Appendix A Conventions

A.1 Dimensional analysis

In relativistic quantum field theory, it is standard to set

$$c = 2.998 \times 10^8 \text{meters/second} = 1, \tag{A.1}$$

which turns meters into seconds and

$$\hbar = \frac{h}{2\pi} = 1.054\,572 \times 10^{-34} \text{joules} \cdot \text{seconds} = 1, \tag{A.2}$$

which turns joules into inverse seconds. This gives all quantities dimensions of energy (or mass, using $E = mc^2$) to some power. Quantities with positive mass dimension (e.g. momentum p) can be thought of as energies, and quantities with negative mass dimension (e.g. position x) can be thought of as lengths.

Sometimes we write the mass dimension of a quantity with brackets, as in $[p] = \left[\frac{1}{x}\right] = 1$, meaning these quantities have mass dimension 1. Other examples are

$$[dx] = [x] = [t] = -1, \tag{A.3}$$

$$[\partial_\mu] = [p_\mu] = 1, \tag{A.4}$$

$$[\text{velocity}] = \left[\frac{x}{t}\right] = [x] - [t] = 0. \tag{A.5}$$

Thus,

$$[d^4x] = -4. \tag{A.6}$$

The action should be a dimensionless quantity:

$$[S] = \left[\int d^4x \mathcal{L}\right] = 0. \tag{A.7}$$

So Lagrangians (really, Lagrangian densities) have dimension 4:

$$[\mathcal{L}] = 4. \tag{A.8}$$

For example, a free scalar field has Lagrangian $\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi)$ so

$$[\phi] = 1, \tag{A.9}$$

and so on. In general, bosons (whose kinetic terms have two derivatives) have mass dimension 1 and fermions (whose kinetic terms have one derivative) have mass dimension $\frac{3}{2}$.

You can always put the $\hbar$ and c factors back by dimensional analysis. For example, a cross section has units of area, which might be measured in picobarns (pb):[1]

$$1 \text{ picobarn} = 10^{-40} \text{ meters}^2. \tag{A.10}$$

A quantum field theory calculation might produce $\sigma = \frac{1}{m_P^2} \sim \frac{1}{\text{GeV}^2}$, where

$$1 \text{ gigaelectronvolt} = 1.602 \times 10^{-10} \text{ joules}. \tag{A.11}$$

So we need a combination of $\hbar$ and c that converts GeV^{-2} into area. The unique answer is $\hbar^2 c^2 = 9.996 \times 10^{-52} \text{ joules}^2 \cdot \text{meters}^2$. Thus,

$$\frac{1}{\text{GeV}^2}\hbar^2 c^2 = 3.894 \times 10^{-32} \text{ meters}^2 = 3.894 \times 10^8 \text{ picobarns}, \tag{A.12}$$

which is a useful conversion factor.

A.1.1 Factors of 2π

Keeping the factors of 2π straight is important. The origin of all the 2π's is the relation

$$\delta(x) = \int_{-\infty}^{\infty} dp\, e^{\pm 2\pi i p x}. \tag{A.13}$$

This identity holds with either sign; our sign convention for quantum fields is discussed below. To remove the 2π from the exponent, we can rescale either x or p. We rescale p. Then

$$\int_{-\infty}^{\infty} dp\, e^{\pm i p x} = 2\pi\, \delta(x). \tag{A.14}$$

Our convention for the Fourier transform is

$$f(x) = \int \frac{d^4 p}{(2\pi)^4} \tilde{f}(p) e^{-ipx} \quad \leftrightarrow \quad \tilde{f}(p) = \int d^4 x\, f(x) e^{ipx}. \tag{A.15}$$

In general, momentum space integrals will have $\frac{1}{2\pi}$ factors while position space integrals have no 2π factors. Thus, you should get used to writing $\frac{d^4 p}{(2\pi)^4}$ in momentum space integrals. Although physical quantities do not care about our 2π convention, the factors of 2π have important physical effects. Our Fourier transform convention is consistent with

$$p_\mu \leftrightarrow i\partial_\mu, \tag{A.16}$$

which has spatial components $\vec{p} \leftrightarrow -i\vec{\nabla}$, as in quantum mechanics.

[1] The origin of the term **barn** comes from the fact that inducing nuclear fission by hitting ^{235}U with neutrons is as easy as hitting the broad side of a barn. The inelastic neutron–^{235}U scattering cross section is around 1 barn $= 10^{-28}\text{m}^2$ at $E \sim 1$ MeV.

A.2 Signs

Although the meat of most calculations is independent of the signs, physical results are very dependent on getting the sign right. Here we tabulate some of the signs in important equations.

First, we will never use curved-space backgrounds, so the metric $g_{\mu\nu}$ and the Minkowski metric $\eta^{\mu\nu}$ are interchangeable. The metric we use has sign convention

$$g^{\mu\nu} = \eta^{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}. \tag{A.17}$$

This convention makes $p^2 = p_0^2 - \vec{p}^2 = m^2 > 0$. The alternative, $g = \text{diag}(-1, 1, 1, 1)$, makes $p^2 < 0$.

The signs of kinetic terms in Lagrangians are set so that the total energy is positive (see Sections 8.2 and 12.5). It is easiest to remember the signs by writing the Lagrangian as $\mathcal{L} = \mathcal{L}_{\text{kin}} - V$, where V is the potential energy, which should be positive in a stable system. For example, for a scalar field, the mass term $\frac{1}{2}m^2\phi^2$ should give positive energy, so $V = \frac{1}{2}m^2\phi^2$ and $\mathcal{L} = -\frac{1}{2}m^2\phi^2$. The kinetic term sign can then be recalled from $p^2 \to -\Box = -\partial_\mu^2$ in Fourier space and $p^2 = m^2$ on-shell, so that the equations of motion should be $(\Box + m^2)\phi = 0$. Therefore, we have

$$\mathcal{L} = -\frac{1}{2}\phi(\Box + m^2)\phi = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}m^2\phi^2. \tag{A.18}$$

The factor of $\frac{1}{2}$ makes the kinetic term contribute $(\Box + m^2)\phi$ to the equations of motion (instead of $2(\Box + m^2)\phi$). For a complex scalar, the Lagrangian is

$$\mathcal{L} = -\phi^\star(\Box + m^2)\phi = (\partial^\mu\phi^\star)(\partial_\mu\phi) - m^2\phi^\star\phi \tag{A.19}$$

without the $\frac{1}{2}$, since now variation with respect to $\phi^\star$ will give $(\Box + m^2)\phi$.

For gauge bosons, the Lagrangian is

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^2 = -\frac{1}{2}\partial_\mu A_\nu \partial_\mu A_\nu + \frac{1}{2}\partial_\mu A_\nu \partial_\nu A_\mu = \frac{1}{2}A_\nu \Box A_\nu - \frac{1}{2}A_\mu(\partial_\mu\partial_\nu)A_\nu, \tag{A.20}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. In this equation and many others we employ the modern summation convention under which contracted indices can be raised or lowered without ambiguity: $x \cdot p = x^\mu p_\mu = x_\mu p^\mu = x_\mu p_\mu$. All of these contractions are equal to $g^{\mu\nu}x_\mu p_\nu = g_{\mu\nu}x^\mu p^\nu$. The sign and normalization of the $-\frac{1}{4}$ factor in Eq. (A.20) can be understood as follows. In Lorenz gauge $\partial_\mu A_\mu = 0$ the Lagrangian is just $\mathcal{L} = \frac{1}{2}A_\nu \Box A_\nu = \frac{1}{2}A_0 \Box A_0 - \frac{1}{2}\vec{A}\Box\vec{A}$. This gives the three spatial components $\vec{A}$, which actually contain the propagating transverse degrees of freedom, the same kinetic terms as for scalars. (That the scalar component A_0 with the wrong sign is not problematic is explained in Section 8.2.)

Dirac fermions are normalized so that

$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - eA\!\!\!/ - m)\psi, \tag{A.21}$$

where $\not{\partial} = \gamma^\mu \partial_\mu$ and $\not{A} = \gamma^\mu A_\mu$. As in the scalar case, the $-m\bar{\psi}\psi$ is fixed so that the corresponding energy density is positive.

The covariant derivative in a non-Abelian gauge theory is

$$D_\mu = \partial_\mu - igT_R^a A_\mu^a, \tag{A.22}$$

with T_R^a the generators in the appropriate representation. Normalization conventions for these generators are discussed in Section 25.1. We write tr for a sum over group generators or a sum over states, while Tr is used exclusively to denote a Dirac trace. For QED, $D_\mu = \partial_\mu - ieQA_\mu$, where e is the strength of the electromagnetic force ($e = 0.303$ in dimensionless units) and Q is a particle's electric charge (its U(1) quantum number). The electron is defined to have $Q = -1$, which leads to

$$D_\mu \psi_e = (\partial_\mu + ieA_\mu)\psi_e. \tag{A.23}$$

We use this simple form of the covariant derivative throughout Parts II and III.

The Feynman propagators in our conventions are

$$\langle 0|T\{\phi(x)\phi(y)\}|0\rangle = \int \frac{d^4p}{(2\pi)^4} e^{ip(x-y)} \frac{i}{p^2 - m^2 + i\epsilon} \tag{A.24}$$

for a real scalar and

$$\langle 0|T\{A_\mu(x)A_\nu(y)\}|0\rangle = \int \frac{d^4p}{(2\pi)^4} e^{ip(x-y)} \frac{-i(g^{\mu\nu} - (1-\xi)\frac{p^\mu p^\nu}{p^2})}{p^2 + i\epsilon} \tag{A.25}$$

for a massless spin-1 field in covariant gauges. The $-i$ in the photon propagator versus the $+i$ in the scalar propagator is the same sign difference as in $\mathcal{L} = -\frac{1}{2}\phi\Box\phi + \frac{1}{2}A_\nu\Box A_\nu$. The Dirac fermion propagator is

$$\langle 0|T\{\psi(x)\bar{\psi}(y)\}|0\rangle = \int \frac{d^4p}{(2\pi)^4} e^{-ip(x-y)} \frac{i}{\not{p} - m + i\varepsilon} = \int \frac{d^4p}{(2\pi)^4} e^{-ip(x-y)} \frac{i(\not{p} + m)}{p^2 - m^2 + i\epsilon}. \tag{A.26}$$

It is conventional to write $\psi(x)\bar{\psi}(y) = \psi(x)_\alpha \bar{\psi}(y)_\beta$ instead of $\bar{\psi}(x)\psi(y)$ so one is not tempted to mistake the spinors as being contracted. $\psi(x)\bar{\psi}(y)$ is a matrix in spinor space, just as $\vec{v}\vec{w}^T$ is a matrix.

When we expand fields in terms of creation and annihilation operators, we write for a single real scalar field

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \left[a_p(t)e^{i\vec{p}\vec{x}} + a_p^\dagger(t)e^{-i\vec{p}\vec{x}}\right], \tag{A.27}$$

where $\omega_p \equiv \sqrt{\vec{p}^2 + m^2}$. Including the free-field time dependence and generalizing to the complex case, this becomes

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \left(a_p e^{-ipx} + b_p^\dagger e^{ipx}\right), \tag{A.28}$$

$$\phi^\star(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \left(a_p^\dagger e^{ipx} + b_p e^{-ipx}\right). \tag{A.29}$$

Similarly, we take

$$\psi(x) = \sum_s \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \left(a_p^s u_p^s e^{-ipx} + b_p^{s\dagger} v_p^s e^{ipx} \right), \tag{A.30}$$

$$\bar{\psi}(x) = \sum_s \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \left(a_p^{s\dagger} \bar{u}_p^s e^{ipx} + b_p^s \bar{v}_p^s e^{-ipx} \right). \tag{A.31}$$

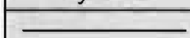
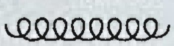
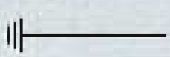
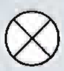
The sign of the phases follows from $a(t) = e^{-i\omega t} a(0)$ for annihilation operators by Heisenberg's equations of motion in any simple harmonic oscillator.

A.3 Feynman rules

The conventions for the Feynman rules follow from the sign conventions above. How the rules are derived is described in Chapter 7. The Feynman rules for various theories covered in the text are given in the appropriate chapter.

For scalar QED, the Feynman rules can be found in Section 9.2, for QED in Section 13.1, for QCD in Section 26.1, for the electroweak theory in Section 29.1, for background fields in Section 34.3.2 and for heavy-quark effective theory in Section 35.2. The notation for various symbols appearing in diagrams throughout the book is shown in Table A.1.

Table A.1 Symbols appearing in Feynman diagrams.

Symbol	Meaning	Symbol	Meaning
	generic particle		fermion
	scalar		charged scalar
	photon or Z boson		ghost
	gluon		W boson
	graviton		heavy quark
	background field		counterterm
	operator or current		generic amplitude
	all one-particle irreducible contributions		alternative generic amplitude

A.4 Dirac algebra

The Dirac matrices satsify $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$. We define

$$\gamma_5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3, \tag{A.32}$$

which leads to $\{\gamma^5, \gamma^\mu\} = 0$. We also define

$$\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]. \tag{A.33}$$

Some useful identities are

$$g^{\mu\nu}g_{\mu\nu} = 4, \tag{A.34}$$

$$\gamma^\mu\gamma_\mu = 4, \tag{A.35}$$

$$\gamma^\mu\gamma^\nu\gamma_\mu = -2\gamma^\nu, \tag{A.36}$$

$$\gamma^\mu\gamma^\nu\gamma^\rho\gamma_\mu = 4g^{\nu\rho}, \tag{A.37}$$

$$\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_\mu = -2\gamma^\sigma\gamma^\rho\gamma^\nu. \tag{A.38}$$

Some useful trace identities are

$$\mathrm{Tr}[\gamma_5] = \mathrm{Tr}[\gamma^\mu] = \mathrm{Tr}[\gamma^\mu\gamma^\alpha\gamma^\nu] = \mathrm{Tr}[\text{odd \# of } \gamma\text{-matrices}] = 0, \tag{A.39}$$

and

$$\mathrm{Tr}[\gamma^\mu\gamma^\nu] = 4g^{\mu\nu}, \tag{A.40}$$

$$\mathrm{Tr}[\gamma^\alpha\gamma^\mu\gamma^\beta\gamma^\nu] = 4(g^{\alpha\mu}g^{\beta\nu} - g^{\alpha\beta}g^{\mu\nu} + g^{\alpha\nu}g^{\mu\beta}), \tag{A.41}$$

$$\mathrm{Tr}\left[\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^5\right] = -4i\varepsilon^{\mu\nu\alpha\beta}. \tag{A.42}$$

The projectors are

$$P_L = \frac{1-\gamma_5}{2}, \quad P_R = \frac{1+\gamma_5}{2}, \tag{A.43}$$

so that left-handed fields satisfy $\gamma_5\psi_L = -\psi_L$ and right-handed fields satisfy $\gamma_5\psi_R = \psi_R$. A Dirac spinor in the $\left(\frac{1}{2},0\right) \oplus \left(0,\frac{1}{2}\right)$ representation is written with the left-handed spinor on top:

$$\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}. \tag{A.44}$$

Spinor sums are, for particles,

$$\sum_{s=1}^{2} u_s(p)\bar{u}_s(p) = \not{p} + m \tag{A.45}$$

and for antiparticles,

$$\sum_{s=1}^{2} v_s(p)\bar{v}_s(p) = \not{p} - m. \tag{A.46}$$

Also,

$$\bar{u}_\sigma(p)\gamma^\mu u_{\sigma'}(p) = 2\delta_{\sigma\sigma'}p^\mu \tag{A.47}$$

is occasionally useful. Left- and right-handed photon polarizations (circularly polarized light) are

$$\epsilon_L^\mu = \frac{1}{\sqrt{2}}(0,1,-i,0), \quad \epsilon_R^\mu = \frac{1}{\sqrt{2}}(0,1,i,0). \tag{A.48}$$

These polarization vectors are consistent with Eq. (A.43) and the representations of the Lorentz group discussed in Chapter 17.

Some other useful identities are

$$\not{D}^2 = D_\mu^2 + \frac{e}{2}F_{\mu\nu}\sigma^{\mu\nu} \tag{A.49}$$

and

$$(\sigma_{\mu\nu}F^{\mu\nu})^2 = 2F_{\mu\nu}^2 + 2i\gamma_5 F_{\mu\nu}\widetilde{F}_{\mu\nu}, \tag{A.50}$$

where

$$\widetilde{F}^{\mu\nu} \equiv \frac{1}{2}\varepsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}. \tag{A.51}$$

Problems

A.1 Dimensional analysis.

(a) A photon coupled to a complex scalar field in d dimensions has action

$$S = \int d^d x \left[-\frac{1}{4}F_{\mu\nu}^2 - \phi^\star\Box\phi + gA_\mu\phi^\star\partial_\mu\phi + \lambda\phi^3 + \cdots \right], \tag{A.52}$$

where $F_{\mu\nu} = (\partial_\mu A_\nu - \partial_\nu A_\mu)$ and $\Box = \partial^\mu\partial_\mu$ as always, but now $\mu = 0, 1, \cdots, d-1$. What are the mass dimensions of A_μ, ϕ, g and λ (as functions of d)?

(b) An interaction is said to be *renormalizable* if its coupling constant is dimensionless. In what dimension d is the electromagnetic interaction renormalizable? How about the ϕ^3 interaction?