Solution to second homework problem.

Physics 524, April 23, 2015

Problem: Show that

$$\overline{E}_\ell \not{D}^\ell E_\ell = [\tfrac{1}{2}(1+\gamma_5)E]^\dagger i\gamma^0 \not{D}^\ell \tfrac{1}{2}(1+\gamma_5)E = \overline{E} \not{D}^\ell \tfrac{1}{2}(1+\gamma_5)E \tag{1}$$

where in Weinberg's notation

$$\overline{\psi} = \psi^\dagger i\gamma^0 = \psi^\dagger \beta = \psi^\dagger \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \quad \text{and} \quad \gamma_5 = \gamma^5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2}$$

in which I is the 2×2 identity matrix.
Solution: First, note that

$$\frac{1}{2}\left(1+\gamma^5\right) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = P_\ell \tag{3}$$

is a projection operator

$$P_\ell^2 = P_\ell \tag{4}$$

on the left-handed two-component spinor

$$\frac{1}{2}\left(1+\gamma^5\right) E = E_\ell. \tag{5}$$

Thus, we have

$$\overline{E}_\ell \not{D}^\ell E_\ell = [\tfrac{1}{2}(1+\gamma_5)E]^\dagger i\gamma^0 \not{D}^\ell \tfrac{1}{2}(1+\gamma_5)E. \tag{6}$$

Next, we note that

$$[\tfrac{1}{2}(1+\gamma_5)E]^\dagger = E^\dagger \tfrac{1}{2}(1+\gamma_5), \tag{7}$$

and so

$$[\tfrac{1}{2}(1+\gamma_5)E]^\dagger i\gamma^0 \not{D}^\ell \tfrac{1}{2}(1+\gamma_5)E = E^\dagger \tfrac{1}{2}(1+\gamma_5) i\gamma^0 \not{D}^\ell \tfrac{1}{2}(1+\gamma_5)E \tag{8}$$

$$= E^\dagger i\gamma^0 \not{D}^\ell \tfrac{1}{2}(1+\gamma_5)\tfrac{1}{2}(1+\gamma_5)E \tag{9}$$

$$= \overline{E}_\ell \not{D}^\ell \tfrac{1}{2}(1+\gamma_5)E = \overline{E}_\ell \not{D}^\ell E_\ell. \tag{10}$$