15
Functional Derivatives

15.1 Functionals

A **functional** $G[f]$ is a map from a space of functions to a set of numbers. For instance, the **action** functional $S[q]$ for a particle in one dimension maps the coordinate $q(t)$, which is a function of the time t, into a number—the action of the process. If the particle has mass m and is moving slowly and freely, then for the interval (t_1, t_2) its action is

$$S_0[q] = \int_{t_1}^{t_2} dt \; \frac{m}{2} \left(\frac{dq(t)}{dt} \right)^2 . \tag{15.1}$$

If the particle is moving in a potential $V(q(t))$, then its action is

$$S[q] = \int_{t_1}^{t_2} dt \; \left[\frac{m}{2} \left(\frac{dq(t)}{dt} \right)^2 - V(q(t)) \right] . \tag{15.2}$$

15.2 Functional Derivatives

A **functional derivative** is a functional

$$\delta G[f][h] = \frac{d}{d\epsilon} G[f + \epsilon h] \bigg|_{\epsilon=0} \tag{15.3}$$

of a functional. For instance, if $G_n[f]$ is the functional

$$G_n[f] = \int dx \, f^n(x) \tag{15.4}$$

then its functional derivative is the functional that maps the pair of functions f, h to the number

$$\begin{aligned} \delta G_n[f][h] &= \frac{d}{d\epsilon} G_n[f+\epsilon h]\Big|_{\epsilon=0} \\ &= \frac{d}{d\epsilon} \int dx\, (f(x)+\epsilon h(x))^n \Big|_{\epsilon=0} \\ &= \int dx\, n f^{n-1}(x) h(x). \end{aligned} \tag{15.5}$$

Physicists often use the less elaborate notation

$$\frac{\delta G[f]}{\delta f(y)} = \delta G[f][\delta_y] \tag{15.6}$$

in which the function $h(x)$ is $\delta_y(x) = \delta(x-y)$. Thus in the preceding example

$$\frac{\delta G[f]}{\delta f(y)} = \int dx\, n f^{n-1}(x) \delta(x-y) = n f^{n-1}(y). \tag{15.7}$$

Functional derivatives of functionals that involve powers of derivatives also are easily dealt with. Suppose that the functional involves the square of the derivative $f'(x)$

$$G[f] = \int dx\, \left(f'(x)\right)^2. \tag{15.8}$$

Then its functional derivative is

$$\begin{aligned} \delta G[f][h] &= \frac{d}{d\epsilon} G[f+\epsilon h]\Big|_{\epsilon=0} \\ &= \frac{d}{d\epsilon} \int dx\, \left(f'(x)+\epsilon h'(x)\right)^2 \Big|_{\epsilon=0} \\ &= \int dx\, 2 f'(x) h'(x) = -2 \int dx\, f''(x) h(x) \end{aligned} \tag{15.9}$$

in which we have integrated by parts and used suitable boundary conditions on $h(x)$ to drop the surface terms. In physics notation, we have

$$\frac{\delta G[f]}{\delta f(y)} = -2 \int dx\, f''(x) \delta(x-y) = -2 f''(y). \tag{15.10}$$

Let's now compute the functional derivative of the action (15.2), which involves the square of the time-derivative $\dot{q}(t)$ and the potential energy $V(q(t))$

$$\begin{aligned}\delta S[q][h] &= \frac{d}{d\epsilon} S[q+\epsilon h]\bigg|_{\epsilon=0} \\ &= \frac{d}{d\epsilon}\int dt \left[\frac{m}{2}\left(\dot{q}(t)+\epsilon\dot{h}(t)\right)^2 - V(q(t)+\epsilon h(t))\right]\bigg|_{\epsilon=0} \\ &= \int dt \left[m\dot{q}(t)\dot{h}(t) - V'(q(t))h(t)\right] \\ &= \int dt \left[-m\ddot{q}(t) - V'(q(t))\right] h(t) \end{aligned} \tag{15.11}$$

where we once again have integrated by parts and used suitable boundary conditions to drop the surface terms. In physics notation, this is

$$\frac{\delta S[q]}{\delta q(t)} = \int dt' \left[-m\ddot{q}(t') - V'(q(t'))\right] \delta(t'-t) = -m\ddot{q}(t) - V'(q(t)). \tag{15.12}$$

In these terms, the stationarity of the action $S[q]$ is the vanishing of its functional derivative either in the form

$$\delta S[q][h] = 0 \tag{15.13}$$

for arbitrary functions $h(t)$ (that vanish at the end points of the interval) or equivalently in the form

$$\frac{\delta S[q]}{\delta q(t)} = 0 \tag{15.14}$$

which is Lagrange's equation of motion

$$m\ddot{q}(t) = -V'(q(t)). \tag{15.15}$$

Physicists also use the compact notation

$$\frac{\delta^2 Z[j]}{\delta j(y)\delta j(z)} \equiv \frac{\partial^2 Z[j+\epsilon\delta_y+\epsilon'\delta_z]}{\partial\epsilon\,\partial\epsilon'}\bigg|_{\epsilon=\epsilon'=0} \tag{15.16}$$

in which $\delta_y(x) = \delta(x-y)$ and $\delta_z(x) = \delta(x-z)$.

Example 15.1 (Shortest Path is a Straight Line) On a plane, the length of the path $(x, y(x))$ from (x_0, y_0) to (x_1, y_1) is

$$L[y] = \int_{x_0}^{x_1} \sqrt{dx^2+dy^2} = \int_{x_0}^{x_1} \sqrt{1+y'^2}\,dx. \tag{15.17}$$

The shortest path $y(x)$ minimizes this length $L[y]$, so

$$\begin{aligned}\delta L[y][h] &= \frac{d}{d\epsilon} L[y+\epsilon h]\bigg|_{\epsilon=0} = \frac{d}{d\epsilon}\int_{x_0}^{x_1} \sqrt{1+(y'+\epsilon h')^2}\,dx\bigg|_{\epsilon=0} \\ &= \int_{x_0}^{x_1} \frac{y'h'}{\sqrt{1+y'^2}}\,dx = -\int_{x_0}^{x_1} h \frac{d}{dx}\frac{y'}{\sqrt{1+y'^2}}\,dx = 0 \end{aligned} \tag{15.18}$$

since $h(x_0) = h(x_1) = 0$. This can vanish for arbitrary $h(x)$ only if

$$\frac{d}{dx}\frac{y'}{\sqrt{1+y'^2}} = 0 \tag{15.19}$$

which implies $y'' = 0$. Thus $y(x)$ is a straight line, $y = mx + b$. □

15.3 Higher-Order Functional Derivatives

The second functional derivative is

$$\delta^2 G[f][h] = \frac{d^2}{d\epsilon^2}\, G[f+\epsilon h]|_{\epsilon=0}\,. \tag{15.20}$$

So if $G_N[f]$ is the functional

$$G_N[f] = \int f^N(x) dx \tag{15.21}$$

then

$$\begin{aligned}\delta^2 G_N[f][h] &= \frac{d^2}{d\epsilon^2}\, G_N[f+\epsilon h]|_{\epsilon=0} \\ &= \frac{d^2}{d\epsilon^2} \int (f(x)+\epsilon h(x))^N\,dx\bigg|_{\epsilon=0} \\ &= \frac{d^2}{d\epsilon^2} \int \binom{N}{2} \epsilon^2 h^2(x) f^{N-2}(x)\,dx\bigg|_{\epsilon=0} \\ &= N(N-1)\int f^{N-2}(x) h^2(x) dx. \end{aligned} \tag{15.22}$$

Example 15.2 ($\delta^2 S_0$) The second functional derivative of the action $S_0[q]$ (15.1) is

$$\begin{aligned}\delta^2 S_0[q][h] &= \frac{d^2}{d\epsilon^2}\int_{t_1}^{t_2} dt\ \frac{m}{2}\left(\frac{dq(t)}{dt} + \epsilon\frac{dh(t)}{dt}\right)^2\bigg|_{\epsilon=0} \\ &= \int_{t_1}^{t_2} dt\ m\left(\frac{dh(t)}{dt}\right)^2 \geq 0 \end{aligned} \tag{15.23}$$

and is positive for all functions $h(t)$. The stationary classical trajectory

$$q(t) = \frac{t - t_1}{t_2 - t_1}\, q(t_2) + \frac{t_2 - t}{t_2 - t_1}\, q(t_1) \tag{15.24}$$

is a **minimum** of the action $S_0[q]$. □

The second functional derivative of the action $S[q]$ (15.2) is

$$\begin{aligned} \delta^2 S[q][h] &= \frac{d^2}{d\epsilon^2} \int_{t_1}^{t_2} dt \left[\frac{m}{2} \left(\frac{dq(t)}{dt} + \epsilon \frac{dh(t)}{dt} \right)^2 - V(q(t) + \epsilon h(t)) \right] \Bigg|_{\epsilon=0} \\ &= \int_{t_1}^{t_2} dt \left[m \left(\frac{dh(t)}{dt} \right)^2 - \frac{\partial^2 V(q(t))}{\partial q^2(t)} h^2(t) \right] \end{aligned} \tag{15.25}$$

and it can be positive, zero, or negative. Chaos sometimes arises in systems of several particles when the second variation of $S[q]$ about a stationary path is negative, $\delta^2 S[q][h] < 0$ while $\delta S[q][h] = 0$.

The nth functional derivative is defined as

$$\delta^n G[f][h] = \frac{d^n}{d\epsilon^n}\, G[f + \epsilon h]\big|_{\epsilon=0} . \tag{15.26}$$

The nth functional derivative of the functional (15.21) is

$$\delta^n G_N[f][h] = \frac{N!}{(N-n)!} \int f^{N-n}(x) h^n(x) dx. \tag{15.27}$$

15.4 Functional Taylor Series

It follows from the Taylor-series theorem (section 4.6) that

$$e^{\delta} G[f][h] = \sum_{n=0}^{\infty} \frac{\delta^n}{n!} G[f][h] = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{d^n}{d\epsilon^n} G[f + \epsilon h] \Bigg|_{\epsilon=0} = G[f + h] \tag{15.28}$$

which illustrates an advantage of the present mathematical notation.

The functional $S_0[q]$ of Eq.(15.1) provides a simple example of the functional Taylor series (15.28):

$$
\begin{aligned}
e^{\delta} S_0[q][h] &= \left(1 + \frac{d}{d\epsilon} + \frac{1}{2}\frac{d^2}{d\epsilon^2}\right) S_0[q+\epsilon h]\Big|_{\epsilon=0} \\
&= \frac{m}{2}\int_{t_1}^{t_2} \left(1 + \frac{d}{d\epsilon} + \frac{1}{2}\frac{d^2}{d\epsilon^2}\right)\left(\dot{q}(t) + \epsilon \dot{h}(t)\right)^2 dt\Big|_{\epsilon=0} \\
&= \frac{m}{2}\int_{t_1}^{t_2} \left(\dot{q}^2(t) + 2\dot{q}(t)\dot{h}(t) + \dot{h}^2(t)\right) dt \\
&= \frac{m}{2}\int_{t_1}^{t_2} \left(\dot{q}(t) + \dot{h}(t)\right)^2 dt = S_0[q+h].
\end{aligned} \tag{15.29}
$$

If the function $q(t)$ makes the action $S_0[q]$ stationary, and if $h(t)$ is smooth and vanishes at the endpoints of the time interval, then

$$
S_0[q+h] = S_0[q] + S_0[h]. \tag{15.30}
$$

More generally, if $q(t)$ makes the action $S[q]$ stationary, and $h(t)$ is any loop from and to the origin, then

$$
S[q+h] = e^{\delta} S[q][h] = S[q] + \sum_{n=2}^{\infty} \frac{1}{n!}\frac{d^n}{d\epsilon^n}\, S[q+\epsilon h]\big|_{\epsilon=0}\,. \tag{15.31}
$$

If further $S_2[q]$ is purely quadratic in q and $\dot{q}$, like the harmonic oscillator, then

$$
S_2[q+h] = S_2[q] + S_2[h]. \tag{15.32}
$$

15.5 Functional Differential Equations

In inner products like $\langle q'|f\rangle$, we represent the momentum operator as

$$
p = \frac{\hbar}{i}\frac{d}{dq'} \tag{15.33}
$$

because then

$$
\langle q'|p\,q|f\rangle = \frac{\hbar}{i}\frac{d}{dq'}\langle q'|q|f\rangle = \frac{\hbar}{i}\frac{d}{dq'}\left(q'\langle q'|f\rangle\right) = \left(\frac{\hbar}{i} + q'\frac{\hbar}{i}\frac{d}{dq'}\right)\langle q'|f\rangle \tag{15.34}
$$

which respects the commutation relation $[q,p] = i\hbar$.

So too in inner products $\langle \phi'|f\rangle$ of eigenstates $|\phi'\rangle$ of $\phi(\boldsymbol{x},t)$

$$
\phi(\boldsymbol{x},t)|\phi'\rangle = \phi'(\boldsymbol{x})|\phi'\rangle \tag{15.35}
$$

we can represent the momentum $\pi(\boldsymbol{x},t)$ canonically conjugate to the field $\phi(\boldsymbol{x},t)$ as the functional derivative

$$\pi(\boldsymbol{x},t) = \frac{\hbar}{i}\frac{\delta}{\delta\phi'(\boldsymbol{x})} \tag{15.36}$$

because then

$$\begin{aligned}
\langle\phi'|\pi(\boldsymbol{x'},t)\phi(\boldsymbol{x},t)|f\rangle &= \frac{\hbar}{i}\frac{\delta}{\delta\phi'(\boldsymbol{x'})}\langle\phi'|\phi(\boldsymbol{x},t)|f\rangle \\
&= \frac{\hbar}{i}\frac{\delta}{\delta\phi'(\boldsymbol{x'})}\big(\phi'(\boldsymbol{x})\langle\phi'|f\rangle\big) \\
&= \frac{\hbar}{i}\frac{\delta}{\delta\phi'(\boldsymbol{x'})}\left(\int \delta(\boldsymbol{x}-\boldsymbol{x'})\,\phi'(\boldsymbol{x'})\,d^3x'\,\langle\phi'|f\rangle\right) \\
&= \frac{\hbar}{i}\left(\delta(\boldsymbol{x}-\boldsymbol{x'}) + \phi'(\boldsymbol{x})\frac{\delta}{\delta\phi'(\boldsymbol{x'})}\right)\langle\phi'|f\rangle \\
&= \langle\phi'| - i\hbar\delta(\boldsymbol{x}-\boldsymbol{x'}) + \phi(\boldsymbol{x},t)\,\pi(\boldsymbol{x'},t)|f\rangle
\end{aligned} \tag{15.37}$$

which respects the equal-time commutation relation

$$[\phi(\boldsymbol{x},t), \pi(\boldsymbol{x'},t)] = i\,\hbar\,\delta(\boldsymbol{x}-\boldsymbol{x'}). \tag{15.38}$$

We can use the representation (15.36) for $\pi(x)$ to find the wave function of the ground state $|0\rangle$ of the hamiltonian

$$H = \frac{1}{2}\int \left[\pi^2 + (\nabla\phi)^2 + m^2\phi^2\right]\,d^3x \tag{15.39}$$

where we set $\hbar = c = 1$. We will use the trick we used in section 2.11 to find the ground state $|0\rangle$ of the harmonic-oscillator hamiltonian

$$H_0 = \frac{p^2}{2m} + \frac{m\omega^2 q^2}{2}. \tag{15.40}$$

In that trick, one writes

$$\begin{aligned}
H_0 &= \frac{1}{2m}(m\omega q - ip)(m\omega q + ip) + \frac{i\omega}{2}[p,q] \\
&= \frac{1}{2m}(m\omega q - ip)(m\omega q + ip) + \frac{1}{2}\hbar\omega
\end{aligned} \tag{15.41}$$

and seeks a state $|0\rangle$ that is annihilated by $m\omega q + ip$

$$\langle q'|m\omega q + ip|0\rangle = \left(m\omega q' + \hbar\frac{d}{dq'}\right)\langle q'|0\rangle = 0. \tag{15.42}$$

The solution to this differential equation

$$\frac{d}{dq'}\langle q'|0\rangle = -\frac{m\omega q'}{\hbar}\langle q'|0\rangle \tag{15.43}$$

is

$$\langle q'|0\rangle = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp\left(-\frac{m\omega q'^2}{2\hbar}\right) \tag{15.44}$$

in which the prefactor is a constant of normalization.

So extending that trick to the hamiltonian (15.39), we factor H

$$H = \frac{1}{2}\int \left[\sqrt{-\nabla^2 + m^2}\,\phi - i\pi\right]\left[\sqrt{-\nabla^2 + m^2}\,\phi + i\pi\right] d^3x + C \tag{15.45}$$

in which C is the (infinite) constant

$$C = \frac{i}{2}\int \left[\pi, \sqrt{-\triangle + m^2}\,\phi\right] d^3x. \tag{15.46}$$

The ground state $|0\rangle$ of H therefore must satisfy the functional differential equation $\langle\phi'|\sqrt{-\nabla^2 + m^2}\,\phi + i\pi|0\rangle = 0$ or

$$\frac{\delta\langle\phi'|0\rangle}{\delta\phi'(\boldsymbol{x})} = -\sqrt{-\nabla^2 + m^2}\,\phi'(\boldsymbol{x})\,\langle\phi'|0\rangle. \tag{15.47}$$

The solution to this equation is

$$\langle\phi'|0\rangle = N\,\exp\left(-\frac{1}{2}\int \phi'(\boldsymbol{x})\,\sqrt{-\nabla^2 + m^2}\,\phi'(\boldsymbol{x})\,d^3x\right) \tag{15.48}$$

in which N is a normalization constant. To see that this functional does satisfy equation (15.47), we compute the derivative

$$\frac{d\langle\phi' + \epsilon h|0\rangle}{d\epsilon} = N\frac{d}{d\epsilon}\exp\left[-\tfrac{1}{2}\int \left(\phi' + \epsilon h\right)\sqrt{-\triangle + m^2}\left(\phi' + \epsilon h\right) d^3x\right] \tag{15.49}$$

which at $\epsilon = 0$ is

$$\begin{aligned} \left.\frac{d\langle\phi' + \epsilon h|0\rangle}{d\epsilon}\right|_{\epsilon=0} = -\frac{1}{2}\Bigg[\int h(\boldsymbol{x})\sqrt{-\triangle + m^2}\,\phi'(\boldsymbol{x})\,\delta^3x \\ + \int \phi'(\boldsymbol{x})\sqrt{-\triangle + m^2}\,h(\boldsymbol{x})\,d^3x\Bigg]\,\langle\phi'|0\rangle. \end{aligned} \tag{15.50}$$

We integrate the second term by parts and drop the surface terms because the smooth function h goes to zero quickly as its arguments go to infinity. We then have

$$\left.\frac{d\langle\phi' + \epsilon h|0\rangle}{d\epsilon}\right|_{\epsilon=0} = -\int h(\boldsymbol{x'})\sqrt{-\triangle + m^2}\,\phi'(\boldsymbol{x'})\,d^3x'\,\langle\phi'|0\rangle. \tag{15.51}$$

Letting $h(\boldsymbol{x'}) = \delta^{(3)}(\boldsymbol{x'} - \boldsymbol{x})$, we arrive at (15.47).

The spatial Fourier transform $\tilde{\phi}'(\boldsymbol{p})$

$$\phi'(\boldsymbol{x}) = \int e^{i\boldsymbol{p}\cdot\boldsymbol{x}}\, \tilde{\phi}'(\boldsymbol{p})\, \frac{d^3p}{(2\pi)^3} \tag{15.52}$$

satisfies $\tilde{\phi}'(-\boldsymbol{p}) = \tilde{\phi}'^*(\boldsymbol{p})$ since ϕ' is real. In terms of it, the ground-state wave function is

$$\langle \phi'|0\rangle = N\, \exp\left(-\frac{1}{2} \int |\tilde{\phi}'(\boldsymbol{p})|^2 \sqrt{\boldsymbol{p}^2 + m^2}\, \frac{d^3p}{(2\pi)^3} \right). \tag{15.53}$$

Example 15.3 (Other Theories, Other Vacua) We can find exact ground states for interacting theories with hamiltonians like

$$H = \frac{1}{2} \int \left[\sqrt{-\nabla^2 + m^2}\, \phi - i c_n \phi^n - i\pi \right] \left[\sqrt{-\nabla^2 + m^2}\, \phi + i c_n \phi^n + i\pi \right] d^3x. \tag{15.54}$$

The state $|\Omega\rangle$ will be an eigenstate of H with eigenvalue zero if

$$\frac{\delta \langle \phi'|\Omega\rangle}{\delta \phi'(\boldsymbol{x})} = - \left[\sqrt{-\nabla^2 + m^2}\, \phi'(\boldsymbol{x}) + i c_n \phi'^n \right] \langle \phi'|\Omega\rangle. \tag{15.55}$$

By extending the argument of equations (15.45–15.51), one may show (exercise 15.4) that the wave functional of the vacuum is

$$\langle \phi'|\Omega\rangle = N\, \exp\left[- \int \left(\tfrac{1}{2}\phi' \sqrt{-\nabla^2 + m^2}\, \phi' + \frac{i c_n}{n+1} \phi'^{n+1} \right) d^3x \right]. \tag{15.56}$$

□

Exercises

15.1 Compute the action $S_0[q]$ (15.1) for the classical path (15.24).

15.2 Use (15.25) to find a formula for the second functional derivative of the action (15.2) of the harmonic oscillator for which $V(q) = m\omega^2 q^2/2$.

15.3 Derive (15.53) from equations (15.48 & 15.52).

15.4 Show that (15.56) satisfies (15.55).