

Solution to first homework problem.

Physics 524, 14 April 2015

Problem: Derive Equation 25.80

$$J_a^\mu = -\bar{\psi}_i \gamma^\mu T_{ij}^a \psi_j + f^{abc} A_\nu^b F_c^{\mu\nu}$$

Solution: The conserved currents are

$$J_a^\mu = \sum_n \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi_n)} \frac{\delta \phi_n}{\delta \alpha_a}$$

summed over the fields ϕ_n of the action density

$$\mathcal{L} = -\frac{1}{4} (\partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f^{abc} A_\mu^b A_\nu^c)^2 + \bar{\psi}_i (i \delta_{ij} \gamma^\mu \partial_\mu + g \gamma^\mu A_\mu^a T_{ij}^a - m \delta_{ij}) \psi_j$$

which is invariant under the global transformation

$$\begin{aligned} \psi_i &\rightarrow \psi_i + i \alpha^a T_{ij}^a \psi_j \\ A_\mu^a &\rightarrow A_\mu^a - f^{abc} \alpha^b A_\mu^c. \end{aligned}$$

One sums over both the matter fields $\phi_n = \psi_i$ and the gauge fields $\phi_n = A_\mu^a$.

First one sums over the fields $\phi_n = \psi_i$:

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial (\partial_\mu \psi_i)} \frac{\delta \psi_i}{\delta \alpha_a} &= \frac{\partial}{\partial (\partial_\mu \psi_i)} [\bar{\psi}_i (i \delta_{ij} \gamma^\mu \partial_\mu + g \gamma^\mu A_\mu^b T_{ij}^b - m \delta_{ij}) \psi_j] \frac{\delta \psi_i}{\delta \alpha_a} \\ &= \frac{\partial}{\partial (\partial_\mu \psi_i)} [i \bar{\psi}_i \delta_{ij} \gamma^\mu \partial_\mu \psi_j + \bar{\psi}_i g \gamma^\mu A_\mu^b T_{ij}^b \psi_j - \bar{\psi}_i m \delta_{ij} \psi_j] \frac{\delta \psi_i}{\delta \alpha_a} \\ &= i \bar{\psi}_i \gamma^\mu \frac{\delta \psi_i}{\delta \alpha_a}. \end{aligned}$$

Then one uses

$$\frac{\delta \psi_i}{\delta \alpha_a} = \frac{\partial}{\partial \alpha_a} (\psi_i + i \alpha^b T_{ij}^b \psi_j) = i T_{ij}^a \psi_j$$

to get

$$\boxed{\frac{\partial \mathcal{L}}{\partial (\partial_\mu \psi_i)} \frac{\delta \psi_i}{\delta \alpha_a} = -\bar{\psi}_i \gamma^\mu T_{ij}^a \psi_j.} \quad (1)$$

Now one sums over the gauge fields, $\phi_n = A_\nu^b$

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial (\partial_\mu A_\nu^b)} \frac{\delta A_\nu^b}{\delta \alpha_a} &= \frac{\partial}{\partial (\partial_\mu A_\nu^b)} \left[-\frac{1}{4} (\partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g f^{bcd} A_\mu^b A_\nu^c) F_b^{\mu\nu} \right] \frac{\delta A_\nu^b}{\delta \alpha_a} \\ &= -F_b^{\mu\nu} \frac{\delta A_\nu^b}{\delta \alpha_a}. \end{aligned}$$

The gauge fields vary as

$$\frac{\delta A_\nu^b}{\delta \alpha_a} = \frac{\delta}{\delta \alpha_a} (A_\nu^b - f^{bcd} \alpha^c A_\nu^d) = -f^{bad} A_\nu^d.$$

Thus one gets

$$\frac{\partial \mathcal{L}}{\partial (\partial_\mu A_\nu^b)} \frac{\delta A_\nu^b}{\delta \alpha_a} = -F_b^{\mu\nu} (-f^{bad} A_\nu^d) = f^{adb} A_\nu^d F_b^{\mu\nu}. \quad (2)$$

Equations 1 and 2 now give Equation 25.80 from Schwartz

$$J^\mu = -\bar{\psi}_i \gamma^\mu T_{ij}^a \psi_j + f^{abc} A_\nu^b F_c^{\mu\nu}.$$