta \end{pmatrix}, \tag{24}
$$

so

$$
v(0,s) = \begin{pmatrix} m & -m \\ -m & m \end{pmatrix} \begin{pmatrix} \xi \\ \zeta \end{pmatrix} = m \begin{pmatrix} \xi - \zeta \\ -\xi + \zeta \end{pmatrix}. \tag{25}
$$

That is, only the difference $\xi - \zeta$ matters, so we may set

$$v(0,s) = \sqrt{m}\begin{pmatrix}\beta(s)\\ -\beta(s)\end{pmatrix}. \tag{26}$$

The lower 2-spinor must be the negative of the upper 2-spinor. Weinberg's choice is

$$\beta(+) = \begin{pmatrix}0\\ 1\end{pmatrix} \qquad \text{and} \qquad \beta(-) = \begin{pmatrix}-1\\ 0\end{pmatrix}. \tag{27}$$

So the $\vec{p}=0$ spinors for antiparticles are

$$v(0,+) = \sqrt{m}\begin{pmatrix}0\\ 1\\ 0\\ -1\end{pmatrix} \qquad \text{and} \qquad v(0,-) = \sqrt{m}\begin{pmatrix}-1\\ 0\\ 1\\ 0\end{pmatrix}. \tag{28}$$

For arbitrary $\vec{p}$, the spin-up spinor for antiparticles is

$$\begin{aligned} v(p,+) &= \frac{1}{\sqrt{2(E+m)}}\begin{pmatrix}m & -p\sigma\\ -p\bar{\sigma} & m\end{pmatrix}\begin{pmatrix}\beta(+)\\ -\beta(+)\end{pmatrix} = \frac{1}{\sqrt{2(E+m)}}\begin{pmatrix}m & -p\sigma\\ -p\bar{\sigma} & m\end{pmatrix}\begin{pmatrix}0\\ 1\\ 0\\ -1\end{pmatrix}\\ &= \frac{1}{\sqrt{2(E+m)}}\begin{pmatrix}m & -E+\vec{p}\cdot\vec{\sigma}\\ -E-\vec{p}\cdot\vec{\sigma} & m\end{pmatrix}\begin{pmatrix}0\\ 1\\ 0\\ -1\end{pmatrix}. \end{aligned} \tag{29}$$

Doing the matrix multiplication, we get

$$\begin{aligned} v(p,+) &= \frac{1}{\sqrt{2(E+m)}}\begin{pmatrix}m & 0 & -E+p_3 & p_1-ip_2\\ 0 & m & p_1+ip_2 & -E-p_3\\ -E-p_3 & -p_1+ip_2 & m & 0\\ -p_1-ip_2 & -E+p_3 & 0 & m\end{pmatrix}\begin{pmatrix}0\\ 1\\ 0\\ -1\end{pmatrix}\\ &= \frac{1}{\sqrt{2(E+m)}}\begin{pmatrix}-p_1+ip_2\\ m+E+p_3\\ -p_1+ip_2\\ p_3-E-m\end{pmatrix}. \end{aligned} \tag{30}$$

In the massless limit, the antiparticle spinor for $p = (0, 0, 0, p)$ and spin up is

$$v(p,+) = \begin{pmatrix} 0 \\ \sqrt{2E} \\ 0 \\ 0 \end{pmatrix}. \tag{31}$$

The antiparticle spinor for spin-down is

$$\begin{aligned}
v(p,-) &= \frac{1}{\sqrt{2(E+m)}} \begin{pmatrix} m & -p\sigma \\ -p\bar{\sigma} & m \end{pmatrix} \begin{pmatrix} \beta(-) \\ -\beta(-) \end{pmatrix} = \frac{1}{\sqrt{2(E+m)}} \begin{pmatrix} m & -p\sigma \\ -p\bar{\sigma} & m \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \\
&= \frac{1}{\sqrt{2(E+m)}} \begin{pmatrix} m & -E + \vec{p}\cdot\vec{\sigma} \\ -E - \vec{p}\cdot\vec{\sigma} & m \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \\
&= \frac{1}{\sqrt{2(E+m)}} \begin{pmatrix} m & 0 & -E+p_3 & p_1 - ip_2 \\ 0 & m & p_1 + ip_2 & -E - p_3 \\ -E - p_3 & -p_1 + ip_2 & m & 0 \\ -p_1 - ip_2 & -E + p_3 & 0 & m \end{pmatrix} \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \\
&= \frac{1}{\sqrt{2(E+m)}} \begin{pmatrix} p_3 - E - m \\ p_1 + ip_2 \\ E + p_3 + m \\ p_1 + ip_2 \end{pmatrix}.
\end{aligned} \tag{32}$$

In the massless limit, the antiparticle spinor for $p = (0, 0, 0, p)$ and spin down is

$$v(p,-) = \begin{pmatrix} 0 \\ 0 \\ \sqrt{2E} \\ 0 \end{pmatrix}. \tag{33}$$

More succinctly, the spinors for particles and antiparticles are

$$\begin{aligned}
u(p,\pm) &= \frac{m + \not{p}}{\sqrt{2(E+m)}} \begin{pmatrix} \alpha(\pm) \\ \alpha(\pm) \end{pmatrix} = \frac{1}{\sqrt{2(E+m)}} \begin{pmatrix} m & p\sigma \\ p\bar{\sigma} & m \end{pmatrix} \begin{pmatrix} \alpha(\pm) \\ \alpha(\pm) \end{pmatrix} \\
v(p,\pm) &= \frac{m - \not{p}}{\sqrt{2(E+m)}} \begin{pmatrix} \beta(\pm) \\ -\beta(\pm) \end{pmatrix} = \frac{1}{\sqrt{2(E+m)}} \begin{pmatrix} m & -p\sigma \\ -p\bar{\sigma} & m \end{pmatrix} \begin{pmatrix} \beta(\pm) \\ -\beta(\pm) \end{pmatrix}
\end{aligned} \tag{34}$$

with

$$\alpha(\pm) = \frac{1}{2}\begin{pmatrix} 1 \pm 1 \\ 1 \mp 1 \end{pmatrix} \qquad \text{and} \qquad \beta(\pm) = \pm\,\alpha(\mp). \tag{35}$$

Their inner products are, since $\gamma^{\mu\dagger}\gamma^0 = \gamma^0\gamma^\mu$,

$$\begin{aligned}
\bar{u}(p,s)u(p,s') &= u^\dagger(p,s)\gamma^0 u(p,s') \\
&= \frac{1}{2(E+m)} \begin{pmatrix} \alpha^\dagger(s) & \alpha^\dagger(s) \end{pmatrix} \left(m + \not{p}^\dagger\right) \gamma^0 \left(m + \not{p}\right) \begin{pmatrix} \alpha(s') \\ \alpha(s') \end{pmatrix} \\
&= \frac{1}{2(E+m)} \begin{pmatrix} \alpha^\dagger(s) & \alpha^\dagger(s) \end{pmatrix} \gamma^0 \left(m + \not{p}\right) \left(m + \not{p}\right) \begin{pmatrix} \alpha(s') \\ \alpha(s') \end{pmatrix} \\
&= \frac{1}{2(E+m)} \begin{pmatrix} \alpha^\dagger(s) & \alpha^\dagger(s) \end{pmatrix} \gamma^0 \left(m^2 + 2m\not{p} + \not{p}^2\right) \begin{pmatrix} \alpha(s') \\ \alpha(s') \end{pmatrix}.
\end{aligned} \tag{36}$$

Now $\not{p}^2 = p^2 = m^2$, and the $\gamma^0\not{p}$ term is

$$\begin{aligned}
&\begin{pmatrix} \alpha^\dagger(s) & \alpha^\dagger(s) \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & p\sigma \\ p\bar{\sigma} & 0 \end{pmatrix} \begin{pmatrix} \alpha(s') \\ \alpha(s') \end{pmatrix} = \begin{pmatrix} \alpha^\dagger(s) & \alpha^\dagger(s) \end{pmatrix} \begin{pmatrix} p\bar{\sigma} & 0 \\ 0 & p\sigma \end{pmatrix} \begin{pmatrix} \alpha(s') \\ \alpha(s') \end{pmatrix} \\
&\qquad = \alpha^\dagger(s)\left(p\bar{\sigma} + p\sigma\right)\alpha(s') = \alpha^\dagger(s) 2E\alpha(s').
\end{aligned} \tag{37}$$

So these inner products are

$$\bar{u}(p,s)u(p,s') = \frac{1}{2(E+m)}\alpha^\dagger(s)\left(4m^2 + 4mE\right)\alpha(s') = 2m\,\delta_{s,s'}. \tag{38}$$

The usual inner product is

$$\begin{aligned}
u^\dagger(p,s)u(p,s') &= \frac{1}{2(E+m)} \begin{pmatrix} \alpha^\dagger(s) & \alpha^\dagger(s) \end{pmatrix} \left(m + \not{p}^\dagger\right) \left(m + \not{p}\right) \begin{pmatrix} \alpha(s') \\ \alpha(s') \end{pmatrix} \\
&= \frac{1}{2(E+m)} \begin{pmatrix} \alpha^\dagger(s) & \alpha^\dagger(s) \end{pmatrix} \left(m + E\gamma^0 + \vec{p}\cdot\vec{\gamma}\right) \left(m + E\gamma^0 - \vec{p}\cdot\vec{\gamma}\right) \begin{pmatrix} \alpha(s') \\ \alpha(s') \end{pmatrix} \\
&= \frac{1}{2(E+m)} \begin{pmatrix} \alpha^\dagger(s) & \alpha^\dagger(s) \end{pmatrix} \left[(m + E\gamma^0)^2 + (\vec{p})^2\right] \begin{pmatrix} \alpha(s') \\ \alpha(s') \end{pmatrix} \\
&= \frac{1}{2(E+m)} \begin{pmatrix} \alpha^\dagger(s) & \alpha^\dagger(s) \end{pmatrix} \begin{pmatrix} 2E^2 & 2mE \\ 2mE & 2E^2 \end{pmatrix} \begin{pmatrix} \alpha(s') \\ \alpha(s') \end{pmatrix} = 2E\,\delta s, s'.
\end{aligned} \tag{39}$$

Once again, the spinors are

$$\begin{aligned} u(p,\pm) &= \frac{m+\not{p}}{\sqrt{2(E+m)}}\begin{pmatrix}\alpha(\pm)\\ \alpha(\pm)\end{pmatrix} \\ v(p,\pm) &= \frac{m-\not{p}}{\sqrt{2(E+m)}}\begin{pmatrix}\beta(\pm)\\ -\beta(\pm)\end{pmatrix} \end{aligned} \tag{40}$$

with

$$\alpha(\pm) = \frac{1}{2}\begin{pmatrix}1\pm 1\\ 1\mp 1\end{pmatrix} \qquad \text{and} \qquad \beta(\pm) = \pm\,\alpha(\mp). \tag{41}$$

Since $\gamma^{\mu\dagger}\gamma^0 = \gamma^0\gamma^\mu$, the spin sum of the outer products of the particle spinors is

$$\sum_s u(p,s)\bar{u}(p,s) = \frac{1}{2(E+m)}\,(m+\not{p})\left[\sum_\pm \begin{pmatrix}\alpha(\pm)\\ \alpha(\pm)\end{pmatrix}\begin{pmatrix}\alpha^\dagger(\pm) & \alpha^\dagger(\pm)\end{pmatrix}\right]\left(m+\not{p}^\dagger\right)\gamma^0. \tag{42}$$

The inner spin sum is

$$\sum_\pm \begin{pmatrix}\alpha(\pm)\\ \alpha(\pm)\end{pmatrix}\begin{pmatrix}\alpha^\dagger(\pm) & \alpha^\dagger(\pm)\end{pmatrix} = \begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}. \tag{43}$$

So

$$\begin{aligned} \sum_s u(p,s)\bar{u}(p,s) &= \frac{1}{2(E+m)}\begin{pmatrix}m & p\sigma\\ p\bar{\sigma} & m\end{pmatrix}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}\begin{pmatrix}m & p\bar{\sigma}\\ p\sigma & m\end{pmatrix}\gamma^0 \\ &= \frac{1}{2(E+m)}\begin{pmatrix}m & p\sigma\\ p\bar{\sigma} & m\end{pmatrix}\begin{pmatrix}m+p\sigma & m+p\bar{\sigma}\\ m+p\sigma & m+p\bar{\sigma}\end{pmatrix}\gamma^0 \\ &= \frac{1}{2(E+m)}\begin{pmatrix}(m+p\sigma)^2 & (m+p\sigma)(m+p\bar{\sigma})\\ (m+p\bar{\sigma})(m+p\sigma) & (m+p\bar{\sigma})^2\end{pmatrix}\gamma^0 \\ &= \frac{1}{2(E+m)}\begin{pmatrix}2(E+m)p\sigma & 2m(E+m)\\ 2m(E+m) & 2(E+m)p\bar{\sigma}\end{pmatrix}\gamma^0 \\ &= \begin{pmatrix}p\sigma & m\\ m & p\bar{\sigma}\end{pmatrix}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} = \begin{pmatrix}m & p\sigma\\ p\bar{\sigma} & m\end{pmatrix} = p_\mu\gamma^\mu + m = \not{p} + m. \end{aligned} \tag{44}$$

The analogous sum for antiparticle spinors is

$$\sum_s v(p,s)\bar{v}(p,s) = p_\mu\gamma^\mu - m = \not{p} - m. \tag{45}$$

2 Charge conjugation

The basic idea is that a unitary operator C turns particle creation operators $a^\dagger(p,s,n)$ for particles of kind n into creation operators $a^\dagger(p,s,n_c)$ for antiparticles of kind n_c

$$C\, a^\dagger(p,s,n)\, C^{-1} = \alpha_n\, a^\dagger(p,s,n_c) \tag{46}$$

in which α_n is a phase factor. If we take the adjoint of both sides, we get

$$C\, a(p,s,n)\, C^{-1} = \alpha_n^*\, a(p,s,n_c). \tag{47}$$

The corresponding relations for kind n_c are

$$\begin{aligned} C\, a^\dagger(p,s,n_c)\, C^{-1} &= \alpha_{n_c}\, a^\dagger(p,s,n) \\ C\, a(p,s,n_c)\, C^{-1} &= \alpha_{n_c}^*\, a(p,s,n). \end{aligned} \tag{48}$$

It will turn out that $\alpha_{n_c} = \alpha_n^*$.

The operation C of charge conjugation turns the field

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3} \sum_{s=-}^{+} \left[u(p,s) a(p,s,n) e^{-ipx} + v(p,s) a^\dagger(p,s,n_c) e^{ipx} \right] \tag{49}$$

into

$$\begin{aligned} C\psi(x)C^{-1} &\equiv \psi_c(x) \\ &= \int \frac{d^3p}{(2\pi)^3} \sum_{s=-}^{+} \left[u(p,s)\alpha_n^*\, a(p,s,n_c) e^{-ipx} + v(p,s)\alpha_{n_c}\, a^\dagger(p,s,n) e^{ipx} \right]. \end{aligned} \tag{50}$$

Dirac's equation for a particle of charge e and mass m in an electromagnetic field A_μ is

$$\left(i\partial_\mu\gamma^\mu - eA_\mu\gamma^\mu - m \right) \psi(x) = 0. \tag{51}$$

Its conjugate is

$$\left(-i\partial_\mu\gamma^{\mu *} - eA_\mu\gamma^{\mu *} - m \right) \psi^*(x) = 0 \tag{52}$$

in which numbers are complex conjugated and operators are hermitian conjugated, but vectors are not transposed. Since Dirac's gamma matrices are defined by

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}, \tag{53}$$

the product $(-\gamma^2)\,\gamma^2$ is the 4×4 identity matrix

$$(-\gamma^2)\,\gamma^2 = I. \tag{54}$$

So the conjugated form (52) of Dirac's equation is equivalent to

$$\gamma^2 \left(-i\partial_\mu \gamma^{\mu *} - eA_\mu \gamma^{\mu *} - m\right)(-\gamma^2)\,\gamma^2 \psi^*(x) = 0. \tag{55}$$

Schwartz's gamma matrices

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \tag{56}$$

are hermitian except for γ^2 which is antihermitian. Thus their anticommutation relations (53) imply

$$\gamma^2 \gamma^{\mu *}(-\gamma^2) = \gamma^\mu (\gamma^2)^2 = -\gamma^\mu. \tag{57}$$

So our conjugated Dirac equation (55) becomes

$$\left(i\partial_\mu \gamma^\mu + eA_\mu \gamma^\mu - m\right)\gamma^2 \psi^*(x) = 0 \tag{58}$$

which is Dirac's equation for a particle of charge $-e$ and mass m.

Thus we would like the image ψ_c of the field ψ under the operation C of charge conjugation to be $\psi_c = \alpha\,\gamma^2\,\psi^*$ in which α is a phase factor. The combination $\alpha\,\gamma^2\,\psi^*(x)$ is

$$\alpha\,\gamma^2\,\psi^*(x) = \alpha\gamma^2 \int \frac{d^3p}{(2\pi)^3} \sum_{s=-}^{+} \left[u^*(p,s)a^\dagger(p,s,n)e^{ipx} + v^*(p,s)a(p,s,n_c)e^{-ipx}\right] \tag{59}$$

while $C\psi(x)C^{-1}$ is from (50)

$$C\psi(x)C^{-1} = \int \frac{d^3p}{(2\pi)^3} \sum_{s=-}^{+} \left[u(p,s)\alpha_n^*\,a(p,s,n_c)e^{-ipx} + v(p,s)\alpha_{n_c}\,a^\dagger(p,s,n)e^{ipx}\right]. \tag{60}$$

Since the gamma matrices are real except for γ^2, which is imaginary, and since they anticommute, one has

$$\gamma^2 \gamma^{\mu *} = -\gamma^\mu \gamma^2. \tag{61}$$

Thus

$$\begin{aligned} \gamma^2 u^*(p,s) &= \gamma^2 \frac{1}{\sqrt{2(E+m)}}\,(m + \not{p}^*) \begin{pmatrix} \alpha(\pm) \\ \alpha(\pm) \end{pmatrix} \\ &= \frac{1}{\sqrt{2(E+m)}}\,(m - \not{p})\,\gamma^2 \begin{pmatrix} \alpha(\pm) \\ \alpha(\pm) \end{pmatrix}. \end{aligned} \tag{62}$$

And

$$\gamma^2 \begin{pmatrix} \alpha(+) \\ \alpha(+) \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ i \\ 0 \\ -i \end{pmatrix} = i \begin{pmatrix} \beta(+) \\ -\beta(+) \end{pmatrix}, \tag{63}$$

so

$$\gamma^2 u^*(p,+) = iv(p,+). \tag{64}$$

Similarly,

$$\gamma^2 u^*(p,-) = iv(p,-). \tag{65}$$

Also,

$$\begin{aligned} \gamma^2 v^*(p,s) &= \gamma^2 \frac{1}{\sqrt{2(E+m)}} (m - \not{p}^*) \begin{pmatrix} \beta(\pm) \\ -\beta(\pm) \end{pmatrix} \\ &= \frac{1}{\sqrt{2(E+m)}} (m + \not{p}) \, \gamma^2 \begin{pmatrix} \beta(\pm) \\ -\beta(\pm) \end{pmatrix}. \end{aligned} \tag{66}$$

And

$$\gamma^2 \begin{pmatrix} \beta(+) \\ -\beta(+) \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} i \\ 0 \\ i \\ 0 \end{pmatrix} = i \begin{pmatrix} \alpha(+) \\ \alpha(+) \end{pmatrix}, \tag{67}$$

so

$$\gamma^2 v^*(p,+) = iu(p,+). \tag{68}$$

Similarly,

$$\gamma^2 v^*(p,-) = iu(p,-). \tag{69}$$

Equivalently,

$$u(p,s) = -i\gamma^2 v^*(p,s), \qquad v(p,s) = -i\gamma^2 u^*(p,s). \tag{70}$$

Thus

$$\begin{aligned} C\psi(x)C^{-1} =& \int \frac{d^3p}{(2\pi)^3} \sum_{s=-}^{+} \left[u(p,s)\alpha_n^* \, a(p,s,n_c) e^{-ipx} + v(p,s)\alpha_{n_c} \, a^\dagger(p,s,n) e^{ipx} \right] \\ =& -i\gamma^2 \int \frac{d^3p}{(2\pi)^3} \sum_{s=-}^{+} \left[v^*(p,s)\alpha_n^* \, a(p,s,n_c) e^{-ipx} + u^*(p,s)\alpha_{n_c} \, a^\dagger(p,s,n) e^{ipx} \right]. \end{aligned} \tag{71}$$

If $\alpha_n^* = \alpha_{n_c}$, then

$$C\psi(x)C^{-1} = \psi_c(x) = -i\gamma^2 \alpha_{n_c} \psi^*(x). \tag{72}$$

The charge-conjugate field ψ_c will be the same as $-i\gamma^2\alpha_{n_c}\psi^*$ only if

$$\alpha_n^* = \alpha_{n_c}. \tag{73}$$

Thus the charge-conjugation phase factor α_n of a particle must be the complex conjugate of the charge-conjugation phase factor α_{n_c} of its antiparticle. Weinberg shows further that unless $\alpha_n = \alpha_{n_c}^*$, the fields $\psi(x)$ and $\psi_c(x) = C\psi(x)C^{-1}$ will not anticommute at spacelike separations.

If $\psi(x)$ is a spin-one-half field whose particles are the same as its antiparticles, then the charge-conjugation phase factor $\alpha_n = \alpha_{n_c}^* = \alpha_n^*$ must be real and therefore ± 1.

3 Parity

The basic idea is that a unitary operator P reverses the 3-momentum of particle creation $a^\dagger(p, s, n)$ and annihilation $a(p, s, n)$ operators

$$\begin{aligned} P\, a^\dagger(p, s, n)\, P^{-1} &= \alpha_n\, a^\dagger(Pp, s, n) \\ P\, a(p, s, n)\, P^{-1} &= \alpha_n^*\, (Pp, s, n) \end{aligned} \tag{74}$$

in which $Pp = (p^0, -\vec{p})$ and α_n is a phase factor. The corresponding relations for the antiparticles of kind n_c are

$$\begin{aligned} P\, a^\dagger(p, s, n_c)\, C^{-1} &= \alpha_{n_c}\, a^\dagger(Pp, s, n_c) \\ C\, a(p, s, n_c)\, C^{-1} &= \alpha_{n_c}^*\, a(Pp, s, n_c). \end{aligned} \tag{75}$$

It will turn out that $\alpha_{n_c} = -\alpha_n^*$.

Dirac's equation for a particle of charge e and mass m in an electromagnetic field A_μ is

$$(i\partial_\mu \gamma^\mu - eA_\mu \gamma^\mu - m)\, \psi(x) = 0. \tag{76}$$

or more simply

$$\left(i\partial_0 \gamma^0 - i\vec{\nabla} \cdot \vec{\gamma} - eA_0 \gamma^0 + e\vec{A} \cdot \vec{\gamma} - m \right) \psi(x) = 0. \tag{77}$$

We insert $(\gamma^0)^2 = I$ and multiply from the left by γ^0:

$$\gamma^0 \left(i\partial_0 \gamma^0 - i\vec{\nabla} \cdot \vec{\gamma} - eA_0 \gamma^0 + e\vec{A} \cdot \vec{\gamma} - m \right) \gamma^0 \gamma^0 \psi(x) = 0. \tag{78}$$

Since

$$\gamma^0 \vec{\gamma} \gamma^0 = -\vec{\gamma}, \tag{79}$$

this is

$$\left(i\partial_0 \gamma^0 + i\vec{\nabla} \cdot \vec{\gamma} - eA_0 \gamma^0 - e\vec{A} \cdot \vec{\gamma} - m \right) \gamma^0 \psi(x) = 0. \tag{80}$$

Thus we would like the operation P of parity to turn the fields

$$\begin{aligned} \psi(x) &= \int \frac{d^3p}{(2\pi)^3} \sum_{s=-}^{+} \left[u(p,s) a(p,s) e^{-ipx} + v(p,s) b^\dagger(p,s) e^{ipx} \right] \\ A_\mu(x) &= \int \frac{d^3k}{(2\pi)^3} \sum_{s=-}^{+} \left[\varepsilon_\mu(p,s) c(p,s) e^{ipx} + \varepsilon^*_\mu(p,s) c^\dagger(p,s) e^{-ipx} \right] \end{aligned} \tag{81}$$

into

$$\begin{aligned} P\psi(t,\vec{x})P^{-1} &= \alpha\, \gamma^0\, \psi^*(Px) = \alpha\, \gamma^0\, \psi^*(t,-\vec{x}) \\ P\left(A_0(t,\vec{x}), \vec{A}(t,\vec{x}) \right) P^{-1} &= PA(Px) = \left(A_0(t,-\vec{x}), -\vec{A}(t,-\vec{x}) \right). \end{aligned} \tag{82}$$

in which α is a phase factor.

Recalling the effect (74, 75) of the unitary parity operator P on the operators $a(p,s)$ and $b^\dagger(p,s)$, we have with $Px = (t,-\vec{x})$ and $Pp = (p^0, -\vec{p})$

$$\begin{aligned} P\psi(t,\vec{x})P^{-1} &= \int \frac{d^3p}{(2\pi)^3} \sum_{s=-}^{+} \left[u(p,s) \alpha^*_n a(Pp,s,n) e^{-ipx} + v(p,s) \alpha_{n_c} a^\dagger(Pp,s,n_c) e^{ipx} \right] \\ &= \int \frac{d^3p}{(2\pi)^3} \sum_{s=-}^{+} \left[u(Pp,s) \alpha^*_n a(p,s,n) e^{-iPp\,x} + v(Pp,s) \alpha_{n_c} a^\dagger(p,s,n_c) e^{iPp\,x} \right]. \end{aligned} \tag{83}$$

Yet again, our spinors are

$$\begin{aligned} u(p,\pm) &= \frac{m + p_\mu \gamma^\mu}{\sqrt{2(E+m)}} \begin{pmatrix} \alpha(\pm) \\ \alpha(\pm) \end{pmatrix} \\ v(p,\pm) &= \frac{m - p_\mu \gamma^\mu}{\sqrt{2(E+m)}} \begin{pmatrix} \beta(\pm) \\ -\beta(\pm) \end{pmatrix}. \end{aligned} \tag{84}$$

So we see that

$$\begin{aligned}
\gamma^0 u(p,\pm) &= \frac{\gamma^0\left(m+p_\mu\gamma^\mu\right)\gamma^0\gamma^0}{\sqrt{2(E+m)}}\begin{pmatrix}\alpha(\pm)\\ \alpha(\pm)\end{pmatrix}\\
&= \frac{\left(m+(Pp)_\mu\gamma^\mu\right)\gamma^0}{\sqrt{2(E+m)}}\begin{pmatrix}\alpha(\pm)\\ \alpha(\pm)\end{pmatrix}\\
&= \frac{\left(m+(Pp)_\mu\gamma^\mu\right)}{\sqrt{2(E+m)}}\begin{pmatrix}\alpha(\pm)\\ \alpha(\pm)\end{pmatrix} = u(Pp,\pm).
\end{aligned} \tag{85}$$

Also

$$\begin{aligned}
\gamma^0 v(p,\pm) &= \frac{\gamma^0\left(m-p_\mu\gamma^\mu\right)\gamma^0\gamma^0}{\sqrt{2(E+m)}}\begin{pmatrix}\beta(\pm)\\ -\beta(\pm)\end{pmatrix}\\
&= \frac{\left(m+(Pp)_\mu\gamma^\mu\right)\gamma^0}{\sqrt{2(E+m)}}\begin{pmatrix}\beta(\pm)\\ -\beta(\pm)\end{pmatrix}\\
&= \frac{\left(m+(Pp)_\mu\gamma^\mu\right)}{\sqrt{2(E+m)}}\begin{pmatrix}-\beta(\pm)\\ \beta(\pm)\end{pmatrix} = -v(Pp,\pm).
\end{aligned} \tag{86}$$

Thus we need $\alpha_n^* = -\alpha_{n_c}$. So for a Majorana fermion, $\alpha_n = \pm i$.

4 Time reversal

Time reversal is represented as an antilinear, antiunitary operator. That is,

$$\begin{aligned}
\langle T\Phi|T\Psi\rangle &= \langle\Phi|\Psi\rangle^* = \langle\Psi|\Phi\rangle\\
T(z|a\rangle + w|b\rangle) &= z^*T|a\rangle + w^*T|b\rangle.
\end{aligned} \tag{87}$$

On create and annihilation operators of type n, it is

$$\begin{aligned}
Ta^\dagger(\vec{p},s,n)T^{-1} &= \beta_n(-1)^{j-s}a^\dagger(-\vec{p},-s,n)\\
Ta(\vec{p},s,n)T^{-1} &= \beta_n^*(-1)^{j-s}a(-\vec{p},-s,n).
\end{aligned} \tag{88}$$

On their antiparticle operators, it is

$$\begin{aligned}
Ta^\dagger(\vec{p},s,n_c)T^{-1} &= \beta_{n_c}(-1)^{j-s}a^\dagger(-\vec{p},-s,n_c)\\
Ta(\vec{p},s,n_c)T^{-1} &= \beta_{n_c}^*(-1)^{j-s}a(-\vec{p},-s,n_c)
\end{aligned} \tag{89}$$

in which $j = 1/2$ for spins-one-half fermions. On a Fermi field it is

$$T\psi(t,\vec{x})T^{-1} = -\beta_n^*\gamma^5\gamma^2\gamma^0\psi(-t,\vec{x}). \tag{90}$$

Thus we would like the operation T of time reversal to turn the field

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3} \sum_{s=-}^{+} \left[u(p,s,n)a(p,s,n)e^{-ipx} + v(p,s,n_c)a^\dagger(p,s,n_c)e^{ipx} \right] \tag{91}$$

into

$$T\psi(t,\vec{x})T^{-1} = c\beta_n^* \gamma^5\gamma^2\gamma^0 \psi(-t,\vec{x}) \tag{92}$$

in which c is a phase factor. Recalling the effect (88–89) of T on the creation and annihilation operators, we find

$$\begin{aligned} T\psi(t,\vec{x})T^{-1} &= T \int \frac{d^3p}{(2\pi)^3} \sum_{s=-}^{+} \left[u(\vec{p},s,n)a(\vec{p},s,n)e^{-ipx} + v(\vec{p},s,n_c)a^\dagger(\vec{p},s,n_c)e^{ipx} \right] T^{-1} \\ &= \int \frac{d^3p}{(2\pi)^3} \sum_{s=-}^{+} (-1)^{1/2-s} \left[u^*(p,s,n)\beta_n^* a(-\vec{p},-s,n)e^{ipx} \right. \\ &\quad \left. + v^*(p,s,n_c)\beta_{n_c} a^\dagger(-\vec{p},-s,n_c)e^{-ipx} \right]. \end{aligned} \tag{93}$$

We now flip the sign of $\vec{p}$ and of s

$$\begin{aligned} T\psi(t,\vec{x})T^{-1} = & \int \frac{d^3p}{(2\pi)^3} \sum_{s=-}^{+} (-1)^{1/2-s} \left[u^*(-\vec{p},-s,n)\beta_n^* a(\vec{p},s,n)e^{ip^0t+i\vec{p}\cdot\vec{x}} \right. \\ & \left. + v^*(-\vec{p},-s,n_c)\beta_{n_c} a^\dagger(\vec{p},s,n_c)e^{-ip^0t-i\vec{p}\cdot\vec{x}} \right]. \end{aligned} \tag{94}$$

So we must compute the effect of $\gamma^5\gamma^2\gamma^0$ on the spinors of the particles:

$$\begin{aligned} \gamma^5\gamma^2\gamma^0 u(\vec{p},s,n) &= \gamma^5\gamma^2\gamma^0 \frac{m + p_\mu\gamma^\mu}{\sqrt{2(E+m)}} \begin{pmatrix} \alpha(\pm) \\ \alpha(\pm) \end{pmatrix} \\ &= \gamma^5\gamma^2\gamma^0 \frac{m + p^0\gamma^0 - \vec{p}\cdot\vec{\gamma}}{\sqrt{2(E+m)}} \begin{pmatrix} \alpha(\pm) \\ \alpha(\pm) \end{pmatrix} \\ &= \gamma^5\gamma^2 \frac{m + p^0\gamma^0 + \vec{p}\cdot\vec{\gamma}}{\sqrt{2(E+m)}} \gamma^0 \begin{pmatrix} \alpha(\pm) \\ \alpha(\pm) \end{pmatrix} \\ &= \gamma^5 \frac{m - p^0\gamma^0 - \vec{p}\cdot\vec{\gamma}^*}{\sqrt{2(E+m)}} \gamma^2\gamma^0 \begin{pmatrix} \alpha(\pm) \\ \alpha(\pm) \end{pmatrix} \\ &= \frac{m + p^0\gamma^0 + \vec{p}\cdot\vec{\gamma}^*}{\sqrt{2(E+m)}} \gamma^5\gamma^2\gamma^0 \begin{pmatrix} \alpha(\pm) \\ \alpha(\pm) \end{pmatrix}. \end{aligned} \tag{95}$$

The product of these gamma matrices is in Schwartz's version of Weyl's notation

$$\begin{aligned} \gamma^5\gamma^2\gamma^0 &= \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sigma^2 & 0 \\ 0 & -\sigma^2 \end{pmatrix} = - \begin{pmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{pmatrix}. \end{aligned} \tag{96}$$

Now

$$\sigma^2\alpha(+) = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = i\alpha(-), \tag{97}$$

and

$$\sigma^2\alpha(-) = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = -i\alpha(+). \tag{98}$$

So

$$\begin{aligned} \gamma^5\gamma^2\gamma^0 u(\vec{p}, \pm, n) &= \frac{m + p^0\gamma^0 + \vec{p}\cdot\vec{\gamma}^*}{\sqrt{2(E+m)}} \gamma^5\gamma^2\gamma^0 \begin{pmatrix} \alpha(\pm) \\ \alpha(\pm) \end{pmatrix} \\ &= \frac{m + p^0\gamma^0 + \vec{p}\cdot\vec{\gamma}^*}{\sqrt{2(E+m)}} \begin{pmatrix} \mp i\alpha(\mp) \\ \mp i\alpha(\mp) \end{pmatrix} \\ &= \mp i \frac{m + p^0\gamma^0 + \vec{p}\cdot\vec{\gamma}^*}{\sqrt{2(E+m)}} \begin{pmatrix} \alpha(\mp) \\ \alpha(\mp) \end{pmatrix} \\ &= \mp i u^*(-\vec{p}, \mp, n). \end{aligned} \tag{99}$$

Thus

$$u^*(-\vec{p}, \mp, n) = \pm i\gamma^5\gamma^2\gamma^0 u(\vec{p}, \pm, n). \tag{100}$$

The factor $(-1)^{1/2-s}$ cancels the sign $\pm$, and so

$$(-1)^{1/2-s} u^*(-\vec{p}, \mp, n) = i\gamma^5\gamma^2\gamma^0 u(\vec{p}, \pm, n). \tag{101}$$

Similarly, we compute the effect of $\gamma^5\gamma^2\gamma^0$ on the antiparticle spinors:

$$\begin{aligned} \gamma^5\gamma^2\gamma^0 v(\vec{p}, s, n) &= \gamma^5\gamma^2\gamma^0 \frac{m - p_\mu\gamma^\mu}{\sqrt{2(E+m)}} \begin{pmatrix} \beta(\pm) \\ -\beta(\pm) \end{pmatrix} \\ &= \frac{m + p^0\gamma^0 + \vec{p}\cdot\vec{\gamma}^*}{\sqrt{2(E+m)}} \gamma^5\gamma^2\gamma^0 \begin{pmatrix} \beta(\pm) \\ -\beta(\pm) \end{pmatrix}. \end{aligned} \tag{102}$$

The product of these gamma matrices in Schwartz's version of Weyl's notation is

$$\gamma^5\gamma^2\gamma^0 = \, = - \begin{pmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{pmatrix}. \tag{103}$$

Now

$$\sigma^2\beta(+) = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -i \\ 0 \end{pmatrix} = -i\alpha(+) = i\beta(-), \tag{104}$$

and

$$\sigma^2\beta(-) = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} -1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -i \end{pmatrix} = -i\alpha(-) = -i\beta(+). \tag{105}$$

So

$$\begin{aligned} \gamma^5\gamma^2\gamma^0 v(\vec{p},\pm,n) &= \frac{m + p^0\gamma^0 + \vec{p}\cdot\vec{\gamma}^*}{\sqrt{2(E+m)}}\gamma^5\gamma^2\gamma^0 \begin{pmatrix} \beta(\pm) \\ -\beta(\pm) \end{pmatrix} \\ &= \frac{m + p^0\gamma^0 + \vec{p}\cdot\vec{\gamma}^*}{\sqrt{2(E+m)}} \begin{pmatrix} \mp i\beta(\mp) \\ \pm i\beta(\mp) \end{pmatrix} \\ &= \mp i \frac{m + p^0\gamma^0 + \vec{p}\cdot\vec{\gamma}^*}{\sqrt{2(E+m)}} \begin{pmatrix} \beta(\mp) \\ -\beta(\mp) \end{pmatrix} \\ &= \mp i v^*(-\vec{p},\mp,n). \end{aligned} \tag{106}$$

Thus,

$$v^*(-\vec{p},\mp,n) = \pm i\gamma^5\gamma^2\gamma^0 v(\vec{p},\pm,n). \tag{107}$$

The factor $(-1)^{1/2-s}$ cancels the sign $\pm$, and so

$$(-1)^{1/2-s} v^*(-\vec{p},\mp,n) = i\gamma^5\gamma^2\gamma^0 v(\vec{p},\pm,n). \tag{108}$$

So by (101 and 108), the effect (94) of T on ψ is

$$\begin{aligned} T\psi(t,\vec{x})T^{-1} = & \int \frac{d^3p}{(2\pi)^3} \sum_{s=-}^{+} (-1)^{1/2-s} \Big[u^*(-\vec{p},-s,n)\beta_n^* a(\vec{p},s,n) e^{ip^0t + i\vec{p}\cdot\vec{x}} \\ & + v^*(-\vec{p},-s,n_c)\beta_{n_c} a^\dagger(\vec{p},s,n_c) e^{-ip^0t - i\vec{p}\cdot\vec{x}} \Big] \\ = & \, i\gamma^5\gamma^2\gamma^0 \int \frac{d^3p}{(2\pi)^3} \sum_{s=-}^{+} \Big[u(\vec{p},\pm,n)\beta_n^* a(\vec{p},s,n) e^{ip^0t + i\vec{p}\cdot\vec{x}} \\ & + v(\vec{p},\pm,n_c)\beta_{n_c} a^\dagger(\vec{p},s,n_c) e^{-ip^0t - i\vec{p}\cdot\vec{x}} \Big]. \end{aligned} \tag{109}$$

The field $T\psi T^{-1}$ will anticommute with ψ at spacelike separations only if

$$\beta_n^* = \beta_{n_c} \tag{110}$$

in which case

$$T\psi(t,\vec{x})T^{-1} = i\beta_n^*\gamma^5\gamma^2\gamma^0 \int \frac{d^3p}{(2\pi)^3} \sum_{s=-}^{+} \Big[u(\vec{p},\pm,n)a(\vec{p},s,n)e^{ip^0t+i\vec{p}\cdot\vec{x}} + v(\vec{p},\pm,n_c)a^\dagger(\vec{p},s,n_c)e^{-ip^0t-i\vec{p}\cdot\vec{x}} \Big] = i\beta_n^*\gamma^5\gamma^2\gamma^0\psi(-t,\vec{x}) \tag{111}$$

which is (94) with $c = i$.