10.31 The Lorentz Group

The Lorentz group $O(3,1)$ is the set of all linear transformations L that leave invariant the Minkowski inner product

$$xy \equiv \boldsymbol{x} \cdot \boldsymbol{y} - x^0 y^0 = x^{\mathsf{T}} \eta y \tag{10.222}$$

in which η is the diagonal matrix

$$\eta = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{10.223}$$

So L is in $O(3,1)$ if for all 4-vectors x and y

$$(Lx)^{\mathsf{T}} \eta L\, y = x^{\mathsf{T}} L^{\mathsf{T}} \eta\, L y = x^{\mathsf{T}} \eta\, y. \tag{10.224}$$

Since x and y are arbitrary, this condition amounts to

$$L^{\mathsf{T}} \eta\, L = \eta. \tag{10.225}$$

Taking the determinant of both sides and using the transpose (1.194) and product (1.207) rules, we have

$$(\det L)^2 = 1. \tag{10.226}$$

So $\det L = \pm 1$, and every Lorentz transformation L has an inverse. Multiplying (10.225) by η, we find

$$\eta L^{\mathsf{T}} \eta L = \eta^2 = I \tag{10.227}$$

which identifies L^{-1} as

$$L^{-1} = \eta L^{\mathsf{T}} \eta. \tag{10.228}$$

The subgroup of $O(3,1)$ with $\det L = 1$ is the proper Lorentz group $SO(3,1)$.

To find its Lie algebra, we consider a Lorentz matrix $L = I + \omega$ that differs from the identity matrix I by a tiny matrix ω and require it to satisfy the condition (10.225) for membership in the Lorentz group

$$\left(I + \omega^{\mathsf{T}}\right) \eta\, (I + \omega) = \eta + \omega^{\mathsf{T}} \eta + \eta\, \omega + \omega^{\mathsf{T}} \omega = \eta. \tag{10.229}$$

Neglecting $\omega^{\mathsf{T}} \omega$, we have $\omega^{\mathsf{T}} \eta = -\eta\, \omega$ or since $\eta^2 = I$

$$\omega^{\mathsf{T}} = -\, \eta\, \omega\, \eta. \tag{10.230}$$

This equation says (exercise 10.29) that under transposition the time-time and space-space elements of ω change sign, while the time-space and space-time elements do not. That is, the tiny matrix ω must be for infinitesimal $\boldsymbol{\theta}$ and $\boldsymbol{\lambda}$ a linear combination

$$\omega = \boldsymbol{\theta} \cdot \boldsymbol{R} + \boldsymbol{\lambda} \cdot \boldsymbol{B} \tag{10.231}$$

of the six matrices

$$R_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad R_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \quad R_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{10.232}$$

and

$$B_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad B_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad B_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \tag{10.233}$$

which satisfy condition (10.230). The three R_j are 4×4 versions of the rotation generators (10.88); the three B_j generate Lorentz boosts.

If we write $L = I + \omega$ as

$$L = I - i\theta_\ell iR_\ell - i\lambda_j iB_j \equiv I - i\theta_\ell J_\ell - i\lambda_j K_j \tag{10.234}$$

then the three matrices $J_\ell = iR_\ell$ are imaginary and antisymmetric, and therefore hermitian. But the three matrices $K_j = iB_j$ are imaginary and symmetric, and so are antihermitian. Thus, the 4×4 matrix L is **not unitary**. The reason is that the Lorentz group is **not compact**.

One may verify (exercise 10.30) that the six generators J_ℓ and K_j satisfy three sets of commutation relations:

$$[J_i, J_j] = i\epsilon_{ijk} J_k \tag{10.235}$$

$$[J_i, K_j] = i\epsilon_{ijk} K_k \tag{10.236}$$

$$[K_i, K_j] = -i\epsilon_{ijk} J_k. \tag{10.237}$$

The first (10.235) says that the three J_ℓ generate the rotation group $SO(3)$; the second (10.236) says that the three boost generators transform as a 3-vector under $SO(3)$; and the third (10.237) implies that four canceling infinitesimal boosts can amount to a rotation. These three sets of commutation relations form the Lie algebra of the Lorentz group $SO(3,1)$. Incidentally, one may show (exercise 10.31) that if $\boldsymbol{J}$ and $\boldsymbol{K}$ satisfy these commutation relations (10.235–10.237), then so do

$$\boldsymbol{J} \quad \text{and} \quad -\boldsymbol{K}. \tag{10.238}$$

The infinitesimal Lorentz transformation (10.234) is the 4×4 matrix

$$L = I + \omega = I + \theta_\ell R_\ell + \lambda_j B_j = \begin{pmatrix} 1 & \lambda_1 & \lambda_2 & \lambda_3 \\ \lambda_1 & 1 & -\theta_3 & \theta_2 \\ \lambda_2 & \theta_3 & 1 & -\theta_1 \\ \lambda_3 & -\theta_2 & \theta_1 & 1 \end{pmatrix}. \tag{10.239}$$

It moves any 4-vector x to $x' = L\,x$ or in components $x'^a = L^a{}_b\, x^b$

$$\begin{aligned} x'^0 &= x^0 + \lambda_1 x^1 + \lambda_2 x^2 + \lambda_3 x^3 \\ x'^1 &= \lambda_1 x^0 + x^1 - \theta_3 x^2 + \theta_2 x^3 \\ x'^2 &= \lambda_2 x^0 + \theta_3 x^1 + x^2 - \theta_1 x^3 \\ x'^3 &= \lambda_3 x^0 - \theta_2 x^1 + \theta_1 x^2 + x^3. \end{aligned} \tag{10.240}$$

More succinctly with $t = x^0$, this is

$$\begin{aligned} t' &= t + \boldsymbol{\lambda} \cdot \boldsymbol{x} \\ \boldsymbol{x}' &= \boldsymbol{x} + t\boldsymbol{\lambda} + \boldsymbol{\theta} \wedge \boldsymbol{x} \end{aligned} \tag{10.241}$$

in which $\wedge \equiv \times$ means cross-product.

For arbitrary real $\boldsymbol{\theta}$ and $\boldsymbol{\lambda}$, the matrices

$$L = e^{-i\theta_\ell J_\ell - i\lambda_\ell K_\ell} \tag{10.242}$$

form the subgroup of $SO(3,1)$ that is connected to the identity matrix I. This subgroup preserves the sign of the time of any time-like vector, that is, if $x^2 < 0$, and $y = Lx$, then $y^0 x^0 > 0$. It is called the proper orthochronous Lorentz group. The rest of the (homogeneous) Lorentz group can be obtained from it by space $\mathcal{P}$, time $\mathcal{T}$, and space-time $\mathcal{PT}$ reflections.

The task of finding all the finite-dimensional irreducible representations of the proper orthochronous homogeneous Lorentz group becomes vastly simpler when we write the commutation relations (10.235–10.237) in terms of the hermitian matrices

$$J_\ell^\pm = \frac{1}{2}\left(J_\ell \pm i K_\ell\right) \tag{10.243}$$

which generate two independent rotation groups

$$\begin{aligned} [J_i^+, J_j^+] &= i\epsilon_{ijk} J_k^+ \\ [J_i^-, J_j^-] &= i\epsilon_{ijk} J_k^- \\ [J_i^+, J_j^-] &= 0. \end{aligned} \tag{10.244}$$

Thus the Lie algebra of the Lorentz group is equivalent to two copies of the Lie algebra (10.100) of $SU(2)$. Its finite-dimensional irreducible representations are the direct products

$$D^{(j,j')}(\boldsymbol{\theta}, \boldsymbol{\lambda}) = e^{-i\theta_\ell J_\ell - i\lambda_\ell K_\ell} = e^{(-i\theta_\ell - \lambda_\ell)J_\ell^+} \, e^{(-i\theta_\ell + \lambda_\ell)J_\ell^-} \qquad (10.245)$$

of the nonunitary representations $D^{(j,0)}(\boldsymbol{\theta}, \boldsymbol{\lambda}) = e^{(-i\theta_\ell - \lambda_\ell)J_\ell^+}$ and $D^{(0,j')}(\boldsymbol{\theta}, \boldsymbol{\lambda}) = e^{(-i\theta_\ell + \lambda_\ell)J_\ell^-}$ generated by the three $(2j+1) \times (2j+1)$ matrices J_ℓ^+ and by the three $(2j'+1) \times (2j'+1)$ matrices J_ℓ^-. Under a Lorentz transformation L, a field $\psi^{(j,j')}_{m,m'}(x)$ that transforms under the $D^{(j,j')}$ representation of the Lorentz group responds as

$$U(L)\, \psi^{(j,j')}_{m,m'}(x)\, U^{-1}(L) = D^{(j,0))}_{mm''}(L^{-1})\, D^{(0,j')}_{m'm'''}(L^{-1})\, \psi^{(j,j')}_{m'',m'''}(Lx). \quad (10.246)$$

Although these representations are not unitary, the $SO(3)$ subgroup of the Lorentz group is represented unitarily by the hermitian matrices

$$\boldsymbol{J} = \boldsymbol{J}^+ + \boldsymbol{J}^-. \qquad (10.247)$$

Thus, the representation $D^{(j,j')}$ describes objects of the spins s that can arise from the direct product of spin-j with spin-j' (Weinberg, 1995, p. 231)

$$s = j + j',\, j + j' - 1, \ldots,\, |j - j'|. \qquad (10.248)$$

For instance, $D^{(0,0)}$ describes a spinless field or particle, while $D^{(1/2,0)}$ and $D^{(0,1/2)}$ respectively describe left-handed and right-handed spin-1/2 fields or particles. The representation $D^{(1/2,1/2)}$ describes objects of spin 1 and spin 0—the spatial and time components of a 4-vector.

The generators K_j of the Lorentz boosts are related to $\boldsymbol{J}^\pm$ by

$$\boldsymbol{K} = -i\boldsymbol{J}^+ + i\boldsymbol{J}^- \qquad (10.249)$$

which like (10.247) follows from the definition (10.243).

The interchange of $\boldsymbol{J}^+$ and $\boldsymbol{J}^-$ replaces the generators $\boldsymbol{J}$ and $\boldsymbol{K}$ with $\boldsymbol{J}$ and $-\boldsymbol{K}$, a substitution that we know (10.238) is legitimate.

10.32 Two-Dimensional Representations of the Lorentz Group

The generators of the representation $D^{(1/2,0)}$ with $j = 1/2$ and $j' = 0$ are given by (10.247 & 10.249) with $\boldsymbol{J}^+ = \boldsymbol{\sigma}/2$ and $\boldsymbol{J}^- = 0$. They are

$$\boldsymbol{J} = \frac{1}{2}\boldsymbol{\sigma} \quad \text{and} \quad \boldsymbol{K} = -i\frac{1}{2}\boldsymbol{\sigma}. \qquad (10.250)$$

The 2×2 matrix $D^{(1/2,0)}$ that represents the Lorentz transformation (10.242)

$$L = e^{-i\theta_\ell J_\ell - i\lambda_\ell K_\ell} \tag{10.251}$$

is

$$D^{(1/2,0)}(\boldsymbol{\theta},\boldsymbol{\lambda}) = \exp\left(-i\boldsymbol{\theta}\cdot\boldsymbol{\sigma}/2 - \boldsymbol{\lambda}\cdot\boldsymbol{\sigma}/2\right). \tag{10.252}$$

And so the generic $D^{(1/2,0)}$ matrix is

$$D^{(1/2,0)}(\boldsymbol{\theta},\boldsymbol{\lambda}) = e^{-\boldsymbol{z}\cdot\boldsymbol{\sigma}/2} \tag{10.253}$$

with $\boldsymbol{\lambda} = \mathrm{Re}\boldsymbol{z}$ and $\boldsymbol{\theta} = \mathrm{Im}\boldsymbol{z}$. It is nonunitary and of unit determinant; it is a member of the group $SL(2,C)$ of complex unimodular 2×2 matrices. The (covering) group $SL(2,C)$ relates to the Lorentz group $SO(3,1)$ as $SU(2)$ relates to the rotation group $SO(3)$.

Example 10.31 (The Standard Left-Handed Boost) For a particle of mass $m > 0$, the "standard" boost that takes the 4-vector $k = (m, \mathbf{0})$ to $p = (p^0, \boldsymbol{p})$, where $p^0 = \sqrt{m^2 + \boldsymbol{p}^2}$, is a boost in the $\hat{\boldsymbol{p}}$ direction

$$B(p) = R(\hat{\boldsymbol{p}})\, B_3(p^0)\, R^{-1}(\hat{\boldsymbol{p}}) = \exp\left(\alpha\, \hat{\boldsymbol{p}}\cdot\boldsymbol{B}\right) \tag{10.254}$$

in which $\cosh\alpha = p^0/m$ and $\sinh\alpha = |\boldsymbol{p}|/m$, as one may show by expanding the exponential (exercise 10.33).

For $\boldsymbol{\lambda} = \alpha\,\hat{\boldsymbol{p}}$, one may show (exercise 10.34) that the matrix $D^{(1/2,0)}(\mathbf{0},\boldsymbol{\lambda})$ is

$$\begin{aligned} D^{(1/2,0)}(\mathbf{0},\alpha\,\hat{\boldsymbol{p}}) &= e^{-\alpha\hat{\boldsymbol{p}}\cdot\boldsymbol{\sigma}/2} = I\, \cosh(\alpha/2) - \hat{\boldsymbol{p}}\cdot\boldsymbol{\sigma}\, \sinh(\alpha/2) \\ &= I\, \sqrt{(p^0+m)/(2m)} - \hat{\boldsymbol{p}}\cdot\boldsymbol{\sigma}\, \sqrt{(p^0-m)/(2m)} \\ &= \frac{p^0 + m - \boldsymbol{p}\cdot\boldsymbol{\sigma}}{\sqrt{2m(p^0+m)}} \end{aligned} \tag{10.255}$$

in the third line of which the 2×2 identity matrix I is suppressed. □

Under $D^{(1/2,0)}$, the vector $(-I,\boldsymbol{\sigma})$ transforms like a 4-vector. For tiny $\boldsymbol{\theta}$ and $\boldsymbol{\lambda}$, one may show (exercise 10.36) that the vector $(-I,\boldsymbol{\sigma})$ transforms as

$$\begin{aligned} D^{\dagger(1/2,0)}(\boldsymbol{\theta},\boldsymbol{\lambda})(-I)D^{(1/2,0)}(\boldsymbol{\theta},\boldsymbol{\lambda}) &= -I + \boldsymbol{\lambda}\cdot\boldsymbol{\sigma} \\ D^{\dagger(1/2,0)}(\boldsymbol{\theta},\boldsymbol{\lambda})\,\boldsymbol{\sigma}\,D^{(1/2,0)}(\boldsymbol{\theta},\boldsymbol{\lambda}) &= \boldsymbol{\sigma} + (-I)\boldsymbol{\lambda} + \boldsymbol{\theta}\wedge\boldsymbol{\sigma} \end{aligned} \tag{10.256}$$

which is how the 4-vector $(t,\boldsymbol{x})$ transforms (10.241). Under a finite Lorentz transformation L, the 4-vector $S^a \equiv (-I,\boldsymbol{\sigma})$ becomes

$$D^{\dagger(1/2,0)}(L)\, S^a\, D^{(1/2,0)}(L) = L^a{}_b S^b. \tag{10.257}$$

A massless field $\xi(x)$ that responds to a unitary Lorentz transformation $U(L)$ like

$$U(L)\,\xi(x)\,U^{-1}(L) = D^{(1/2,0)}(L^{-1})\,\xi(Lx) \tag{10.258}$$

is called a **left-handed Weyl spinor**. We will see in example 10.32 why the action density for such spinors

$$\mathcal{L}_\ell(x) = i\,\xi^\dagger(x)\,(\partial_0 I - \boldsymbol{\nabla}\cdot\boldsymbol{\sigma})\,\xi(x) \tag{10.259}$$

is Lorentz covariant, that is

$$U(L)\,\mathcal{L}_\ell(x)\,U^{-1}(L) = \mathcal{L}_\ell(Lx). \tag{10.260}$$

Example 10.32 (Why $\mathcal{L}_\ell$ Is Lorentz Covariant) We first note that the derivatives ∂'_b in $\mathcal{L}_\ell(Lx)$ are with respect to $x' = Lx$. Since the inverse matrix L^{-1} takes x' back to $x = L^{-1}x'$ or in tensor notation $x^a = L^{-1a}{}_b\,x'^b$, the derivative ∂'_b is

$$\partial'_b = \frac{\partial}{\partial x'^b} = \frac{\partial x^a}{\partial x'^b}\frac{\partial}{\partial x^a} = L^{-1a}{}_b\,\frac{\partial}{\partial x^a} = \partial_a\,L^{-1a}{}_b. \tag{10.261}$$

Now using the abbreviation $\partial_0 I - \boldsymbol{\nabla}\cdot\boldsymbol{\sigma} \equiv -\partial_a S^a$ and the transformation laws (10.257 & 10.258), we have

$$\begin{aligned} U(L)\,\mathcal{L}_\ell(x)\,U^{-1}(L) &= i\,\xi^\dagger(Lx)D^{(1/2,0)\dagger}(L^{-1})(-\partial_a S^a)D^{(1/2,0)}(L^{-1})\,\xi(Lx) \\ &= i\,\xi^\dagger(Lx)(-\partial_a L^{-1a}{}_b S^b)\,\xi(Lx) \\ &= i\,\xi^\dagger(Lx)(-\partial'_b S^b)\,\xi(Lx) = \mathcal{L}_\ell(Lx) \end{aligned} \tag{10.262}$$

which shows that $\mathcal{L}_\ell$ is Lorentz covariant. □

Incidentally, the rule (10.261) ensures, among other things, that the divergence $\partial_a V^a$ is invariant

$$(\partial_a V^a)' = \partial'_a V'^a = \partial_b\,L^{-1b}{}_a\,L^a{}_c V^c = \partial_b\,\delta^b{}_c\,V^c = \partial_b\,V^b. \tag{10.263}$$

Example 10.33 (Why ξ is Left Handed) The space-time integral S of the action density $\mathcal{L}_\ell$ is stationary when $\xi(x)$ satisfies the wave equation

$$(\partial_0 I - \boldsymbol{\nabla}\cdot\boldsymbol{\sigma})\,\xi(x) = 0 \tag{10.264}$$

or in momentum space

$$(E + \boldsymbol{p}\cdot\boldsymbol{\sigma})\,\xi(p) = 0. \tag{10.265}$$

Multiplying from the left by $(E - \boldsymbol{p}\cdot\boldsymbol{\sigma})$, we see that the energy of a particle created or annihilated by the field ξ is the same as its momentum $E = |\boldsymbol{p}|$ in accord with the absence of a mass term in the action density $\mathcal{L}_\ell$. And because the spin of the particle is represented by the matrix $\boldsymbol{J} = \boldsymbol{\sigma}/2$, the momentum-space relation (10.265) says that $\xi(p)$ is an eigenvector of $\hat{\boldsymbol{p}} \cdot \boldsymbol{J}$

$$\hat{\boldsymbol{p}} \cdot \boldsymbol{J}\, \xi(p) = -\frac{1}{2}\, \xi(p) \tag{10.266}$$

with eigenvalue $-1/2$. A particle whose spin is opposite to its momentum is said to have **negative helicity** or to be **left handed**. Nearly massless neutrinos are nearly left handed. □

One may add to this action density the **Majorana mass term**

$$\mathcal{L}_M(x) = -\tfrac{1}{2} m \left(\xi^\dagger(x)\, \sigma_2\, \xi^*(x) + \xi^\mathsf{T}(x)\, \sigma_2\, \xi(x) \right) \tag{10.267}$$

which is Lorentz covariant because the matrices σ_1 and σ_3 anti-commute with σ_2 which is antisymmetric (exercise 10.39). This term would vanish if $\xi_1 \xi_2$ were equal to $\xi_2 \xi_1$. Since charge is conserved, only neutral fields like neutrinos can have Majorana mass terms.

The generators of the representation $D^{(0,1/2)}$ with $j = 0$ and $j' = 1/2$ are given by (10.247 & 10.249) with $\boldsymbol{J}^+ = 0$ and $\boldsymbol{J}^- = \boldsymbol{\sigma}/2$; they are

$$\boldsymbol{J} = \frac{1}{2}\boldsymbol{\sigma} \quad \text{and} \quad \boldsymbol{K} = i\frac{1}{2}\boldsymbol{\sigma}. \tag{10.268}$$

Thus 2×2 matrix $D^{(0,1/2)}(\boldsymbol{\theta}, \boldsymbol{\lambda})$ that represents the Lorentz transformation (10.242)

$$L = e^{-i\theta_\ell J_\ell - i\lambda_j K_j} \tag{10.269}$$

is

$$D^{(0,1/2)}(\boldsymbol{\theta}, \boldsymbol{\lambda}) = \exp\left(-i\boldsymbol{\theta} \cdot \boldsymbol{\sigma}/2 + \boldsymbol{\lambda} \cdot \boldsymbol{\sigma}/2\right) = D^{(1/2,0)}(\boldsymbol{\theta}, -\boldsymbol{\lambda}) \tag{10.270}$$

which differs from $D^{(1/2,0)}(\boldsymbol{\theta}, \boldsymbol{\lambda})$ only by the sign of $\boldsymbol{\lambda}$. The generic $D^{(0,1/2)}$ matrix is the complex unimodular 2×2 matrix

$$D^{(0,1/2)}(\boldsymbol{\theta}, \boldsymbol{\lambda}) = e^{\boldsymbol{z}^* \cdot \boldsymbol{\sigma}/2} \tag{10.271}$$

with $\boldsymbol{\lambda} = \mathrm{Re}\boldsymbol{z}$ and $\boldsymbol{\theta} = \mathrm{Im}\boldsymbol{z}$.

Example 10.34 (The Standard Right-Handed Boost) For a particle of mass $m > 0$, the "standard" boost (10.254) that transforms $k = (m, \boldsymbol{0})$ to $p = (p^0, \boldsymbol{p})$ is the 4×4 matrix $B(p) = \exp\left(\alpha\, \hat{\boldsymbol{p}} \cdot \boldsymbol{B}\right)$ in which $\cosh\alpha = p^0/m$ and $\sinh\alpha = |\boldsymbol{p}|/m$. This Lorentz transformation with $\boldsymbol{\theta} = \boldsymbol{0}$ and $\boldsymbol{\lambda} = \alpha\, \hat{\boldsymbol{p}}$

is represented by the matrix (exercise 10.35)

$$
\begin{aligned}
D^{(0,1/2)}(\mathbf{0}, \alpha\,\hat{\boldsymbol{p}}) = e^{\alpha \hat{\boldsymbol{p}}\cdot\boldsymbol{\sigma}/2} &= I\,\cosh(\alpha/2) + \hat{\boldsymbol{p}}\cdot\boldsymbol{\sigma}\,\sinh(\alpha/2) \\
&= I\,\sqrt{(p^0+m)/(2m)} + \hat{\boldsymbol{p}}\cdot\boldsymbol{\sigma}\,\sqrt{(p^0-m)/(2m)} \\
&= \frac{p^0 + m + \boldsymbol{p}\cdot\boldsymbol{\sigma}}{\sqrt{2m(p^0+m)}}
\end{aligned} \tag{10.272}
$$

in the third line of which the 2×2 identity matrix I is suppressed. □

Under $D^{(0,1/2)}$, the vector $(I, \boldsymbol{\sigma})$ transforms as a 4-vector; for tiny $\boldsymbol{z}$

$$
\begin{aligned}
D^{\dagger(0,1/2)}(\boldsymbol{\theta},\boldsymbol{\lambda})\,I\,D^{(0,1/2)}(\boldsymbol{\theta},\boldsymbol{\lambda}) &= I + \boldsymbol{\lambda}\cdot\boldsymbol{\sigma} \\
D^{\dagger(0,1/2)}(\boldsymbol{\theta},\boldsymbol{\lambda})\,\boldsymbol{\sigma}\,D^{(0,1/2)}(\boldsymbol{\theta},\boldsymbol{\lambda}) &= \boldsymbol{\sigma} + I\boldsymbol{\lambda} + \boldsymbol{\theta}\wedge\boldsymbol{\sigma}
\end{aligned} \tag{10.273}
$$

as in (10.241).

A massless field $\zeta(x)$ that responds to a unitary Lorentz transformation $U(L)$ as

$$
U(L)\,\zeta(x)\,U^{-1}(L) = D^{(0,1/2)}(L^{-1})\,\zeta(Lx) \tag{10.274}
$$

is called a **right-handed Weyl spinor**. One may show (exercise 10.38) that the action density

$$
\mathcal{L}_r(x) = i\,\zeta^\dagger(x)\,(\partial_0 I + \boldsymbol{\nabla}\cdot\boldsymbol{\sigma})\,\zeta(x) \tag{10.275}
$$

is Lorentz covariant

$$
U(L)\,\mathcal{L}_r(x)\,U^{-1}(L) = \mathcal{L}_r(Lx). \tag{10.276}
$$

Example 10.35 (Why ζ Is Right Handed) An argument like that of example (10.33) shows that the field $\zeta(x)$ satisfies the wave equation

$$
(\partial_0 I + \boldsymbol{\nabla}\cdot\boldsymbol{\sigma})\,\zeta(x) = 0 \tag{10.277}
$$

or in momentum space

$$
(E - \boldsymbol{p}\cdot\boldsymbol{\sigma})\,\zeta(p) = 0. \tag{10.278}
$$

Thus, $E = |\boldsymbol{p}|$, and $\zeta(p)$ is an eigenvector of $\hat{\boldsymbol{p}}\cdot\boldsymbol{J}$

$$
\hat{\boldsymbol{p}}\cdot\boldsymbol{J}\,\zeta(p) = \frac{1}{2}\,\zeta(p) \tag{10.279}
$$

with eigenvalue 1/2. A particle whose spin is parallel to its momentum is said to have **positive helicity** or to be **right handed**. Nearly massless antineutrinos are nearly right handed. □

The Majorana mass term

$$\mathcal{L}_M(x) = -\tfrac{1}{2}m\left(\zeta^\dagger(x)\,\sigma_2\,\zeta^*(x) + \zeta^{\mathsf{T}}(x)\,\sigma_2\,\zeta(x)\right) \tag{10.280}$$

like (10.267) is Lorentz covariant.

10.33 The Dirac Representation of the Lorentz Group

Dirac's representation of $SO(3,1)$ is the direct sum $D^{(1/2,0)} \oplus D^{(0,1/2)}$ of $D^{(1/2,0)}$ and $D^{(0,1/2)}$. Its generators are the 4×4 matrices

$$\boldsymbol{J} = \frac{1}{2}\begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix} \quad \text{and} \quad \boldsymbol{K} = \frac{i}{2}\begin{pmatrix} -\boldsymbol{\sigma} & 0 \\ 0 & \boldsymbol{\sigma} \end{pmatrix}. \tag{10.281}$$

Dirac's representation uses the **Clifford algebra** of the gamma matrices γ^a which satisfy the anticommutation relation

$$\{\gamma^a, \gamma^b\} \equiv \gamma^a\,\gamma^b + \gamma^b\,\gamma^a = 2\eta^{ab} I \tag{10.282}$$

in which η is the 4×4 diagonal matrix (10.223) with $\eta^{00} = -1$ and $\eta^{jj} = 1$ for $j = 1$, 2, and 3, and I is the 4×4 identity matrix.

Remarkably, the generators of the Lorentz group

$$J^{ij} = \epsilon_{ijk} J_k \quad \text{and} \quad J^{0j} = K_j \tag{10.283}$$

may be represented as commutators of gamma matrices

$$J^{ab} = -\frac{i}{4}\,[\gamma^a, \gamma^b]. \tag{10.284}$$

They transform the gamma matrices as a 4-vector

$$[J^{ab}, \gamma^c] = -i\gamma^a\,\eta^{bc} + i\gamma^b\,\eta^{ac} \tag{10.285}$$

(exercise 10.40) and satisfy the commutation relations

$$i[J^{ab}, J^{cd}] = \eta^{bc}\,J^{ad} - \eta^{ac}\,J^{bd} - \eta^{da}\,J^{cb} + \eta^{db}\,J^{ca} \tag{10.286}$$

of the Lorentz group (Weinberg, 1995, p. 213–217) (exercise 10.41).

The gamma matrices γ^a are not unique; if S is any 4×4 matrix with an inverse, then the matrices $\gamma'^a \equiv S\gamma^a S^{-1}$ also satisfy the definition (10.282). The choice

$$\gamma^0 = -i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \boldsymbol{\gamma} = -i\begin{pmatrix} 0 & \boldsymbol{\sigma} \\ -\boldsymbol{\sigma} & 0 \end{pmatrix} \tag{10.287}$$

makes $\boldsymbol{J}$ and $\boldsymbol{K}$ block diagonal (10.281) and lets us assemble a left-handed spinor ξ and a right-handed spinor ζ neatly into a 4-component spinor

$$\psi = \begin{pmatrix} \xi \\ \zeta \end{pmatrix}. \tag{10.288}$$

Dirac's action density for a 4-spinor is

$$\mathcal{L} = -\overline{\psi}\,(\gamma^a \partial_a + m)\,\psi \equiv -\overline{\psi}\,(\not{\partial} + m)\,\psi \tag{10.289}$$

in which

$$\overline{\psi} \equiv i\psi^\dagger \gamma^0 = \psi^\dagger \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \zeta^\dagger & \xi^\dagger \end{pmatrix}. \tag{10.290}$$

The kinetic part is the sum of the left-handed $\mathcal{L}_\ell$ and right-handed $\mathcal{L}_r$ action densities (10.259 & 10.275)

$$-\overline{\psi}\,\gamma^a \partial_a \psi = i\xi^\dagger\,(\partial_0 I - \boldsymbol{\nabla}\cdot\boldsymbol{\sigma})\,\xi + i\,\zeta^\dagger\,(\partial_0 I + \boldsymbol{\nabla}\cdot\boldsymbol{\sigma})\,\zeta. \tag{10.291}$$

If ξ is a left-handed spinor transforming as (10.258), then the spinor

$$\zeta = \sigma_2\,\xi^* \equiv \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} \xi_1^\dagger \\ \xi_2^\dagger \end{pmatrix} \tag{10.292}$$

transforms as a right-handed spinor (10.274), that is (exercise 10.42)

$$e^{\boldsymbol{z}^*\cdot\boldsymbol{\sigma}/2}\sigma_2\,\xi^* = \sigma_2 \left(e^{-\boldsymbol{z}\cdot\boldsymbol{\sigma}/2}\xi\right)^*. \tag{10.293}$$

Similarly, if ζ is right handed, then $\xi = \sigma_2\,\zeta^*$ is left handed.

The simplest 4-spinor is the Majorana spinor

$$\psi_M = \begin{pmatrix} \xi \\ \sigma_2 \xi^* \end{pmatrix} = \begin{pmatrix} \sigma_2 \zeta^* \\ \zeta \end{pmatrix} = -i\gamma^2 \psi_M^* \tag{10.294}$$

whose particles are the same as its antiparticles.

If two Majorana spinors $\psi_M^{(1)}$ and $\psi_M^{(2)}$ have the same mass, then one may combine them into a Dirac spinor

$$\psi_D = \frac{1}{\sqrt{2}}\left(\psi_M^{(1)} + i\psi_M^{(2)}\right) = \frac{1}{\sqrt{2}}\begin{pmatrix} \xi^{(1)} + i\xi^{(2)} \\ \zeta^{(1)} + i\zeta^{(2)} \end{pmatrix} = \begin{pmatrix} \xi_D \\ \zeta_D \end{pmatrix}. \tag{10.295}$$

The **Dirac mass term**

$$-m\,\overline{\psi}_D \psi_D = -m\left(\zeta_D^\dagger \xi_D + \xi_D^\dagger \zeta_D\right) \tag{10.296}$$

conserves charge, and since $\exp(\boldsymbol{z}^*\cdot\sigma/2)^\dagger \exp(-\boldsymbol{z}\cdot\sigma/2) = I$ it also is Lorentz invariant. For a Majorana field, it reduces to

$$\begin{aligned} -\tfrac{1}{2}m\overline{\psi}_M\psi_M &= -\tfrac{1}{2}m\left(\zeta^\dagger\xi + \xi^\dagger\zeta\right) = -\tfrac{1}{2}m\left(\xi^\dagger\sigma_2\xi^* + \xi^{\mathsf{T}}\sigma_2\xi\right) \\ &= -\tfrac{1}{2}m\left(\zeta^\dagger\sigma_2\zeta^* + \zeta^{\mathsf{T}}\sigma_2\zeta\right) \end{aligned} \tag{10.297}$$

a Majorana mass term (10.267 or 10.280).

10.34 The Poincaré Group

The elements of the Poincaré group are products of Lorentz transformations and translations in space and time. The Lie algebra of the Poincaré group therefore includes the generators $\boldsymbol{J}$ and $\boldsymbol{K}$ of the Lorentz group as well as the hamiltonian H and the momentum operator $\boldsymbol{P}$ which respectively generate translations in time and space.

Suppose $T(y)$ is a translation that takes a 4-vector x to $x+y$ and $T(z)$ is a translation that takes a 4-vector x to $x+z$. Then $T(z)T(y)$ and $T(y)T(z)$ both take x to $x+y+z$. So if a translation $T(y) = T(t,\boldsymbol{y})$ is represented by a unitary operator $U(t,\boldsymbol{y}) = \exp(iHt - i\boldsymbol{P}\cdot\boldsymbol{y})$, then the hamiltonian H and the momentum operator $\boldsymbol{P}$ commute with each other

$$[H, P^j] = 0 \quad \text{and} \quad [P^i, P^j] = 0. \tag{10.298}$$

We can figure out the commutation relations of H and $\boldsymbol{P}$ with the angular-momentum $\boldsymbol{J}$ and boost $\boldsymbol{K}$ operators by realizing that $P^a = (H, \boldsymbol{P})$ is a 4-vector. Let

$$U(\boldsymbol{\theta},\boldsymbol{\lambda}) = e^{-i\boldsymbol{\theta}\cdot\boldsymbol{J} - i\boldsymbol{\lambda}\cdot\boldsymbol{K}} \tag{10.299}$$

be the (infinite-dimensional) unitary operator that represents (in Hilbert space) the infinitesimal Lorentz transformation

$$L = I + \boldsymbol{\theta}\cdot\boldsymbol{R} + \boldsymbol{\lambda}\cdot\boldsymbol{B} \tag{10.300}$$

where $\boldsymbol{R}$ and $\boldsymbol{B}$ are the six 4×4 matrices (10.232 & 10.233). Then because P is a 4-vector under Lorentz transformations, we have

$$U^{-1}(\boldsymbol{\theta},\boldsymbol{\lambda})PU(\boldsymbol{\theta},\boldsymbol{\lambda}) = e^{+i\boldsymbol{\theta}\cdot\boldsymbol{J}+i\boldsymbol{\lambda}\cdot\boldsymbol{K}}Pe^{-i\boldsymbol{\theta}\cdot\boldsymbol{J}-i\boldsymbol{\lambda}\cdot\boldsymbol{K}} = (I + \boldsymbol{\theta}\cdot\boldsymbol{R} + \boldsymbol{\lambda}\cdot\boldsymbol{B})\,P \tag{10.301}$$

or using (10.273)

$$\begin{aligned} (I + i\boldsymbol{\theta}\cdot\boldsymbol{J} + i\boldsymbol{\lambda}\cdot\boldsymbol{K})\,H\,(I - i\boldsymbol{\theta}\cdot\boldsymbol{J} - i\boldsymbol{\lambda}\cdot\boldsymbol{K}) &= H + \boldsymbol{\lambda}\cdot\boldsymbol{P} \\ (I + i\boldsymbol{\theta}\cdot\boldsymbol{J} + i\boldsymbol{\lambda}\cdot\boldsymbol{K})\,\boldsymbol{P}\,(I - i\boldsymbol{\theta}\cdot\boldsymbol{J} - i\boldsymbol{\lambda}\cdot\boldsymbol{K}) &= \boldsymbol{P} + H\boldsymbol{\lambda} + \boldsymbol{\theta}\wedge\boldsymbol{P}. \end{aligned} \tag{10.302}$$

Thus, one finds (exercise 10.42) that H is invariant under rotations, while $\boldsymbol{P}$ transforms as a 3-vector

$$[J_i, H] = 0 \quad \text{and} \quad [J_i, P_j] = i\epsilon_{ijk} P_k \tag{10.303}$$

and that

$$[K_i, H] = -iP_i \quad \text{and} \quad [K_i, P_j] = -i\delta_{ij} H. \tag{10.304}$$

By combining these equations with (10.286), one may write (exercise 10.44) the Lie algebra of the Poincaré group as

$$\begin{aligned} i[J^{ab}, J^{cd}] &= \eta^{bc}\, J^{ad} - \eta^{ac}\, J^{bd} - \eta^{da}\, J^{cb} + \eta^{db}\, J^{ca} \\ i[P^a, J^{bc}] &= \eta^{ab} P^c - \eta^{ac} P^b \\ [P^a, P^b] &= 0. \end{aligned} \tag{10.305}$$

Further Reading

The classic *Lie Algebras in Particle Physics* (Georgi, 1999), which inspired much of this chapter, is outstanding.

Exercises

10.1 Show that all $n \times n$ (real) orthogonal matrices O leave invariant the quadratic form $x_1^2 + x_2^2 + \cdots + x_n^2$, that is, that if $x' = Ox$, then $x'^2 = x^2$.

10.2 Show that the set of all $n \times n$ orthogonal matrices forms a group.

10.3 Show that all $n \times n$ unitary matrices U leave invariant the quadratic form $|x_1|^2 + |x_2|^2 + \cdots + |x_n|^2$, that is, that if $x' = Ux$, then $|x'|^2 = |x|^2$.

10.4 Show that the set of all $n \times n$ unitary matrices forms a group.

10.5 Show that the set of all $n \times n$ unitary matrices with unit determinant forms a group.

10.6 Show that the matrix $D^{(j)}_{m'm}(g) = \langle j, m'|U(g)|j, m\rangle$ is unitary because the rotation operator $U(g)$ is unitary $\langle j, m'|U^\dagger(g)U(g)|j, m\rangle = \delta_{m'm}$.

10.7 Invent a group of order 3 and compute its multiplication table. For extra credit, prove that the group is unique.

10.8 Show that the relation (10.20) between two equivalent representations is an isomorphism.

10.9 Suppose that D_1 and D_2 are equivalent, finite-dimensional, irreducible representations of a group G so that $D_2(g) = SD_1(g)S^{-1}$ for all $g \in G$. What can you say about a matrix A that satisfies $D_2(g)\, A = A\, D_1(g)$ for all $g \in G$?

10.10 Find all components of the matrix $\exp(i\alpha A)$ in which

$$A = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}. \tag{10.306}$$

10.11 If $[A, B] = B$, find $e^{i\alpha A} B e^{-i\alpha A}$. Hint: what are the α-derivatives of this expression?

10.12 Show that the tensor-product matrix (10.31) of two representations D_1 and D_2 is a representation.

10.13 Find a 4×4 matrix S that relates the tensor-product representation $D_{\frac{1}{2}\otimes\frac{1}{2}}$ to the direct sum $D_1 \oplus D_0$.

10.14 Find the generators in the adjoint representation of the group with structure constants $f_{abc} = \epsilon_{abc}$ where a, b, c run from 1 to 3. *Hint:* The answer is three 3×3 matrices t_a, often written as L_a.

10.15 Show that the generators (10.90) satisfy the commutation relations (10.93).

10.16 Show that the demonstrated equation (10.98) implies the commutation relation (10.99).

10.17 Use the Cayley-Hamilton theorem (1.264) to show that the 3×3 matrix (10.96) that represents a right-handed rotation of θ radians about the axis $\boldsymbol{\theta}$ is given by (10.97).

10.18 Verify the mixed Jacobi identity (10.142).

10.19 For the group $SU(3)$, find the structure constants f_{123} and f_{231}.

10.20 Show that every 2×2 unitary matrix of unit determinant is a quaternion of unit norm.

10.21 Show that the quaternions as defined by (10.175) are closed under addition and multiplication and that the product xq is a quaternion if x is real and q is a quaternion.

10.22 Show that the one-sided derivative $f'(q)$ (10.185) of the quaternionic function $f(q) = q^2$ depends upon the direction along which $q' \to 0$.

10.23 Show that the generators (10.189) of $Sp(2n)$ obey commutation relations of the form (10.190) for some real structure constants f_{abc} and a suitably extended set of matrices A, A', ... and S_k, S'_k, ...

10.24 Show that for $0 < \epsilon \ll 1$, the real $2n \times 2n$ matrix $T = \exp(\epsilon JS)$ in which S is symmetric satisfies $T^{\mathsf{T}} J T = J$ (at least up to terms of order ϵ^2) and so is in $Sp(2n, R)$.

10.25 Show that the matrix T of (10.198) is in $Sp(2, R)$.

10.26 Use the parametrization (10.218) of the group $SU(2)$, show that the parameters $\boldsymbol{a}(\boldsymbol{c}, \boldsymbol{b})$ that describe the product $g(\boldsymbol{a}(\boldsymbol{c}, \boldsymbol{b})) = g(\boldsymbol{c})\, g(\boldsymbol{b})$ are those of (10.220).

10.27 Use formulas (10.220) and (10.213) to show that the left-invariant measure for $SU(2)$ is given by (10.221).

10.28 In tensor notation, which is explained in chapter 11, the condition (10.230) that $I + \omega$ be an infinitesimal Lorentz transformation reads $\left(\omega^{\mathsf{T}}\right)_b{}^a = \omega^a{}_b = -\,\eta_{bc}\,\omega^c{}_d\,\eta^{da}$ in which sums over c and d from 0 to 3 are understood. In this notation, the matrix η_{ef} lowers indices and η^{gh} raises them, so that $\omega_b{}^a = -\,\omega_{bd}\,\eta^{da}$. (Both η_{ef} and η^{gh} are numerically equal to the matrix η displayed in equation (10.223).) Multiply both sides of the condition (10.230) by $\eta_{ae} = \eta_{ea}$ and use the relation $\eta^{da}\,\eta_{ae} = \eta^d{}_e \equiv \delta^d{}_e$ to show that the matrix ω_{ab} with both indices lowered (or raised) is antisymmetric, that is,

$$\omega_{ba} = -\,\omega_{ab} \quad \text{and} \quad \omega^{ba} = -\,\omega^{ab}. \tag{10.307}$$

10.29 Show that the six matrices (10.232) and (10.233) satisfy the $SO(3,1)$ condition (10.230).

10.30 Show that the six generators $\boldsymbol{J}$ and $\boldsymbol{K}$ obey the commutations relations (10.235–10.237).

10.31 Show that if $\boldsymbol{J}$ and $\boldsymbol{K}$ satisfy the commutation relations (10.235–10.237) of the Lie algebra of the Lorentz group, then so do $\boldsymbol{J}$ and $-\boldsymbol{K}$.

10.32 Show that if the six generators $\boldsymbol{J}$ and $\boldsymbol{K}$ obey the commutation relations (10.235–10.237), then the six generators $\boldsymbol{J}^+$ and $\boldsymbol{J}^-$ obey the commutation relations (10.244).

10.33 Relate the parameter α in the definition (10.254) of the standard boost $B(p)$ to the 4-vector p and the mass m.

10.34 Derive the formulas for $D^{(1/2,0)}(\mathbf{0}, \alpha\,\hat{\boldsymbol{p}})$ given in equation (10.255).

10.35 Derive the formulas for $D^{(0,1/2)}(\mathbf{0}, \alpha\,\hat{\boldsymbol{p}})$ given in equation (10.272).

10.36 For infinitesimal complex $\boldsymbol{z}$, derive the 4-vector properties (10.256 & 10.273) of $(-I, \boldsymbol{\sigma})$ under $D^{(1/2,0)}$ and of $(I, \boldsymbol{\sigma})$ under $D^{(0,1/2)}$.

10.37 Show that under the unitary Lorentz transformation (10.258), the action density (10.259) is Lorentz covariant (10.260).

10.38 Show that under the unitary Lorentz transformation (10.274), the action density (10.275) is Lorentz covariant (10.276).

10.39 Show that under the unitary Lorentz transformations (10.258 & 10.274), the Majorana mass terms (10.267 & 10.280) are Lorentz covariant.

10.40 Show that the definitions of the gamma matrices (10.282) and of the generators (10.284) imply that the gamma matrices transform as a 4-vector under Lorentz transformations (10.285).

10.41 Show that (10.284) and (10.285) imply that the generators J^{ab} satisfy the commutation relations (10.286) of the Lorentz group.

10.42 Show that the spinor $\zeta = \sigma_2 \xi^*$ defined by (10.292) is right handed (10.274) if ξ is left handed (10.258).

10.43 Use (10.302) to get (10.303 & 10.304).

10.44 Derive (10.305) from (10.286, 10.298, 10.303, & 10.304).