Scalar renormalization

$$\mathcal{L} = -\tfrac{1}{2}\partial_\mu\phi_b\,\partial^\mu\phi_b - \tfrac{1}{2}m_b^2\phi_b^2 - V_b(\phi_b)$$

$$\phi_b = \sqrt{Z}\,\phi \qquad m_b^2 = m^2 - \delta m^2$$

$$\mathcal{L} = -\tfrac{1}{2}Z\,\partial_\mu\phi\,\partial^\mu\phi - \tfrac{1}{2}(m^2 - \delta m^2)Z\phi^2 - V_b(\sqrt{Z}\phi)$$

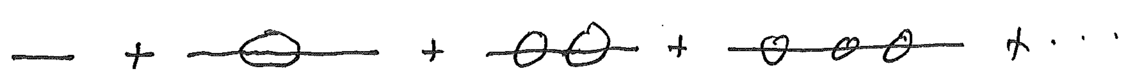
$$= \mathcal{L}_0 + \mathcal{L}_1$$

$$\mathcal{L}_0 = -\tfrac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \tfrac{1}{2}m^2\phi^2$$

$$\mathcal{L}_1 = -\tfrac{1}{2}(Z-1)\left[\partial_\mu\phi\,\partial^\mu\phi + m^2\phi^2\right] + \tfrac{1}{2}Z\,\delta m^2\phi^2 - V(\phi)$$

where $V(\phi) \equiv V_b(\sqrt{Z}\,\phi)$.

If $\Delta'(q)$ is the real propagator, and Π^* is the sum of all one-particle-irreducible graphs, then in the ϕ^4 theory

$\Delta' =$ 

$$= \frac{1}{q^2+m^2-i\epsilon} + \frac{1}{q^2+m^2-i\epsilon}\,\Pi^*\,\frac{1}{q^2+m^2-i\epsilon} + \frac{1}{q^2+m^2-i\epsilon}\left(\Pi^*\frac{1}{q^2+m^2-i\epsilon}\right)^2 + \dots$$

$$= \frac{1}{q^2+m^2-i\epsilon}\;\frac{1}{1-\dfrac{\Pi^*}{q^2+m^2-i\epsilon}} = \frac{1}{q^2+m^2-\Pi^*-i\epsilon}$$

SW uses
$$\phi(x) = \int\frac{d^3p}{\sqrt{(2\pi)^3 2p^0}}\left(a(\vec{p})e^{ipx} + a^\dagger(p)\,e^{-ipx}\right)$$

$$[a(p), a^\dagger(q)] = \delta^{(3)}(\vec{p}-\vec{q}) \qquad |p\rangle = a^\dagger(\vec{p})|0\rangle.$$

so $\langle q' | T e^{\frac{i}{2} Z \delta m^2 \int \phi^2(x) d^4x} | q \rangle$

$$\approx \frac{i}{2} Z \delta m^2 \langle q' | \int \phi^2(x) d^4x | q \rangle$$

$$= \frac{i}{2} Z \delta m^2 \langle 0 | a(q') \int \phi^2(x) \, d^4x \; a^\dagger(q) | 0 \rangle$$

$$= i Z \delta m^2 \int \frac{e^{-ixq'} \, e^{ixq}}{(2\pi)^3 \sqrt{2q'^0 \, 2q^0}} d^4x$$

$$= i Z \delta m^2 \frac{2\pi}{2q^0} \delta^4(q'-q) = \frac{i\pi}{q^0} Z \delta m^2 \delta^4(q'-q) .$$

And $\langle q' | T e^{-\frac{i}{2}(Z-1) \int \partial_\mu \phi \partial^\mu \phi + m^2 \phi^2 d^4x} | q \rangle$

$$= -i(Z-1) \frac{\left[(-iq'_\mu)(iq^\mu) + m^2\right]}{(2\pi)^3 \, 2q^0} \delta^4(q'-q') (2\pi)^4$$

$$= -i \frac{\pi}{q^0} \left[q^2 + m^2\right](Z-1) \, \delta^4(q'-q) .$$

So apart from $-i\frac{\Pi}{q^0}\delta^4(q'-q)$ we have

$$\Pi^*(q^2) = -(Z-1)(q^2+m^2) + Z\delta m^2 + \Pi^*_{loop}(q^2)$$

we want $\Pi^*(-m^2)=0$ and $\frac{d\Pi^*}{dq^2}(-m^2) = 0$.

So

$$Z\,\delta m^2 + \Pi^*_{loop}(-m^2) = 0$$

and

$$-(Z-1) + \frac{d\Pi^*}{dq^2}(-m^2) = 0$$

or

$$Z = 1 + \left.\frac{d\,\Pi^*_{loop}(q^2)}{dq^2}\right|_{q^2=-m^2}.$$

We'd have to do all this even if all the integrals converged.

Note that as $q^2 \to -m^2$

$$\Delta'(q^2) \longrightarrow \Delta(q^2) = \frac{1}{q^2+m^2-i\epsilon}.$$

Also

$$\langle 0|\,\phi(x)\,|\vec{p}\,\rangle = \frac{e^{ipx}}{\sqrt{(2\pi)^3\,2p^0}}.$$