Symmetry Redux

Say under $\psi_i \to \psi_i' = \psi_i + \delta\psi_i$, the Lagrange density is invariant. Then

$$0 = \delta \mathcal{L} = \frac{\partial \mathcal{L}}{\partial \psi_i}\delta\psi_i + \frac{\partial \mathcal{L}}{\partial \partial_\mu \psi_i}\delta \partial_\mu \psi_i \qquad (1)$$

But Lagrange's equations give

$$\frac{\partial \mathcal{L}}{\partial \psi_i} = \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \psi_i}. \qquad (2)$$

So (1) & (2) imply that

$$0 = \left(\partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \psi_i}\right)\delta\psi_i + \frac{\partial \mathcal{L}}{\partial \partial_\mu \psi_i}\delta \partial_\mu \psi_i \qquad (3)$$

But $\partial_\mu \psi_i' = \partial_\mu \psi_i + \partial_\mu \delta\psi_i$, so $\qquad (4)$

$\delta \partial_\mu \psi_i = \partial_\mu \delta \psi_i$ whence

$$0 = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \partial_\mu \psi_i}\right)\delta\psi_i + \frac{\partial \mathcal{L}}{\partial \partial_\mu \psi_i}\partial_\mu \delta\psi_i$$

$$= \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \partial_\mu \psi_i}\delta\psi_i\right) = \partial_\mu J^\mu$$

where $J^\mu = \dfrac{\partial \mathcal{L}}{\partial \partial_\mu \psi_i}\delta\psi_i$.