Solitons

Consider again the theory

$$\mathcal{L} = \frac{1}{2}(\partial\varphi)^2 - V(\varphi)$$

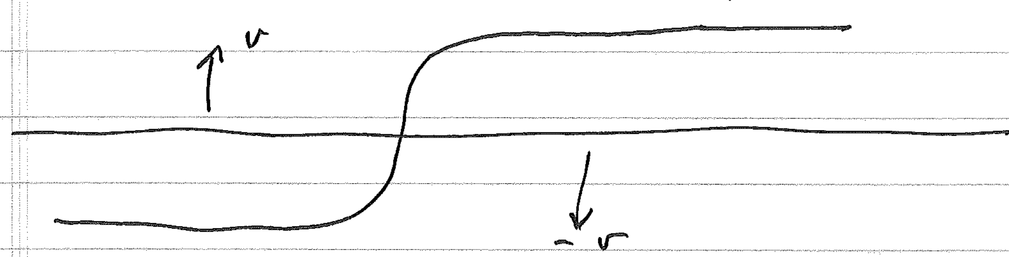
$$= \frac{1}{2}(\partial\varphi)^2 - \frac{\lambda}{4}(\varphi^2 - v^2)^2$$

but now in 1+1 space-time dimensions.

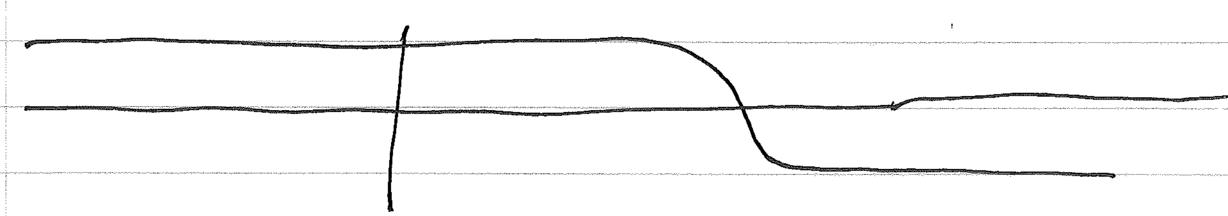
Now look just at time-independent solutions. The energy or mass is

$$M = \int dx \; \frac{1}{2}\varphi'^2 + \frac{\lambda}{4}(\varphi^2 - v^2)^2$$

A kink solution is possible:



or an anti-kink



$M = M_K + M_V \qquad M_K \sim \ell\left(\frac{v}{\ell}\right)^2 = \frac{v^2}{\ell}$

while $M_V \sim \ell\lambda v^4$ so roughly we have

$$M \sim \frac{v^2}{\ell} + \ell \lambda v^4$$

$$0 = \frac{\partial M}{\partial \ell} = -\frac{v^2}{\ell^2} + \lambda v^4$$

so

$$\frac{1}{\ell^2} = \lambda v^2$$

or $$\ell = \frac{1}{v\sqrt{\lambda}} .$$

Recall also that the bosons of the theory have mass we get by setting

$$\varphi = v + \chi \qquad \text{so that}$$

$$\mathcal{L} = \tfrac{1}{2}(\partial\chi)^2 - \frac{\lambda}{4}\left[(v+\chi)^2 - v^2\right]^2$$

$$= \tfrac{1}{2}(\partial\chi)^2 - \frac{\lambda}{4}\left[2v\chi + \chi^2\right]^2$$

$$\approx \tfrac{1}{2}(\partial\chi)^2 - \lambda v^2 \chi^2 + \cdots$$

so $\tfrac{1}{2}\mu^2 = \lambda v^2$ $\qquad \mu = v\sqrt{2\lambda}$.

So $\quad \ell = \dfrac{1}{v\sqrt{\lambda}} = \dfrac{\sqrt{2}}{\mu}$.

The size of the kink is the Compton wavelength of the boson of mass μ.

The mass of the kink is

$$M \doteq \frac{v^2}{\ell} + \ell \lambda v^4$$

$$\approx v^2 v\sqrt{\lambda} + \frac{\lambda v^4}{v\sqrt{\lambda}} = 2v^3\sqrt{\lambda}$$

So

$$M \approx 2v^3\sqrt{\lambda} = 2v^3 \frac{\mu}{v\sqrt{2}} = \sqrt{2}\,\mu v^2.$$

or

$$M \approx \sqrt{2}\,\mu \frac{\mu^2}{2\lambda} = \frac{\mu^3}{\lambda\sqrt{2}}.$$

The mass of a kink is the minimum of

$$M = \int dx \left[\frac{1}{2}\phi'^2 + \frac{\lambda}{4}(\phi^2 - v^2)^2 \right].$$

This is of the form

$$M = \int dx \; a^2_{(x)} + b^2_{(x)},$$

Since $\int dx \, (a \pm b)^2 \geq 0$, we have

$$\int dx \; a^2 + b^2 \geq 2 \left| \int dx \, a b \right|.$$

Thus

$$M \geq 2 \left| \int dx \frac{1}{\sqrt{2}} \phi' \frac{\sqrt{\lambda}}{2} (\phi^2 - v^2) \right|$$

$$\geq \frac{2\sqrt{\lambda}}{2\sqrt{2}} \left| \int dx \; \phi^2 \phi' - v^2 \phi' \right|$$

$$\geq \sqrt{\frac{\lambda}{2}} \left| \left[\frac{\phi^3}{3} - v^2 \phi \right]_{-\infty}^{\infty} \right|$$

or

$$M \geq \sqrt{\frac{\lambda}{2}} \, 2 \left| \frac{v^3}{3} - v^3 \right| = \frac{4}{3} v^3 \sqrt{\frac{\lambda}{2}}$$

Since $\mu = v\sqrt{2\lambda}$, $v = \mu/\sqrt{2\lambda}$, so $M \geq \frac{4}{3} \frac{\mu^3}{2\sqrt{2}\,\lambda^{3/2}} \sqrt{\frac{\lambda}{2}}$

or $M \geq \frac{2}{2} \frac{\mu^3}{3\lambda} = \frac{2\mu^3}{6\lambda} = \frac{\mu^3}{3\lambda}$

We can solve for the shape of a kink or lump or soliton by minimizing the mass

$$M = \int dx \ \tfrac{1}{2}(\phi')^2 + U(\phi)$$

$$0 = \delta M = \int dx \ \phi' \delta\phi' + U'(\phi)\,\delta\phi$$

$$= \int dx \left(-\phi'' + U'(\phi)\right)\delta\phi$$

So

$$\frac{d^2\phi}{dx^2} = \phi'' = U'(\phi) = \frac{dU(\phi)}{d\phi}$$

$$\phi''\phi' = \frac{dU}{d\phi}\phi'$$

$$\tfrac{1}{2}(\phi')^2 = U(\phi) + C$$

$$\phi'^2 = 2U + 2C$$

$$\phi' = \sqrt{2U + 2C}$$

$$\frac{d\phi}{\sqrt{2U+2C}} = dx$$

For the kink $u = \frac{\lambda}{4}(\phi^2 - v^2)^2$

Choose c so that $\phi' = 0$ when $\phi = v$.

Then

$$dx = \frac{d\phi}{\sqrt{\frac{\lambda}{2}(\phi^2-v^2)^2}} = \frac{d\phi}{\sqrt{\frac{\lambda}{2}}\,(\phi^2-v^2)}$$

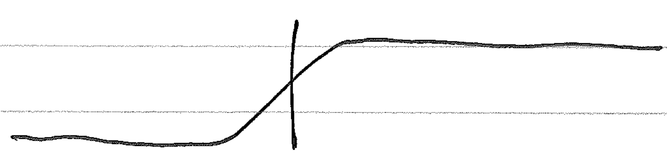
$$x - x_0 = \sqrt{\frac{2}{\lambda}}\int\frac{d\phi}{\phi^2-v^2} = -\sqrt{\frac{2}{\lambda}}\,\frac{\tanh^{-1}\left(\frac{\phi}{v}\right)}{v}$$

$$-v\sqrt{\frac{\lambda}{2}}\,(x-x_0) = \tanh^{-1}\frac{\phi}{v}$$

$$\phi_c(x) = v\tanh\left[-v\sqrt{\frac{\lambda}{2}}\,(x-x_0)\right]$$

Actually, this is an anti-kink, hence the c. So the kink is

$$\phi(x) = v\tanh\left[v\sqrt{\frac{\lambda}{2}}\,(x-x_0)\right].$$



x_0

Similarly, for

$$U(\phi) = \frac{\alpha}{\beta^2}\left(1 - \cos\beta\phi\right)$$

we get

$$\phi(x) = \frac{4}{\beta}\tan^{-1}\exp(\sqrt{\alpha}\,x)$$

or rather

$$\phi(x) = \frac{4}{\beta}\tan^{-1}\left[\exp\left(\sqrt{\alpha}\,(x - x_0)\right)\right].$$

as $x \to \infty$ $\quad \phi \to \frac{4}{\beta}\frac{\pi}{2} = \frac{2\pi}{\beta}$

as $x \to -\infty$ $\quad \phi \to 0$.

The energy or mass of the sine-Gordon lump is

$$E = \frac{8\sqrt{\alpha}}{\beta^2}.$$