Superfluids

$$\mathcal{L} = i\varphi^\dagger \partial_0 \varphi - \frac{1}{2m}\nabla\varphi^\dagger \cdot \nabla\varphi - g^2(\varphi^\dagger\varphi - \bar{\rho})^2$$

Let $\varphi = \sqrt{\rho}\, e^{i\theta}$, drop $\frac{i}{2}\partial_0\rho$ because it's a total derivative and get

$$\mathcal{L} = -\rho\dot{\theta} - \frac{1}{2m}\left[\frac{1}{4\rho}(\nabla\rho)^2 + \rho(\nabla\theta)^2\right] - g^2(\rho - \bar{\rho})^2.$$

Now let $\sqrt{\rho} = \sqrt{\bar{\rho}} + h \qquad \langle\varphi\rangle = \sqrt{\bar{\rho}} \gg h$

$$\mathcal{L} \simeq -2\sqrt{\bar{\rho}}\, h\, \dot{\theta} - \frac{\bar{\rho}}{2m}(\nabla\theta)^2 - \frac{1}{2m}(\nabla h)^2 - 4g^2\bar{\rho}h^2 \cdots$$

Now recall that

$$\int e^{-\frac{1}{2}\varphi K \varphi + J\varphi} D\varphi = e^{\frac{1}{2}JK^{-1}J}.$$

So effectively

$$\mathcal{L} = \bar{\rho}\,\dot{\theta}\, \frac{1}{4g^2\bar{\rho} - \frac{\nabla^2}{2m}}\, \dot{\theta} - \frac{\bar{\rho}}{2m}(\nabla\theta)^2 + \cdots$$

neglect

$$\simeq \frac{1}{4g^2}\dot{\theta}^2 - \frac{\bar{\rho}}{2m}(\nabla\theta)^2 + \cdots$$

So $\frac{1}{4g^2}\ddot{\theta} = \frac{\bar{\rho}}{2m}\nabla^2\theta$ is the eq. of motion.

So θ is a massless particle as long as

$$k << \sqrt{8g^2\bar{\rho}\, m}$$

so that $\frac{k^2}{2m} << 4g^2\rho$.

θ describes a massless phonon with

$$\omega^2 = \frac{2g^2\bar{\rho}}{m}\,\vec{k}^{\,2}$$

a result due to Bogoliubov.

Landau: Consider mass M of fluid moving at velocity v. Suppose it loses some momentum $\hbar k$

$$Mv = Mv' + \hbar\vec{k}$$

We need

$$\tfrac{1}{2}Mv^2 \geq \tfrac{1}{2}Mv'^2 + \hbar\omega(k)$$

so since M is macroscopic we get

$$\vec{v}\cdot\vec{k} \geq \omega$$

which means $|\vec{v}| \geq \frac{\omega}{|\vec{k}|}$ ie $v \geq \frac{\omega}{k}$.

So here $\omega = \sqrt{\frac{2\bar{\rho}}{m}}\, g k$, $v \geq g\sqrt{\frac{2\bar{\rho}}{m}} \equiv v_c$.

Fluids moving at less than v_c can't lose momentum!

Rescale space. Rewrite $\mathcal{L}$ as

$$\mathcal{L} = \frac{1}{4g^2}(\partial_\mu \theta)^2$$

So the field θ is massless.

It's an angle: $\theta(x)$ and $\theta(x) + 2\pi$ are the same.

There are too few low-energy modes for a superfluid to lose energy and momentum to.