11

One-Loop Radiative Corrections in Quantum Electrodynamics

In this chapter we shall proceed to carry out some of the classic one-loop calculations in the theory of charged leptons — massive spin $\frac{1}{2}$ particles that interact only with the electromagnetic field. There are three known species or 'flavors' of leptons: the electron and muon, and the heavier, more recently discovered tauon. For definiteness we shall refer to the charged particles in our calculations here as 'electrons,' though most of our calculations will apply equally to muons and tauons. After some generalities in Section 11.1, we will move on to the calculation of the vacuum polarization in Section 11.2, the anomalous magnetic moment of the electron in Section 11.3, and the electron self-energy in Section 11.4. Along the way, we will introduce a number of the mathematical techniques that prove useful in such calculations, including the use of Feynman parameters, Wick rotation, and both the dimensional regularization of 't Hooft and Veltman and the older regularization method of Pauli and Villars. Although we shall encounter infinities, it will be seen that the final results are finite if expressed in terms of the renormalized charge and mass. In the next chapter we shall extend what we have learned here about renormalization to general theories in arbitrary orders of perturbation theory.

11.1 Counterterms

The Lagrangian density for electrons and photons is taken in the form*

$$\mathscr{L} = -\tfrac{1}{4}F_{\rm B}^{\mu\nu}F_{{\rm B}\,\mu\nu} - \bar{\psi}_{\rm B}\Big[\gamma_\mu\,[\partial^\mu + ie_{\rm B}A_{\rm B}^\mu] + m_{\rm B}\Big]\psi_{\rm B} \qquad (11.1.1)$$

where $F_{\rm B}^{\mu\nu} \equiv \partial^\mu A_{\rm B}^\nu - \partial^\nu A_{\rm B}^\mu$; $A_{\rm B}^\mu$ and $\psi_{\rm B}$ are the bare (i.e., unrenormalized) fields of the photon and electron, and $-e_{\rm B}$ and $m_{\rm B}$ are the bare charge and

* In this chapter we will not be making transformations between Heisenberg- and interaction-picture operators, so we shall return to a conventional notation, in which an upper case A and a lower case ψ are used to denote the photon and charged particle fields, respectively.

mass of the electron. As described in the previous chapter, we introduce renormalized fields and charge and mass:

$$\psi \equiv Z_2^{-1/2}\psi_B , \tag{11.1.2}$$

$$A^\mu \equiv Z_3^{-1/2}A_B^\mu , \tag{11.1.3}$$

$$e \equiv Z_3^{+1/2}e_B , \tag{11.1.4}$$

$$m \equiv m_B + \delta m , \tag{11.1.5}$$

with the constants Z_2, Z_3, and δm adjusted so that the propagators of the renormalized fields have poles in the same position and with the same residues as the propagators of the free fields in the absence of interactions. The Lagrangian may then be written in terms of renormalized quantities, as

$$\mathscr{L} = \mathscr{L}_0 + \mathscr{L}_1 + \mathscr{L}_2 , \tag{11.1.6}$$

where

$$\mathscr{L}_0 = -\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} - \bar{\psi}\left[\gamma_\mu\partial^\mu + m\right]\psi , \tag{11.1.7}$$

$$\mathscr{L}_1 = -ieA_\mu\bar{\psi}\gamma^\mu\psi , \tag{11.1.8}$$

and $\mathscr{L}_2$ is a sum of 'counterterms'

$$\begin{aligned}\mathscr{L}_2 = &-\tfrac{1}{4}(Z_3-1)F^{\mu\nu}F_{\mu\nu} - (Z_2-1)\bar{\psi}\left[\gamma_\mu\partial^\mu + m\right]\psi \\ &+Z_2\delta m\bar{\psi}\psi - ie(Z_2-1)A_\mu\bar{\psi}\gamma^\mu\psi .\end{aligned} \tag{11.1.9}$$

It will turn out that all of the terms in $\mathscr{L}_2$ are of second order and higher order in e, and that these terms just suffice to cancel the ultraviolet divergences that arise from loop graphs.

11.2 Vacuum Polarization

We now begin our first calculation of a radiative correction involving loop graphs, the so-called vacuum polarization effect, consisting of the corrections to the propagator associated with an internal photon line. Vacuum polarization produces measurable shifts in the energy levels of hydrogen, and makes an important correction to the energies of muons bound in atomic orbits around heavy nuclei. Also, as we shall see in Volume II, the calculation of the vacuum polarization provides a key

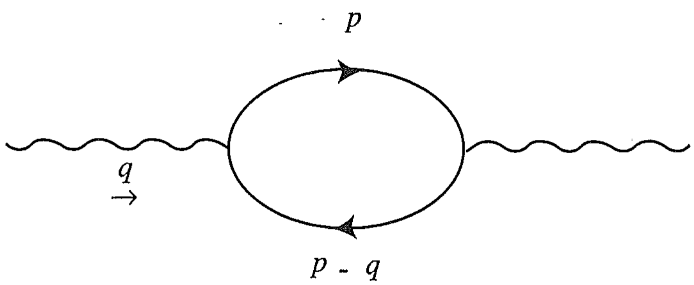

Figure 11.1. The one-loop diagram for the vacuum polarization in quantum electrodynamics. Here wavy lines represent photons; lines carrying arrows represent electrons.

element in the calculation of the high energy behavior of electrodynamics and other gauge theories.

As in Section 10.5, we define $i(2\pi)^4\Pi^{*\mu\nu}(q)$ as the sum of all connected graphs with two external photon lines with polarization indices μ and ν and carrying four-momentum q into and out of the diagram, not including photon propagators for the two external lines, and with the asterisk indicating that we exclude diagrams that can be disconnected by cutting through some internal photon line. The complete photon propagator $\Delta'^{\mu\nu}(q)$ is given by Eq. (10.5.13):

$$\Delta' = \Delta\left[1 - \Pi^*\Delta\right]^{-1}, \tag{11.2.1}$$

where $\Delta^{\mu\nu}(q)$ is the photon propagator without radiative corrections. Our task here is to calculate the leading contributions to $\Pi^{*\rho\sigma}(q)$.

In lowest order there is a one-loop contribution to Π^*, corresponding to the diagram in Figure 11.1:

$$\begin{aligned} i(2\pi)^4\Pi^{*\rho\sigma}_{1\,\text{loop}}(q) = -\int d^4p\,\text{Tr}\Bigg\{&\left[\frac{-i}{(2\pi)^4}\,\frac{-i\,\not{p}+m}{p^2+m^2-i\epsilon}\right] \\ \times\left[(2\pi)^4 e\gamma^\rho\right]&\left[\frac{-i}{(2\pi)^4}\,\frac{-i(\not{p}-\not{q})+m}{(p-q)^2+m^2-i\epsilon}\right]\left[(2\pi)^4 e\gamma^\sigma\right]\Bigg\} \end{aligned} \tag{11.2.2}$$

with the first minus sign on the right required by the presence of a fermion loop. More simply, this is

$$\Pi^{*\rho\sigma}_{1\,\text{loop}}(q) = \frac{-ie^2}{(2\pi)^4}\int d^4p\,\frac{\text{Tr}\left\{[-i\not{p}+m]\gamma^\rho\left[-i(\not{p}-\not{q})+m\right]\gamma^\sigma\right\}}{(p^2+m^2-i\epsilon)\left((p-q)^2+m^2-i\epsilon\right)}\,. \tag{11.2.3}$$

The first step in doing this integral is to use a trick introduced by Feynman.[1] We use the elementary formula

$$\frac{1}{AB} = \int_0^1 \frac{dx}{[(1-x)A+xB]^2} \tag{11.2.4}$$

to write the product of scalar propagators in Eq. (11.2.3) as

$$\frac{1}{(p^2+m^2-i\epsilon)((p-q)^2+m^2-i\epsilon)} = \int_0^1 \Big[(p^2+m^2-i\epsilon)(1-x) + \Big((p-q)^2+m^2-i\epsilon\Big)x\Big]^{-2} dx$$
$$= \int_0^1 \Big[p^2+m^2-i\epsilon-2p\cdot qx+q^2x\Big]^{-2} dx$$
$$= \int_0^1 \Big[(p-qx)^2+m^2-i\epsilon+q^2x(1-x)\Big]^{-2} dx\,.$$

(This is a special case of a class of integrals given in the Appendix to this chapter.) We can now shift the variable of integration in momentum space*

$$p \to p+qx\,,$$

so that Eq. (11.2.3) becomes

$$\Pi^{*\rho\sigma}_{1\,\text{Loop}}(q) = \frac{-ie^2}{(2\pi)^4}\int_0^1 dx \int d^4p\Big[p^2+m^2-i\epsilon+q^2x(1-x)\Big]^{-2}$$
$$\times\,\text{Tr}\{[-i(\not{p}+\not{q}x)+m]\,\gamma^\rho\,[-i\,(\not{p}-\not{q}(1-x))+m]\,\gamma^\sigma\}\,. \quad (11.2.5)$$

Using the results of the Appendix to Chapter 8, the trace here can easily be calculated as

$$\text{Tr}\{[-i(\not{p}+\not{q}x)+m]\,\gamma^\rho\,[-i\,(\not{p}-\not{q}(1-x))+m]\,\gamma^\sigma\}$$
$$=4\Big[-(p+qx)^\rho\,(p-q(1-x))^\sigma+(p+qx)\cdot(p-q(1-x))\eta^{\rho\sigma}$$
$$-(p+qx)^\sigma\,(p-q(1-x))^\rho+m^2\eta^{\rho\sigma}\Big]\,. \quad (11.2.6)$$

Our next step is called a *Wick rotation.*[2] As long as $-q^2 < 4m^2$, the quantity $m^2+q^2x(1-x)$ is positive for all x between 0 and 1, so the poles in the integrand of Eq. (11.2.5) are at $p^0 = \pm\sqrt{\mathbf{p}^2+m^2+q^2x(1-x)-i\epsilon}$, i.e., just above the negative real axis and just below the positive real axis. (See Figure 11.2.) We can rotate the contour of integrations of p^0 counterclockwise without crossing either of these poles, so that instead of integrating p^0 on the real axis from $-\infty$ to $+\infty$, we integrate it on the imaginary axis from $-i\infty$ to $+i\infty$. That is, we can write $p^0 = ip^4$, with p^4 integrated over real values from $-\infty$ to $+\infty$. (If an $i\epsilon$ instead of $-i\epsilon$ had appeared in the denominator of the propagator, then we would have been setting $p^0 = -ip^4$, with p^4 again integrated over real values from $-\infty$ to

* Strictly speaking, this step is only valid in convergent integrals. In principle, in order to justify the shift of variables, we should introduce some regulator scheme to make all integrals converge, such as the dimensional regularization scheme discussed below.

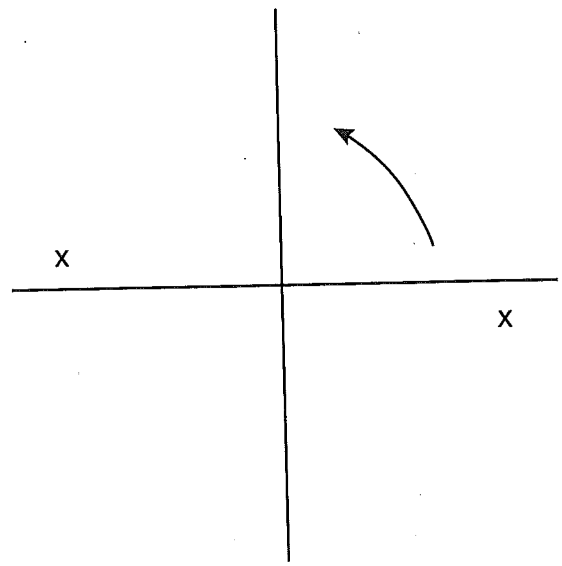

Figure 11.2. Wick rotation of the p^0 contour of integration. Small x's mark the poles in the p^0 complex plane; the arrow indicates the direction of rotation of the contour of integration, from the real to the imaginary p^0-axes.

$+\infty$. The effect would be a change of sign of $\Pi^{*\rho\sigma}_{1\,\mathrm{loop}}(q)$.) Eq. (11.2.5) now becomes

$$\Pi^{*\rho\sigma}_{1\,\mathrm{loop}}(q) = \frac{4e^2}{(2\pi)^4}\int_0^1 dx \int (d^4p)_E\, \left[p^2+m^2+q^2x(1-x)\right]^{-2}$$
$$\times\Big[-(p+qx)^\rho(p-q(1-x))^\sigma + (p+qx)\cdot(p-q(1-x))\,\eta^{\rho\sigma}$$
$$-(p+qx)^\sigma(p-q(1-x))^\rho + m^2\eta^{\rho\sigma}\Big], \tag{11.2.7}$$

where

$$(d^4p)_E = dp^1dp^2dp^3dp^4$$

and all scalar products are evaluated using the Euclidean norm

$$a\cdot b = a^1b^1+a^2b^2+a^3b^3+a^4b^4$$

with the understanding that $q^4 \equiv -iq^0$. Also, $\eta^{\rho\sigma}$ can be taken as either the Kronecker delta, with the indices running over 1, 2, 3, 4, or as the usual Minkowski tensor, with the indices running over 1, 2, 3, 0.

The integral (11.2.7) is badly divergent. Eventually all infinities will cancel, but to see this it is necessary at intermediate stages of the calculation to use some sort of regularization technique that makes the integrals finite. It would not do simply to cut off the integrals at some maximum momentum Λ, integrating only over p^μ with $p^2 < \Lambda^2$, because this would amount to introducing a step function $\theta(\Lambda^2 - p^2)$ into the electron propagator, and the Ward identity (10.4.25) shows that in order to maintain gauge invariance, any modification of the electron propagator must be accompanied with a modification of the electron–photon vertex. In fact, with an ordinary cutoff Λ, radiative corrections would induce a photon mass, a clear violation of the requirements of gauge invariance.

Experience has shown that the most convenient method for regulating divergent integrals without impairing gauge invariance is the dimensional regularization technique introduced by 't Hooft and Veltman[3] in 1972, based on a continuation from four to an arbitrary number d of spacetime dimensions. This amounts to carrying out angular averages in integrals like (11.2.7) by dropping all terms that are odd in p, and replacing the terms that have even numbers of p-factors with**

$$p^\mu p^\nu \to p^2 \eta^{\mu\nu} \big/ d\,, \tag{11.2.8}$$

$$p^\mu p^\nu p^\rho p^\sigma \to (p^2)^2 [\eta^{\mu\nu}\eta^{\rho\sigma} + \eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho}] \big/ d(d+2)\,, \tag{11.2.9}$$

Also, after writing the integrand in this way as a function only of p^2, the volume element d^4p_E is to be replaced with $\Omega_d \kappa^{d-1} d\kappa$, where $\kappa \equiv \sqrt{p^2}$, and Ω_d is the area of a unit sphere in d dimensions

$$\Omega_d = 2\pi^{d/2} \big/ \Gamma(d/2)\,. \tag{11.2.10}$$

The integral (11.2.7) now converges for complex spacetime dimensionality d. We can continue the integral through complex d-values to $d = 4$, the infinities then reappearing as factors $(d-4)^{-1}$.

For the integral (11.2.7), dimensional regularization gives

$$\Pi^{*\rho\sigma}_{1\,\text{loop}}(q) = \frac{4e^2\Omega_d}{(2\pi)^4} \int_0^1 dx \int_0^\infty \kappa^{d-1} d\kappa \left[\kappa^2 + m^2 + q^2 x(1-x)\right]^{-2}$$
$$\times \left[\frac{-2\kappa^2}{d}\eta^{\rho\sigma} + 2q^\rho q^\sigma x(1-x) + \left(\kappa^2 - q^2 x(1-x)\right)\eta^{\rho\sigma} + m^2\eta^{\rho\sigma}\right].$$

The integrals over κ can be carried out for any complex d (or for any real d, aside from the even integers). We use the well-known formulas (given in greater generality in the Appendix to this chapter):

$$\int_0^\infty \kappa^{d-1} [\kappa^2 + \nu^2]^{-2} d\kappa = \tfrac{1}{2} (\nu^2)^{\frac{d}{2}-2}\, \Gamma(d/2)\, \Gamma(2 - d/2)\,, \tag{11.2.11}$$

$$\int_0^\infty \kappa^{d+1} [\kappa^2 + \nu^2]^{-2} d\kappa = \tfrac{1}{2} (\nu^2)^{\frac{d}{2}-1} \Gamma(1 + d/2)\, \Gamma(1 - d/2)\,, \tag{11.2.12}$$

** These expressions may most easily be derived by noting that their form is dictated by Lorentz invariance and the symmetry among the indices μ, ν, ρ, etc., while the factors may be found by requiring that both sides give the same result when contracted with ηs.

and find

$$\Pi^{*\rho\sigma}_{1\text{ loop}}(q) = \frac{2e^2\Omega_d}{(2\pi)^4}$$

$$\times \int_0^1 dx \Big[(1-2/d)\,\eta^{\rho\sigma}\left(m^2+q^2x(1-x)\right)^{\frac{d}{2}-1}\Gamma(1+d/2)\,\Gamma(1-d/2)$$

$$+ \left(2q^\rho q^\sigma x(1-x) - q^2\eta^{\rho\sigma}x(1-x) + m^2\eta^{\rho\sigma}\right)\left(m^2+q^2x(1-x)\right)^{\frac{d}{2}-2}$$

$$\times\,\Gamma(d/2)\,\Gamma(2-d/2)\Big].$$

The two terms in the integrand can be combined, using

$$(1-2/d)\,\Gamma(1+d/2)\,\Gamma(1-d/2) = -\Gamma(d/2)\,\Gamma(2-d/2)\,.$$

We find

$$\Pi^{*\rho\sigma}_{1\text{ loop}}(q) = \frac{4e^2\Omega_d}{(2\pi)^4}\,\Gamma(d/2)\,\Gamma(2-d/2)\,(q^\rho q^\sigma - q^2\eta^{\rho\sigma})$$
$$\times \int_0^1 dx\; x(1-x)(m^2+q^2x(1-x))^{\frac{d}{2}-2}\,. \tag{11.2.13}$$

We note the very important result that this contribution to $\Pi^{*\rho\sigma}$ satisfies the relation

$$q_\rho \Pi^{*\rho\sigma}_{1\text{ loop}}(q) = 0 \tag{11.2.14}$$

that was derived in Section 10.5 on the basis of the conservation and neutrality of the electric current. It was precisely to achieve this result that we adopted the dimensional regularization scheme. The reason that dimensional regularization gives this result is that the conservation of current does not depend on the dimensionality of spacetime.

The gamma function $\Gamma(2-d/2)$ in Eq. (11.2.13) blows up for $d \to 4$. Fortunately, as we saw in Section 11.1, there is another term that must be added to $\Pi^{*\rho\sigma}(q)$, arising from the term $-\frac{1}{4}(Z_3-1)F_{\mu\nu}F^{\mu\nu}$ in the interaction Lagrangian. This term has a structure like Eq. (11.2.13)

$$\Pi^{*\rho\sigma}_{\mathscr{L}_2}(q) = -(Z_3-1)(q^2\eta^{\rho\sigma} - q^\rho q^\sigma)\,, \tag{11.2.15}$$

so to order e^2, the complete Π^* has the form

$$\Pi^{*\rho\sigma}(q) = (q^2\eta^{\rho\sigma} - q^\rho q^\sigma)\pi(q^2)\,, \tag{11.2.16}$$

with

$$\pi(q^2) = -\frac{4e^2\Omega_d}{(2\pi)^4}\,\Gamma\left(\tfrac{d}{2}\right)\Gamma\left(2-\tfrac{d}{2}\right)\int_0^1 dx\; x(1-x)(m^2+q^2x(1-x))^{\frac{d}{2}-2}$$
$$-(Z_3-1)\,. \tag{11.2.17}$$

As we saw in Section 10.5, the definition of the renormalized electromagnetic field requires that $\pi(0) = 0$ (in order that the residue of the pole in the complete photon propagator at $q^2 = 0$ should be the same as for the bare propagator, aside from gauge-dependent terms). Therefore, to order e^2,

$$Z_3 = 1 - \frac{4e^2\Omega_d}{(2\pi)^4}\Gamma\left(\tfrac{d}{2}\right)\Gamma\left(2-\tfrac{d}{2}\right)(m^2)^{\frac{d}{2}-2}\int_0^1 x(1-x)dx\,, \tag{11.2.18}$$

so that, to order e^2,

$$\pi(q^2) = -\frac{4e^2\Omega_d}{(2\pi)^4}\Gamma\left(\tfrac{d}{2}\right)\Gamma\left(2-\tfrac{d}{2}\right)\int_0^1 dx\, x(1-x)$$
$$\times\left[\left(m^2+q^2x(1-x)\right)^{\frac{d}{2}-2} - (m^2)^{\frac{d}{2}-2}\right]. \tag{11.2.19}$$

Now we can remove the regularization, allowing d to approach its physical value $d = 4$. As mentioned before, there is an infinity in the one-loop contribution, arising from the limiting behavior of the Gamma function

$$\Gamma\left(2-\tfrac{d}{2}\right) \to \frac{1}{(2-d/2)} - \gamma\,,$$

where γ is the Euler constant, $\gamma = 0.5772157$. The infinite part of $Z_3 - 1$ is given by using $1/(2-d/2)$ for $\Gamma(2-d/2)$, and replacing d everywhere else by 4:

$$(Z_3 - 1)_\infty = -\frac{4e^2\cdot 2\pi^2}{6(2\pi)^4}\,\frac{1}{2-d/2} = \frac{e^2}{6\pi^2}\,\frac{1}{d-4}\,. \tag{11.2.20}$$

We shall see in Volume II that this result may be used to derive the leading term in the renormalization group equation for the electric charge.

The poles at $d = 4$ obviously cancel in $\pi(q^2)$, because for $d = 4$ both $(m^2+q^2x(1-x))^{\frac{d}{2}-2}$ and $(m^2)^{\frac{d}{2}-2}$ have the same limit, unity. For the same reason, the term $-\gamma$ in $\Gamma(2-d/2)$ cancels in the total $\pi(q^2)$, though it does make a finite contribution to Z_3-1. There are other finite contributions to Z_3-1, that arise from the product of the pole in $\Gamma(2-d/2)$ with the linear terms in the expansion of $\Omega_d\Gamma(d/2)$ around $d = 4$, but these also cancel in the total $\pi(q^2)$. Indeed, in carrying out our dimensional regularization, we might have replaced $(2\pi)^{-4}$ with $(2\pi)^{-d}$, and the factor $\mathrm{Tr}\,1 = 4$ might have been replaced with the dimensionality $2^{d/2}$ of gamma matrices in arbitrary even spacetime dimensionalities d, and these too would have contributed to the finite part of $Z_3 - 1$, but not of $\pi(q^2)$. Moreover, e^2 cannot be supposed to be d-independent, because as shown by inspection of Eq. (11.2.13), it has the d-dependent dimensionality $[\text{mass}]^{4-d}$. If we take $e^2 \propto \mu^{4-d}$, where μ is some quantity with the units of mass, then there are additional finite terms in $Z_3 - 1$, arising from the product of the pole in $\Gamma(2 - d/2)$ with the term $(4 - d)\ln\mu$ in the expansion of μ^{4-d} in powers of $4 - d$, but again, these cancel between $Z_3 - 1$ and the one-loop contributions to $\pi(q^2)$.

The only terms that *do* contribute to $\pi(q^2)$ in the limit $d \to 4$ are those arising from the product of the pole in $\Gamma(2 - d/2)$ with the linear terms in the expansion of $(m^2 + q^2x(1-x))^{\frac{d}{2}-2}$ and $(m^2)^{\frac{d}{2}-2}$ in powers of $d - 4$:

$$(m^2+q^2x(1-x))^{\frac{d}{2}-2}-(m^2)^{\frac{d}{2}-2} \to \left(\tfrac{d}{2}-2\right)\ln\left(1+\frac{q^2x(1-x)}{m^2}\right) . \quad (11.2.21)$$

This gives at last

$$\pi(q^2) = \frac{e^2}{2\pi^2}\int_0^1 x(1-x)\ln\left(1+\frac{q^2x(1-x)}{m^2}\right)dx . \quad (11.2.22)$$

The physical significance of the vacuum polarization can be explored by considering its effect on the scattering of two charged particles of spin $\frac{1}{2}$. The Feynman diagrams of Figure 11.3 make contributions to the scattering S-matrix element of the form

$$S_a(1,2 \to 1',2') = (2\pi)^{-12/2}\delta^4(p_{1'}+p_{2'}-p_1-p_2)\Big[e_1(2\pi)^4\bar{u}_{1'}\gamma^\mu u_1\Big]$$
$$\times\left[-i(2\pi)^{-4}\frac{1}{q^2}\right]\Big[e_2(2\pi)^4\bar{u}_{2'}\gamma_\mu u_2\Big] ,$$

$$S_b(1,2 \to 1',2') = (2\pi)^{-12/2}\delta^4(p_{1'}+p_{2'}-p_1-p_2)\Big[e_1(2\pi)^4\bar{u}_{1'}\gamma^\mu u_1\Big]$$
$$\times\left[-i(2\pi)^{-4}\frac{1}{q^2}\right]^2\Big[i(2\pi)^4(q^2\eta_{\mu\nu}-q_\mu q_\nu)\pi(q^2)\Big]\Big[e_2(2\pi)^4\bar{u}_{2'}\gamma^\nu u_2\Big] ,$$

where e_1 and e_2 are the charges of the two particles being scattered; $\pi(q^2)$ is calculated using for e in Eq. (11.2.22) the magnitude of the charge of the particle circulating in the loop in Figure 11.3; and q^μ is the momentum transfer $q \equiv p_1 - p_{1'} = p_{2'} - p_2$. Using the conservation property $q_\mu\bar{u}_{1'}\gamma^\mu u_1 = 0$ the two diagrams together yield an S-matrix element:

$$S_{a+b}(1,2 \to 1',2') = \frac{-ie_1e_2}{4\pi^2q^2}\left[1+\pi(q^2)\right]\delta^4(p_{1'}+p_{2'}-p_1-p_2)$$
$$\times\Big[\bar{u}_{1'}\gamma^\mu u_1\Big]\Big[\bar{u}_{2'}\gamma_\mu u_2\Big] . \quad (11.2.23)$$

In the non-relativistic limit, $\bar{u}_{1'}\gamma^0 u_1 \simeq -i\delta_{\sigma_1'\sigma_1}$ while $\bar{u}_{1'}\gamma^i u_1 \simeq 0$, and likewise for particle 2. Also, in this limit q^0 is negligible compared with $|\mathbf{q}|$. Eq. (11.2.23) in this limit becomes

$$S_{a+b}(1,2 \to 1',2') = \frac{-ie_1e_2}{4\pi^2\mathbf{q}^2}\left[1+\pi(\mathbf{q}^2)\right]\delta^4(p_{1'}+p_{2'}-p_1-p_2)\delta_{\sigma_1'\sigma_1}\delta_{\sigma_2'\sigma_2} . \quad (11.2.24)$$

This may be compared with the S-matrix in the Born approximation due to a local spin-independent central potential $V(r)$:

$$S_{\rm Born}(1,2\to 1',2') = -2\pi i\delta(E_{1'}+E_{2'}-E_1-E_2)T_{\rm Born}(1,2\to 1',2')\,. \tag{11.2.25}$$

$$T_{\rm Born}(1,2\to 1',2') = \delta_{\sigma_1'\sigma_1}\delta_{\sigma_2'\sigma_2}\int d^3x_1\int d^3x_2\; V\Big(|\mathbf{x}_1-\mathbf{x}_2|\Big)$$
$$\times\,(2\pi)^{-12/2}e^{-i\mathbf{p}_{1'}\cdot\mathbf{x}_1}e^{-i\mathbf{p}_{2'}\cdot\mathbf{x}_2}e^{i\mathbf{p}_1\cdot\mathbf{x}_1}e^{i\mathbf{p}_2\cdot\mathbf{x}_2}\,. \tag{11.2.26}$$

Setting $\mathbf{x}_1 = \mathbf{x}_2+\mathbf{r}$, this gives

$$S_{\rm Born} = \frac{-i}{4\pi^2}\delta^4(p_{1'}+p_{2'}-p_1-p_2)\delta_{\sigma_1'\sigma_1}\delta_{\sigma_2'\sigma_2}$$
$$\times\int d^3r\; V(r)\,e^{-i\mathbf{q}\cdot\mathbf{r}}\,. \tag{11.2.27}$$

Comparing this with Eq. (11.2.23) shows that in the non-relativistic limit the diagrams of Figure 11.3 yield the same S-matrix element as a potential $V(r)$ such that

$$\int d^3r\; V(r)\,e^{-i\mathbf{q}\cdot\mathbf{r}} = e_1e_2\frac{1+\pi(\mathbf{q}^2)}{\mathbf{q}^2}$$

or, inverting the Fourier transform,

$$V(r) = \frac{e_1e_2}{(2\pi)^3}\int d^3q\; e^{i\mathbf{q}\cdot\mathbf{r}}\left[\frac{1+\pi(\mathbf{q}^2)}{\mathbf{q}^2}\right]\,. \tag{11.2.28}$$

Eq. (11.2.28) is to first order in the radiative correction the same potential energy that would be produced by the electrostatic interaction of two extended charge distributions $e_1\eta(\mathbf{x})$ and $e_2\eta(\mathbf{y})$ at a distance r:

$$V(|\mathbf{r}|) = e_1e_2\int d^3x\int d^3y\;\frac{\eta(\mathbf{x})\eta(\mathbf{y})}{4\pi|\mathbf{x}-\mathbf{y}+\mathbf{r}|}\,, \tag{11.2.29}$$

where

$$\eta(\mathbf{r}) = \delta^3(\mathbf{r}) + \frac{1}{2(2\pi)^3}\int d^3q\;\pi(\mathbf{q}^2)\,e^{i\mathbf{q}\cdot\mathbf{r}}\,. \tag{11.2.30}$$

Note that

$$\int d^3r\;\eta(\mathbf{r}) = 1+\tfrac{1}{2}\pi(0) = 1\,, \tag{11.2.31}$$

so the total charges of particles 1 and 2, as determined from the long-range part of the Coulomb potential, are the same constants e_1 and e_2 that govern the interactions of the renormalized electromagnetic field.

For $|\mathbf{r}|\neq 0$ the integral (11.2.30) can be carried out by a straightforward

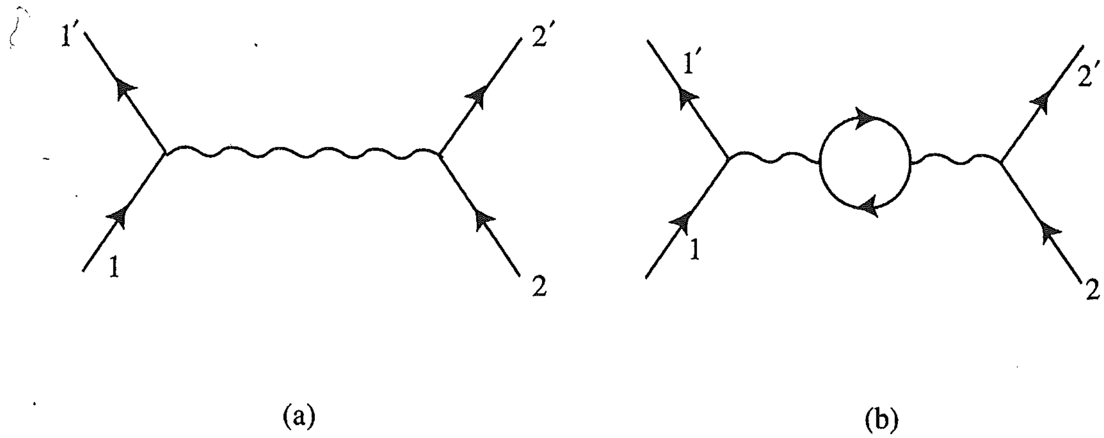

Figure 11.3. Two diagrams for the scattering of charged particles. Here lines carrying arrows are charged particles; wavy lines are photons. Diagram (b) represents the lowest-order vacuum polarization correction to the tree approximation graph (a).

contour integration:

$$\eta(\mathbf{r}) = -\frac{e^2}{8\pi^3 r^3}\int_0^1 x(1-x)\,dx\left[1+\frac{mr}{\sqrt{x(1-x)}}\right]\exp\left(\frac{-mr}{\sqrt{x(1-x)}}\right) .$$

This expression is negative everywhere. However, we have seen that the integral of $\eta(\mathbf{r})$ over all $\mathbf{r}$ equals $+1$. Therefore, $\eta(\mathbf{r})$ must contain a term $(1+L)\delta^3(\mathbf{r})$ that is singular at $\mathbf{r}=0$, with L chosen to satisfy Eq. (11.2.31):

$$L = \frac{e^2}{8\pi^3}\int\frac{d^3r}{r^3}\int_0^1 x(1-x)\,dx\left[1+\frac{mr}{\sqrt{x(1-x)}}\right]\exp\left(\frac{-mr}{\sqrt{x(1-x)}}\right) . \tag{11.2.32}$$

The complete expression for the charge distribution function is then

$$\eta(\mathbf{r}) = (1+L)\delta^3(\mathbf{r}) - \frac{e^2}{8\pi^3 r^3}\int_0^1 x(1-x)\,dx \times\left[1+\frac{mr}{\sqrt{x(1-x)}}\right]\exp\left(\frac{-mr}{\sqrt{x(1-x)}}\right) . \tag{11.2.33}$$

The physical interpretation of this result is that a bare point charge attracts particles of charge of opposite sign out of the vacuum, repelling their antiparticles to infinity, so that the bare charge is partially shielded, yielding a renormalized charge smaller by a factor $1/(1+L)$. As a check, we may note that if we cut off the divergent integral (11.2.32) by taking the integral to extend only over $r \geq a$, we find that the part that is divergent for $a \to 0$ is

$$L_\infty = \frac{e^2}{12\pi^2}\ln a^{-1} . \tag{11.2.34}$$

Hence if we identify the momentum space cutoff Λ with a^{-1}, the divergent part of L is related to the divergent part of $Z_3 - 1$ by

$$(Z_3 - 1)_\infty = -2L_\infty \,, \tag{11.2.35}$$

because to order e^2 the renormalized charge (10.4.18) is given by

$$e_\ell = Z_3^{1/2} e_{B\ell} \simeq \left(1 + \tfrac{1}{2}(Z_3 - 1)\right) e_{B\ell} \simeq (1 + L)^{-1} e_{B\ell} \,. \tag{11.2.36}$$

Eq. (11.2.35) is confirmed below.

Vacuum polarization has a measurable effect on muonic atomic energy levels. As we shall see in Chapter 14, the effect of Feynman graph (b) in Figure 11.3 is to shift the energy of an atomic state with wave function $\psi(\mathbf{r})$ by

$$\Delta E = \int d^3r \, \Delta V(\mathbf{r}) \, |\psi(\mathbf{r})|^2 \,, \tag{11.2.37}$$

where $\Delta V(\mathbf{r})$ is the perturbation in the potential (11.2.28):

$$\Delta V(r) = \frac{e_1 e_2}{(2\pi)^3} \int d^3q \, e^{i\mathbf{q}\cdot\mathbf{r}} \left[\frac{\pi(\mathbf{q}^2)}{\mathbf{q}^2}\right] . \tag{11.2.38}$$

This perturbation falls off exponentially for $r \gg m^{-1}$. On the other hand, the wave function of electrons in ordinary atoms will generally be confined within a much larger radius $a \gg m^{-1}$; for instance, for hydrogenic orbits of electrons around a nucleus of charge Ze we have $a = 137/Zm$ (where here $m = m_e$). The energy shift will then depend only on the behavior of the wave function for $r \ll a$. For orbital angular momentum ℓ, the wave function behaves like r^ℓ for $r \ll a$, so Eq. (11.2.37) gives ΔE proportional to a factor $(ma)^{-(2\ell+1)}$. The effect of vacuum polarization is therefore very much larger for $\ell = 0$ than for higher orbital angular momenta. For $\ell = 0$ the wave function is approximately equal to the constant $\psi(0)$ for r less than or of the order of m^{-1}, so Eq. (11.2.37) becomes

$$\Delta E = |\psi(0)|^2 \int d^3r \, \Delta V(\mathbf{r}) \,. \tag{11.2.39}$$

Using Eqs. (11.2.38) and (11.2.22), the integral of the shift in the potential (for $e_1 e_2 = -Ze^2$) is

$$\int d^3r \, \Delta V(r) = -Ze^2 \pi'(0) = -\frac{4Z\alpha^2}{15m^2} \,. \tag{11.2.40}$$

Also, in states of hydrogenic atoms with $\ell = 0$ and principal quantum number n the wave function at the origin is

$$\psi(0) = \frac{2}{\sqrt{4\pi}} \left(\frac{Z\alpha m}{n}\right)^{3/2} \,, \tag{11.2.41}$$

so the energy shift (11.2.39) is

$$\Delta E = -\frac{4Z^4\alpha^5 m}{15\pi n^3} \,. \tag{11.2.42}$$

For instance, in the 2*s* state of hydrogen this energy shift is -1.122×10^{-7} eV, corresponding to a frequency shift $\Delta E/2\pi\hbar$ of $-$ 27.13 MHz. This is sometimes called the *Uehling effect*.[4] As discussed in Chapter 1, such tiny energy shifts became measurable because in the absence of various radiative corrections the pure Dirac theory would predict exact degeneracy of the 2*s* and 2*p* states of hydrogen. As we shall see in Chapter 14, most of the +1058 MHz 'Lamb shift' between the 2*s* and the 2*p* states comes from other radiative corrections, but the agreement between theory and experiment is good enough to verify the presence of the -37.13 MHz shift due to vacuum polarization.

Although vacuum polarization contributes only a small part of the radiative corrections in ordinary atoms, it dominates the radiative corrections in muonic atoms, in which a muon takes the place of the orbiting electron. This is because most radiative corrections give energy shifts in muonic atoms that on dimensional grounds are proportional to m_μ, while the integrated vacuum polarization energy $\int d^3r\, \Delta V$ due to an *electron* loop is still proportional to m_e^{-2} as in Eq. (11.2.40), giving an energy shift proportional to $m_\mu^3 m_e^{-2} = (210)^2 m_\mu$. However, in this case the muonic atomic radius is not much larger than the electron Compton wavelength, so the approximate result (11.2.39) only gives the order of magnitude of the energy shift due to vacuum polarization.

* * *

For the purposes of comparison with later calculations, note that if we had cut off the integral (11.2.7) at $\kappa = \Lambda$, then in place of Eq. (11.2.20) we would have encountered an integral of the form

$$(Z_3 - 1)_\infty = -\frac{e^2}{6\pi^2}\int_\mu^\Lambda \kappa^{d-5}\, d\kappa = \frac{e^2}{6\pi^2}\,\frac{\mu^{d-4} - \Lambda^{d-4}}{d-4} \,,$$

where μ is an infrared effective cutoff of the order of the mass of the charged particle circulating in the loop of Figure 11.1. (The easiest way to find the constant factor here is to require that the limit of this expression for $d < 3$ and $\Lambda \to \infty$ matches Eq. (11.2.20).) With such an ultraviolet cutoff in place, we can pass to the limit $d \to 4$, and obtain

$$(Z_3 - 1)_\infty = -\frac{e^2}{6\pi^2}\ln(\Lambda/\mu) \,. \tag{11.2.43}$$

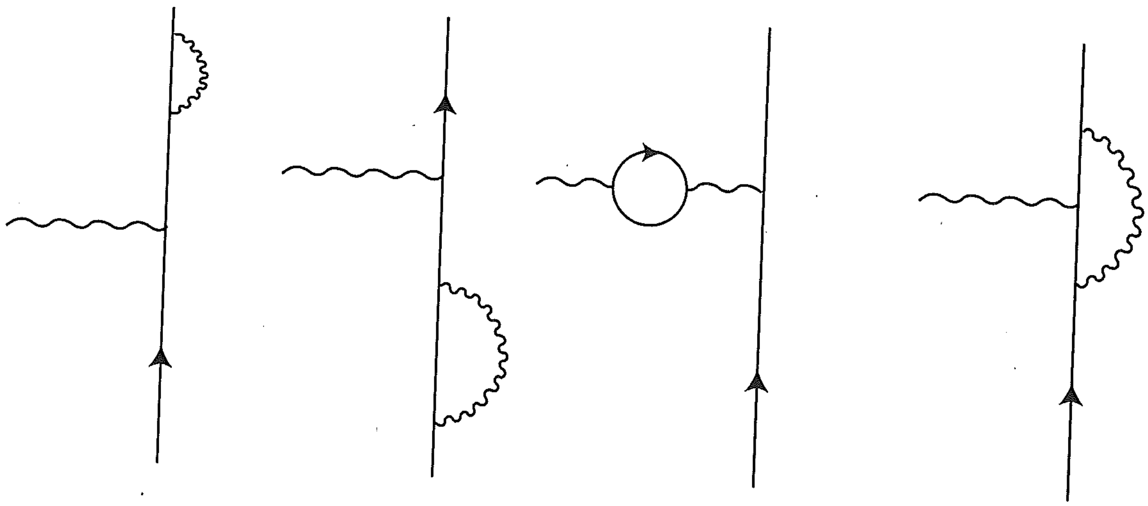

Figure 11.4. One-loop diagrams for the photon–lepton vertex function Γ^μ. Here wavy lines represent photons; other lines represent electrons or muons. Diagrams (a) and (b) are cancelled by lepton field renormalization terms; diagram (c) arises from the vacuum polarization calculated in Section 11.2; and (d) is the term calculated in Section 11.3.

11.3 Anomalous Magnetic Moments and Charge Radii

For our next example, we shall calculate the shift in the magnetic moment and the charge radius of an electron or muon due to lowest-order radiative corrections. The one-loop graphs and renormalization corrections for the photon–lepton vertex are shown in Figure 11.4. Of these graphs, those involving insertions in incoming or outgoing lepton lines vanish because the lepton is on the mass shell, as discussed in Section 10.3. The graph involving an insertion in the external photon line is the vacuum polarization effect, discussed in the previous section. This leaves one one-loop graph (the last in Figure 11.4) that needs to be calculated here:

$$\Gamma^\mu_{1\,\text{loop}}(p',p) = \int d^4k \left[e\gamma^\rho(2\pi)^4\right]\left[\frac{-i}{(2\pi)^4}\,\frac{-i(\not p'-\not k)+m}{(p'-k)^2+m^2-i\epsilon}\right][\gamma^\mu]$$
$$\times\left[\frac{-i}{(2\pi)^4}\,\frac{-i(\not p-\not k)+m}{(p-k)^2+m^2-i\epsilon}\right]\left[e\gamma_\rho(2\pi)^4\right]\left[\frac{-i}{(2\pi)^4}\,\frac{1}{k^2-i\epsilon}\right], \quad (11.3.1)$$

where p' and p are the final and initial lepton four-momenta, respectively. (The contribution of the vertex connecting the external photon line and the internal lepton line is taken as γ^μ, because a factor $e(2\pi)^4$ was extracted in defining Γ^μ.)

This integral has an obvious ultraviolet divergence, roughly like $\int d^4k/(k^2)^2$. Unlike the case of the vacuum polarization, here we do not need a fancy regularization procedure like dimensional regularization to maintain the structure required by gauge invariance, because the photon is a neutral particle and so the integral may be rendered finite by suitable modifications of the photon propagator (for instance by including a factor $M^2/(k^2 + M^2)$ with a large cutoff mass M), without having to introduce modifications elsewhere to maintain gauge invariance. In any case, as we shall see the anomalous magnetic moment and charge radii can be calculated without encountering any ultraviolet divergences at all. In what follows we shall leave the integrals for the vertex function in their infinite form, with it being understood that if necessary any divergent integrals can be expressed in terms of a cutoff mass M.

We start by combining denominators, using a repeated version of the Feynman trick described in the Appendix to this chapter

$$\frac{1}{ABC} = 2\int_0^1 dx \int_0^x dy \left[Ay + B(x-y) + C(1-x)\right]^{-3} . \qquad (11.3.2)$$

Applied to the denominators in Eq. (11.3.1), this gives

$$\frac{1}{(p'-k)^2 + m^2 - i\epsilon} \frac{1}{(p-k)^2 + m^2 - i\epsilon} \frac{1}{k^2 - i\epsilon}$$
$$= 2\int_0^1 dx \int_0^x dy \Big[\Big((p'-k)^2 + m^2 - i\epsilon\Big)y + \Big((p-k)^2 + m^2 - i\epsilon\Big)(x-y)$$
$$+\Big(k^2 - i\epsilon\Big)(1-x)\Big]^{-3}$$
$$= 2\int_0^1 dx \int_0^x dy \left[\Big(k - p'y - p(x-y)\Big)^2 + m^2x^2 + q^2y(x-y) - i\epsilon\right]^{-3} , \qquad (11.3.3)$$

where $q \equiv p - p'$ is the momentum transferred to the photon. Shifting the variable of integration

$$k \to k + p'y + p(x-y)$$

the integral (11.3.1) becomes

$$\Gamma^\mu_{1\,\mathrm{loop}}(p',p) = \frac{2ie^2}{(2\pi)^4} \int_0^1 dx \int_0^x dy \int \frac{d^4k}{\left[k^2 + m^2x^2 + q^2y(x-y) - i\epsilon\right]^3}$$
$$\times \gamma^\rho \Big[-i\Big(\not{p}'(1-y) - \not{k} - \not{p}(x-y)\Big) + m\Big]\gamma^\mu$$
$$\times \Big[-i\Big(\not{p}(1-x+y) - \not{k} - \not{p}'y\Big) + m\Big]\gamma_\rho . \qquad (11.3.4)$$

Our next step is a Wick rotation. As explained in the previous section, the $-i\epsilon$ in the denominator dictates that when we rotate the k^0 contour of integration to the imaginary axis we must rotate counterclockwise, so that the integral over k^0 from $-\infty$ to $+\infty$ is replaced with an integral over imaginary values from $-i\infty$ to $+i\infty$, or equivalently over real values of $k^4 \equiv -ik^0$ from $-\infty$ to $+\infty$. We also exploit the rotational symmetry of the denominator in Eq. (11.3.4); we drop terms in the numerator of odd order in k, replace $k^\lambda k^\sigma$ with $\eta^{\lambda\sigma}k^2/4$, and replace the volume element $d^4k = idk^1dk^2dk^3dk^4$ with $2i\pi^2\kappa^3 d\kappa$, where κ is the Euclidean length of the four-vector k. Putting this all together, Eq. (11.3.4) now becomes

$$\Gamma^\mu_{1\,\text{loop}}(p',p) = \frac{-4\pi^2 e^2}{(2\pi)^4}\int_0^1 dx \int_0^x dy \int_0^\infty \kappa^3 d\kappa \Big\{ -\kappa^2\gamma^\rho\gamma^\sigma\gamma^\mu\gamma_\sigma\gamma_\rho/4$$
$$+\gamma^\rho\Big[-i\Big(\not{p}'(1-y) - \not{p}(x-y)\Big) + m\Big]\gamma^\mu$$
$$\times\Big[-i\Big(\not{p}(1-x+y) - \not{p}'y\Big) + m\Big]\gamma_\rho\Big\}$$
$$\times\Big[\kappa^2 + m^2x^2 + q^2y(x-y)\Big]^{-3}. \tag{11.3.5}$$

We are interested here only in the matrix element $\bar{u}'\Gamma^\mu u$ of the vertex function between Dirac spinors that satisfy the relations

$$\bar{u}'[i\not{p}' + m] = 0\,, \qquad [i\not{p} + m]u = 0\,.$$

We can therefore simplify this expression by using the anticommutation relations of the Dirac matrices to move all factors $\not{p}'$ to the left and all factors $\not{p}$ to the right, replacing them when they arrive on the right or left with im. After a straightforward but tedious calculation, Eq. (11.3.5) then becomes

$$\bar{u}'\Gamma^\mu_{\text{one loop}}(p',p)u = \frac{-4\pi^2e^2}{(2\pi)^4}\int_0^1 dx\int_0^x dy\int_0^\infty \kappa^3 d\kappa$$
$$\bar{u}'\Big\{\gamma^\mu\Big[-\kappa^2 + 2m^2(x^2-4x+2) + 2q^2(y(x-y)+1-x)\Big]$$
$$+4im\,p'^\mu(y-x+xy) + 4im\,p^\mu(x^2-xy-y)\Big\}u$$
$$\times\Big[\kappa^2 + m^2x^2 + q^2y(x-y)\Big]^{-3}. \tag{11.3.6}$$

We next exploit the symmetry of the final factor under the reflection $y \to x-y$. Under this reflection, the functions $y-x+xy$ and x^2-xy-y that multiply p'^μ and p^μ are interchanged, so both may be replaced with their average:

$$\tfrac{1}{2}(y-x+xy) + \tfrac{1}{2}(x^2-xy-y) = -\tfrac{1}{2}x(1-x)\,.$$

This gives finally

$$\bar{u}'\Gamma^{\mu}_{\text{one loop}}(p',p)u = \frac{-4\pi^2 e^2}{(2\pi)^4}\int_0^1 dx \int_0^x dy \int_0^\infty \kappa^3 d\kappa$$
$$\times\, \bar{u}'\Big\{\gamma^\mu\Big[-\kappa^2 + 2m^2(x^2-4x+2) + 2q^2(y(x-y)+1-x)\Big]$$
$$-2im\,(p'^\mu + p^\mu)x(1-x)\Big\}u$$
$$\times \Big[\kappa^2 + m^2x^2 + q^2y(x-y)\Big]^{-3} . \tag{11.3.7}$$

Note that p^μ and p'^μ now enter only in the combination $p^\mu + p'^\mu$, as required by current conservation.

There are other diagrams that need to be taken into account. Of course, there is the zeroth-order term γ^μ in Γ^μ. The term proportional to $Z_2 - 1$ in the correction term (11.1.9) yields a term in Γ^μ

$$\Gamma^\mu_{\mathscr{L}_2} = (Z_2 - 1)\gamma^\mu . \tag{11.3.8}$$

Also, the effect of insertions of corrections to the external photon propagator is a term:

$$\Gamma^\mu_{\text{vac pol}}(p',p) = \frac{1}{(p'-p)^2 - i\epsilon}\Pi^{\mu\nu}(p'-p)\,\gamma_\nu . \tag{11.3.9}$$

The form of each of these terms is in agreement with the general result (10.6.10) (with $H(q^2) = 0$)

$$\bar{u}'\Gamma^\mu(p',p)u = \bar{u}'\left[\gamma^\mu F(q^2) - \frac{i}{2m}(p+p')^\mu G(q^2)\right]u . \tag{11.3.10}$$

To order e^2, the form factors are

$$F(q^2) = Z_2 + \pi(q^2) + \frac{4\pi^2 e^2}{(2\pi)^4}\int_0^1 dx \int_0^x dy \int_0^\infty \kappa^3 d\kappa$$
$$\times \frac{\Big[\kappa^2 - 2m^2(x^2-4x+2) - 2q^2(y(x-y)+1-x)\Big]}{\Big[\kappa^2 + m^2x^2 + q^2y(x-y)\Big]^3} , \tag{11.3.11}$$

$$G(q^2) = \frac{-4\pi^2 e^2}{(2\pi)^4}\int_0^1 dx \int_0^x dy \int_0^\infty \frac{4m^2x(1-x)\,\kappa^3\,d\kappa}{\Big[\kappa^2 + m^2x^2 + q^2y(x-y)\Big]^3} , \tag{11.3.12}$$

where $\pi(q^2)$ is the vacuum polarization function (11.2.22).

The integral for the form factor $G(q^2)$ is finite as it stands:

$$G(q^2) = \frac{-e^2m^2}{4\pi^2}\int_0^1 dx \int_0^x dy\, \frac{x(1-x)}{m^2x^2 + q^2y(x-y)} . \tag{11.3.13}$$

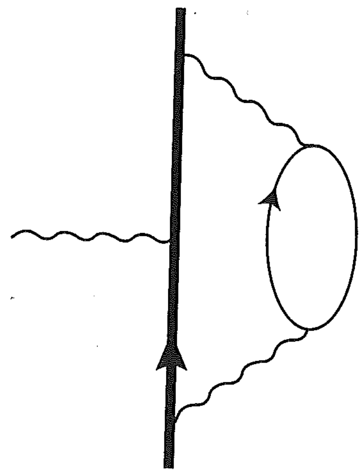

Figure 11.5. A two-loop diagram for the muon magnetic moment. Here the heavy straight line represents a muon; the light wavy lines are photons; and the other light lines are electrons. This diagram makes a relatively large contribution to the fourth-order muon gyromagnetic ratio, proportional to $\ln(m_\mu/m_e)$.

This makes it easy to calculate the anomalous magnetic moment. As noted in Section 10.6, it is only the γ^μ term that contributes to the magnetic moment, so the effect of radiative corrections is to multiply the Dirac value $e/2m$ of the magnetic moment by a factor $F(0)$. But the definition of e as the true lepton charge requires that

$$F(0) + G(0) = 1\,, \tag{11.3.14}$$

so the magnetic moment may be expressed as

$$\mu = \frac{e}{2m}\Big(1 - G(0)\Big)\,. \tag{11.3.15}$$

From Eq. (11.3.13), we find

$$-G(0) = \frac{e^2}{8\pi^2} = 0.001161\,. \tag{11.3.16}$$

This is the famous $\alpha/2\pi$ correction first calculated by Schwinger.[5]

Of course, this is only the first term in the radiative corrections to the magnetic moment. Even in just the next order, fourth order in e, there are so many terms that the calculations become quite complicated. However, because of the large muon–electron mass ratio, there is one fourth-order term in the magnetic moment of the *muon* that is somewhat larger than any of the others. It arises from the insertion of an *electron* loop in the virtual photon line of the second-order diagram, as shown in Figure 11.5. The effect of this electron loop is to change the photon propagator $1/k^2$ in Eq. (11.3.1) to $(1 + \pi_e(k^2))/k^2$, where $\pi_e(k^2)$ is given by Eq. (11.2.22), but with the mass m taken as the *electron* mass:

$$\pi_e(k^2) = \frac{e^2}{2\pi^2} \int_0^1 x(1-x) \ln\left(1 + \frac{k^2 x(1-x)}{m_e^2}\right) dx\,.$$

Inspection of Eq. (11.3.12) shows that in calculating the muon magnetic moment the effective cutoff on the virtual photon momentum k is m_μ. The ratio m_μ/m_e is so large that for k^2 of order m_μ^2 we may approximate

$$\pi_e(k^2) \simeq \frac{e^2}{2\pi^2} \int_0^1 dx\, x(1-x) \ln(m_\mu^2/m_e^2) = \frac{e^2}{12\pi^2} \ln(m_\mu^2/m_e^2) \qquad (11.3.17)$$

with the neglected terms having coefficients of order unity in place of $\ln(m_\mu^2/m_e^2)$. Since this is a constant, the change in $-G(0)$ produced by adding an electron loop in the virtual photon line is simply given by multiplying our previous result (11.3.16) for $-G(0)$ by Eq. (11.3.17), so that now

$$\mu_\mu = \frac{e}{2m_\mu}\left(1 + \frac{e^2}{8\pi^2} + \frac{e^4}{96\pi^4}\left[\ln\frac{m_\mu^2}{m_e^2} + O(1)\right]\right)\,. \qquad (11.3.18)$$

(As we shall see in Volume II, this argument is a primitive version of the method of the renormalization group.) The result (11.3.18) may be compared with the full fourth-order result:[6]

$$\mu_\mu = \frac{e}{2m_\mu}\left(1 + \frac{e^2}{8\pi^2} + \frac{e^4}{96\pi^4}\left[\ln\frac{m_\mu^2}{m_e^2} - \frac{25}{6} + \frac{197}{24} + \frac{\pi^2}{2} + \frac{9\zeta(3)}{2} - 3\pi^2 \ln 2 + O\left(\frac{m_e}{m_\mu}\right)\right]\right)\,. \qquad (11.3.19)$$

It turns out that the '$O(1)$' terms multiplying $e^4/96\pi^4$ add up to -6.137, which is not very much smaller than $\ln(m_\mu^2/m_e^2) = 10.663$, so the approximation (11.3.18) gives the fourth-order terms only to a factor of order 2. The correct fourth-order result (11.3.19) gives $\mu_\mu = 1.00116546\, e/2m_\mu$, in comparison with the second-order result $\mu_\mu = 1.001161\, e/2m_\mu$ and the current experimental value,[7] $\mu_\mu = 1.001165923(8)e/2m_\mu$.

Now let us turn to the other form factor. The integral in Eq. (11.3.11) for $F(q^2)$ has an ultraviolet divergence. However, in order to satisfy the charge-non-renormalization condition (11.3.14), it is necessary that Z_2 take the value

$$Z_2 = 1 + \frac{e^2}{8\pi^2} - \frac{4\pi^2 e^2}{(2\pi)^4} \int_0^1 dx \int_0^x dy \int_0^\infty \kappa^3 d\kappa \times \frac{\kappa^2 - 2m^2(x^2 - 4x + 2)}{\left[\kappa^2 + m^2 x^2\right]^3}\,. \qquad (11.3.20)$$

(Recall that $\pi(0) = 0$.) This is itself ultraviolet divergent, with an infinite part

$$(Z_2 - 1)_\infty = -\frac{e^2}{8\pi^2}\int^\infty \frac{d\kappa}{\kappa}\,. \tag{11.3.21}$$

Inserting Eq. (11.3.20) back into Eq. (11.3.11) gives

$$F(q^2) = 1 + \frac{e^2}{8\pi^2} + \pi(q^2) + \frac{4\pi^2 e^2}{(2\pi)^4}\int_0^1 dx \int_0^x dy \int_0^\infty \kappa^3 d\kappa \times \left\{ \frac{\left[\kappa^2 - 2m^2(x^2 - 4x + 2) - 2q^2(y(x-y) + 1 - x)\right]}{\left[\kappa^2 + m^2x^2 + q^2y(x-y)\right]^3} - \frac{\left[\kappa^2 - 2m^2(x^2 - 4x + 2)\right]}{\left[\kappa^2 + m^2x^2\right]^3} \right\}\,. \tag{11.3.22}$$

The integral over κ is now convergent:

$$F(q^2) = 1 + \frac{e^2}{8\pi^2} + \pi(q^2) + \frac{2\pi^2 e^2}{(2\pi)^4}\int_0^1 dx \int_0^x dy \times \left\{ \frac{-m^2[x^2 - 4x + 2] - q^2[y(x-y) + 1 - x]}{m^2x^2 + q^2y(x-y)} + \frac{x^2 - 4x + 2}{x^2} - \ln\left[\frac{m^2x^2 + q^2y(x-y)}{m^2x^2}\right] \right\}\,. \tag{11.3.23}$$

However, we see that the integral over x and y now diverges logarithmically at $x = 0$ and $y = 0$, because there are two powers of x and/or y in the denominators, and just two differentials $dx\,dy$ in the numerator. This divergence can be traced to the vanishing of the denominator $[\kappa^2 + m^2x^2 + q^2y(x-y)]^3$ in Eq. (11.3.11) at $x = 0$, $y = 0$, and $\kappa = 0$. Because this infinity comes from the region of small rather than large κ, it is termed an *infrared divergence* rather than an ultraviolet divergence.

We shall give a comprehensive treatment of the infrared divergences in Chapter 13. It will be shown there that infrared divergences in the cross-section for processes like electron–electron scattering, such as those that are introduced by the infrared divergence in the electron form factor $F(q^2)$, are cancelled when we include the emission of low-energy photons as well as elastic scattering. Also, as we shall see in Chapter 14, when we calculate radiative corrections to atomic energy levels the infrared divergence in $F(q^2)$ is cut off because the bound electron is not exactly on the free-particle mass shell. For the present we shall continue our calculation by simply introducing a fictitious photon mass μ to cut off the infrared divergence in $F(q^2)$, leaving it for Chapter 14 to see how to use this result.

With a photon mass μ, the denominator $k^2 - i\epsilon$ in Eq. (11.3.1) would be replaced with $k^2 + \mu^2 - i\epsilon$. The effect would then be to add a term $\mu^2(1-x)$ to the cubed quantity in the denominators of Eqs. (11.3.3)–(11.3.7), (11.3.11), (11.3.20), and (11.3.22). Eq. (11.3.23) then is replaced with

$$F(q^2) = 1 + \frac{e^2}{8\pi^2} + \pi(q^2) + \frac{2\pi^2 e^2}{(2\pi)^4}\int_0^1 dx \int_0^x dy \times \left\{ \frac{-m^2[x^2-4x+2] - q^2[y(x-y)+1-x]}{m^2x^2 + q^2 y(x-y) + \mu^2(1-x)} + \frac{m^2[x^2-4x+2]}{m^2x^2+\mu^2(1-x)} - \ln\left[\frac{m^2x^2+q^2y(x-y)+\mu^2(1-x)}{m^2x^2+\mu^2(1-x)}\right]\right\} . \tag{11.3.24}$$

This integral is now completely convergent. It can be expressed in terms of Spence functions, but the result is not particularly illuminating. For our purposes in Chapter 14, it will be sufficient to calculate the behavior of $F(q^2)$ for small q^2. We already know from the Ward identity that $F(0) = 1 - G(0) = 1 + e^2/8\pi^2$, so let us consider the first derivative $F'(q^2)$ at $q^2 = 0$. According to Eq. (11.3.24), this is

$$F'(0) = \pi'(0) + \frac{2\pi^2 e^2}{(2\pi)^4}\int_0^1 dx \int_0^x dy \times \left\{ -\frac{2y(x-y)+1-x}{m^2x^2+\mu^2(1-x)} + \frac{m^2[x^2-4x+2]y(x-y)}{[m^2x^2+\mu^2(1-x)]^2}\right\} . \tag{11.3.25}$$

The vacuum polarization contribution is given by Eq. (11.2.22) as

$$\pi'(0) = \frac{e^2}{60\pi^2 m^2} . \tag{11.3.26}$$

Dropping all terms proportional to powers of μ/m in Eq. (11.3.25), we then have*

$$F'(0) = \frac{e^2}{24\pi^2 m^2}\left[\ln\left(\frac{\mu^2}{m^2}\right) + \frac{2}{5} + \frac{1}{4}\right] \tag{11.3.27}$$

with the term $\frac{2}{5}$ the contribution of vacuum polarization. On the other hand, Eq. (11.3.13) shows that $G(q^2)$ has a finite derivative at $q^2 = 0$,

$$G'(0) = \frac{e^2}{48\pi^2 m^2} . \tag{11.3.28}$$

---

* The y-integral is trivial. The x-integral is most easily calculated in the limit $\mu \ll m$ by dividing the range of integration into two parts, one from 0 to s, where $\mu/m \ll s \ll 1$, and the second from s to 1.

These results are most conveniently expressed in terms of the charge form factor $F_1(q^2)$, defined by the alternative representation (10.6.15) of the vertex function

$$\bar{u}(\mathbf{p}',\sigma')\Gamma^\mu(p',p)u(\mathbf{p},\sigma)$$
$$= \bar{u}(\mathbf{p}',\sigma')\left[\gamma^\mu F_1(q^2) + \tfrac{1}{2}i[\gamma^\mu,\gamma^\nu](p'-p)_\nu F_2(q^2)\right]u(\mathbf{p},\sigma)\,. \quad (11.3.29)$$

According to Eqs. (10.6.17) and (10.6.18),

$$F_1(q^2) = F(q^2) + G(q^2)\,. \quad (11.3.30)$$

For $|q^2| \ll m^2$, this form factor is approximately

$$F_1(q^2) \simeq 1 + \frac{e^2}{24\pi^2}\left(\frac{q^2}{m^2}\right)\left[\ln\left(\frac{\mu^2}{m^2}\right) + \frac{2}{5} + \frac{3}{4}\right]\,. \quad (11.3.31)$$

This may be expressed in terms of a *charge radius a*, defined by the limiting behavior of the charge form factor for $q^2 \to 0$:

$$F_1(q^2) \to 1 - q^2a^2/6\,. \quad (11.3.32)$$

(This definition is motivated by the fact that the average of $\exp(i\mathbf{q}\cdot\mathbf{x})$ over a spherical shell of radius a goes as $1 - \mathbf{q}^2a^2/6$ for $\mathbf{q}^2a^2 \ll 1$.) We see that the charge radius of the electron is given by

$$a^2 = -\frac{e^2}{4\pi^2m^2}\left[\ln\left(\frac{\mu^2}{m^2}\right) + \frac{2}{5} + \frac{3}{4}\right]\,. \quad (11.3.33)$$

We will see in Chapter 14 that for electrons in atoms the role of the photon mass is played by an effective infrared cutoff that is much less than m, so the logarithm here is large and negative, yielding a positive value for a^2.

11.4 Electron Self-Energy

We conclude this chapter with a calculation of the electron self-energy function. This by itself does not have any direct experimental implications, but some of the results here will be useful in Chapter 14 and Volume II.

As in Section 10.3, we define $i(2\pi)^4[\Sigma^*(p)]_{\beta,\alpha}$ as the sum of all graphs with one incoming and one outgoing electron line carrying momenta p and Dirac indices α and β respectively, with the asterisk indicating that we exclude diagrams that can be disconnected by cutting through some internal electron line, and with propagators omitted for the two external lines. The complete electron propagator is then given by the sum

$$[-i(2\pi)^{-4}S'(p)] = [-i(2\pi)^{-4}S(p)]$$
$$+ [-i(2\pi)^{-4}S(p)][i(2\pi)^4\Sigma^*(p)][-i(2\pi)^{-4}S(p)] + \ldots\,, \quad (11.4.1)$$

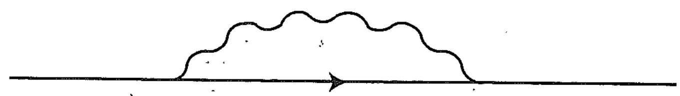

Figure 11.6. The one-loop diagram for the electron self-energy function. As usual, the straight line represents an electron; the wavy line is a photon.

where

$$S(p) \equiv \frac{-i \not{p} + m_e}{p^2 + m_e^2 - i\epsilon} . \tag{11.4.2}$$

The sum is trivial, and gives

$$S'(p) = [i \not{p} + m_e - \Sigma^*(p) - i\epsilon]^{-1} . \tag{11.4.3}$$

In lowest order there is a one-loop contribution to Σ^*, given by Figure 11.6:

$$i(2\pi)^4 \Sigma^*_{1\text{ loop}}(p) = \int d^4k \, [\frac{-i}{(2\pi)^4} \frac{\eta_{\rho\sigma}}{k^2 - i\epsilon}]$$
$$\times [(2\pi)^4 e\gamma^\rho] \left[\frac{-i}{(2\pi)^4} \frac{-i \not{p} + i \not{k} + m_e}{(p-k)^2 + m_e^2 - i\epsilon}\right] [(2\pi)^4 e\gamma^\sigma]$$

or more simply

$$\Sigma^*_{1\text{ loop}}(p) = \frac{ie^2}{(2\pi)^4} \int d^4k \, [\frac{1}{k^2 - i\epsilon}]$$
$$\times \left[\frac{\gamma^\rho(-i \not{p} + i \not{k} + m_e)\gamma_\rho}{(p-k)^2 + m_e^2 - i\epsilon}\right] . \tag{11.4.4}$$

(This is in Feynman gauge; amplitudes with charged particles off the mass shell are not gauge-invariant.) For use in our calculation of the Lamb shift, it will be convenient to use a method of regularization introduced by Pauli and Villars.[8] We replace the photon propagator $(k^2 - i\epsilon)^{-1}$ with

$$\frac{1}{k^2 - i\epsilon} - \frac{1}{k^2 + \mu^2 - i\epsilon} ,$$

so that the electron self-energy function is

$$\Sigma^*_{1\text{ loop}}(p) = \frac{ie^2}{(2\pi)^4} \int d^4k \left[\frac{1}{k^2 - i\epsilon} - \frac{1}{k^2 + \mu^2 - i\epsilon}\right]$$
$$\times \left[\frac{\gamma^\rho(-i \not{p} + i \not{k} + m_e)\gamma_\rho}{(p-k)^2 + m_e^2 - i\epsilon}\right] . \tag{11.4.5}$$

Later we can drop the regulator by letting the regulator mass μ go to infinity. In Chapter 14 we will also be interested in the case where $\mu \ll m_e$.

We again use the Feynman trick to combine denominators, and recall that $\gamma^\rho\gamma^\kappa\gamma_\rho = -2\gamma^\kappa$ and $\gamma^\rho\gamma_\rho = 4$. This gives

$$\Sigma^*_{1\,\text{loop}}(p) = \frac{ie^2}{(2\pi)^4}\int d^4k\;[2i(\not{p}-\not{k})+4m_e] \times \int_0^1 dx\left[\frac{1}{((k-px)^2+p^2x(1-x)+m_e^2x-i\epsilon)^2} - \frac{1}{((k-px)^2+p^2x(1-x)+m_e^2x+\mu^2(1-x)-i\epsilon)^2}\right] \tag{11.4.6}$$

Shifting the variable of integration $k \to k + px$ and rotating the contour of integration gives

$$\Sigma^*_{1\,\text{loop}}(p) = \frac{-2\pi^2e^2}{(2\pi)^4}\int_0^1 dx\;[2i(1-x)\not{p}+4m_e]\int_0^\infty d\kappa\,\kappa^3 \times\left[\frac{1}{(\kappa^2+p^2x(1-x)+m_e^2x)^2} - \frac{1}{(\kappa^2+p^2x(1-x)+m_e^2x+\mu^2(1-x))^2}\right]. \tag{11.4.7}$$

The κ-integral is trivial:

$$\Sigma^*_{1\,\text{loop}}(p) = \frac{-\pi^2e^2}{(2\pi)^4}\int_0^1 dx\;[2i(1-x)\not{p}+4m_e] \times \ln\left(\frac{p^2x(1-x)+m_e^2x+\mu^2(1-x)}{p^2x(1-x)+m_e^2x}\right). \tag{11.4.8}$$

The interaction (11.1.9) also contributes a renormalization counterterm $-(Z_2-1)(i\not{p}+m_e)+Z_2\delta m_e$ in $\Sigma^*(p)$, with Z_2 and δm_e determined by the condition that the complete propagator $S'(p)$ regarded as a function of $i\not{p}$ should have a pole at $i\not{p} = -m_e$ with residue unity. (As we shall see in the next chapter, this makes Σ^* finite as $\mu \to \infty$ to all orders in e.) In lowest order, this gives

$$\delta m_e = -\,\Sigma^*_{1\,\text{loop}}\Big|_{i\not{p}=-m_e} = \frac{2m_e\pi^2e^2}{(2\pi)^4}\int_0^1 dx\;[1+x]\;\ln\left(\frac{m_e^2x^2+\mu^2(1-x)}{m_e^2x^2}\right), \tag{11.4.9}$$

$$Z_2 - 1 = -i\,\frac{\partial\Sigma^*_{1\,\text{loop}}}{\partial\not{p}}\Bigg|_{i\not{p}=-m_e} = -\frac{2\pi^2e^2}{(2\pi)^4}\int_0^1 dx\;\left\{(1-x)\ln\left(\frac{m_e^2x^2+\mu^2(1-x)}{m_e^2x^2}\right) - \frac{2\mu^2(1-x)^2(1+x)}{x(m_e^2x^2+\mu^2(1-x))}\right\}. \tag{11.4.10}$$

(To this order, we do not distinguish between δm_e and $Z_2\delta m_e$.) Dropping terms that vanish for $\mu^2 \to \infty$, Eqs. (11.4.8)–(11.4.10) yield

$$\Sigma^*_{1\text{ loop}}(p) = \frac{-\pi^2 e^2}{(2\pi)^4}\int_0^1 dx\,[2i(1-x)\,\not p + 4m_e]\,\ln\left(\frac{\mu^2(1-x)}{p^2x(1-x)+m_e^2x}\right), \tag{11.4.11}$$

$$\delta m_e = \frac{2m_e\pi^2 e^2}{(2\pi)^4}\int_0^1 dx\,[1+x]\,\ln\left(\frac{\mu^2(1-x)}{m_e^2x^2}\right), \tag{11.4.12}$$

$$Z_2 - 1 = \frac{-2\pi^2 e^2}{(2\pi)^4}\int_0^1 dx\left\{(1-x)\ln\left(\frac{\mu^2(1-x)}{m_e^2x^2}\right) - \frac{2(1-x^2)}{x}\right\}. \tag{11.4.13}$$

Inspection then shows that in the complete self-energy function the $\ln\mu^2$ terms cancel, leaving us with

$$\begin{aligned}\Sigma^*_{\text{order } e^2}(p) &= \Sigma^*_{1\text{ loop}}(p) - (Z_2-1)(i\,\not p + m_e) + Z_2\delta m_e \\ &= \frac{-2\pi^2 e^2}{(2\pi)^4}\int_0^1 dx\left\{[i(1-x)\,\not p + 2m_e]\ln\left(\frac{m_e^2(1-x)}{p^2x(1-x)+m_e^2x}\right)\right. \\ &\quad -m_e[1+x]\ln\left(\frac{1-x}{x^2}\right) \\ &\quad \left.-(i\,\not p + m_e)\left[(1-x)\ln\left(\frac{1-x}{x^2}\right) - \frac{2(1-x^2)}{x}\right]\right\}. \end{aligned} \tag{11.4.14}$$

There is still a divergence from the behavior of the last term as $x \to 0$, which can be traced to the singular behavior of the integral over the photon momentum k in Eq. (11.4.5) at $k^2 = 0$, when we take p^2 at the point $p^2 = -m_e^2$ where we evaluated $Z_2 - 1$. Such infrared divergences will be discussed in detail in Chapter 13. For the present, the point that concerns us is that the ultraviolet divergence has cancelled.

* * *

The result (11.4.9) for δm_e is of some interest in itself. Note that $\delta m_e/m_e > 0$, as we would expect for the electromagnetic self energy due to the interaction of a charge with its own field. But unlike the classical estimates of electromagnetic self-energy by Poincaré, Abraham, and others,[9] Eq. (11.4.9) is only logarithmically divergent in the limit $\mu \to \infty$ where the cutoff is removed. In this limit:

$$\delta m_e \to \frac{6m_e\pi^2 e^2}{(2\pi)^4}\ln\left(\frac{\mu}{m_e}\right). \tag{11.4.15}$$

In our calculation of the Lamb shift in Section 14.3 we will be interested in the opposite limit, $\mu \ll m_e$. Here Eq. (11.4.9) gives

$$\delta m_e \to \frac{e^2\mu}{8\pi}\left[1 - \frac{3\mu}{2\pi m_e} + \ldots\right] . \tag{11.4.16}$$

Appendix Assorted Integrals

In order to combine the denominators of N propagators, we need to replace a product like $D_1^{-1}D_2^{-1}\ldots D_N^{-1}$ with an integral of a function that involves a linear combination of $D_1, D_2, \ldots D_N$. For this purpose it is often convenient to make use of the formula

$$\frac{1}{D_1 D_2 \ldots D_N} = (N-1)! \int_0^1 dx_1 \int_0^{x_1} dx_2 \cdots \int_0^{x_{N-2}} dx_{N-1}$$
$$\times \left[D_1 x_{N-1} + D_2(x_{N-2} - x_{N-1}) + \cdots + D_N(1 - x_1)\right]^{-N} . \tag{11.A.1}$$

In this chapter we have used special cases of this formula for $N = 2$ and $N = 3$.

After combining denominators, shifting the four-momentum variable of integration, Wick rotating, and using four-dimensional rotational invariance, we commonly encounter integrals of the form

$$\int d^4k \, \frac{(k^2)^n}{(k^2 + v^2)^m}$$

with $(k^2 + v^2)^m$ coming from the combined propagator denominators, and $(k^2)^n$ coming from the propagator numerators and vertex momentum factors. This is divergent for $2n + 4 \geq 2m$, but the integral can be given a finite value by analytically continuing the spacetime dimensionality from 4 to a complex value d. To evaluate the resulting integral, we use the well-known formula

$$\int_0^\infty d\kappa \frac{\kappa^{\ell-1}}{(\kappa^2 + v^2)^m} = v^{\ell - 2m} \frac{\Gamma\left(\ell/2\right)\Gamma\left(m - \ell/2\right)}{2\,\Gamma(m)} , \tag{11.A.2}$$

where $\ell = d + 2n$. We used this formula in the special cases $n = 0, m = 2$ and $n = 1, m = 2$ in Section 11.2.

Ultraviolet divergences manifest themselves in Eq. (11.A.2) as poles in the factor $\Gamma(m - \ell/2) = \Gamma(m - n - d/2)$ as $d \to 4$ with fixed integer n. For $2 + n = m$, this factor goes as

$$\Gamma\left(\frac{4-d}{2}\right) \to \frac{2}{d-4} + \gamma , \tag{11.A.3}$$

where $\gamma = 0.5772157\cdots$ is the Euler constant. The limiting behavior for $2 + n > m$ can be obtained from (11.A.3) and the recursion relation for Gamma functions.

Problems

1. Calculate the contributions to the vacuum polarization function $\pi(q^2)$ and to Z_3 of one-loop graphs involving a charged spinless particle of mass m_s. What effect does this have on the energy shift of the 2s state of hydrogen, if $m_s >> Z\alpha m_e$?
2. Suppose that a neutral scalar field ϕ of mass m_ϕ has an interaction $g\phi\bar{\psi}\psi$ with the electron field. To one-loop order, what effect does this have on the magnetic moment of the electron? On Z_2?
3. Consider a neutral scalar field ϕ with mass m_ϕ and self-interaction $g\phi^3/6$. To one-loop order, calculate the S-matrix element for scalar–scalar scattering.
4. To one-loop order, calculate the effect of the neutral scalar field of Problem 2 on the mass shift δm_e of the electron.

References

1. R. P. Feynman, *Phys. Rev.* **76**, 769 (1949).
2. G. C. Wick, *Phys. Rev.* **96**, 1124 (1954).
3. G. 't Hooft and M. Veltman, *Nucl. Phys.* **B44**, 189 (1972).
4. E. A. Uehling, *Phys. Rev.* **48**, 55 (1935). The one-loop function $\pi(q^2)$ was first given for $q^2 \neq 0$ by J. Schwinger, *Phys. Rev.* **75**, 651 (1949).
5. J. Schwinger, *Phys. Rev.* **73**, 416 (1948).
6. This is calculated (including terms that vanish for $m_e \ll m_\mu$) by H. Suura and E. Wichmann, *Phys. Rev.* **105**, 1930 (1957); A. Petermann, *Phys. Rev.* **105**, 1931 (1957); H. H. Elend, *Phys. Lett.* **20**, 682 (1966); **21**, 720 (1966); G. W. Erickson and H. H. T. Liu, UCD-CNL-81 report (1968).
7. J. Bailey *et al.* (CERN–Mainz–Daresbury Collaboration), *Nucl. Phys.* **B150**, 1 (1979). These experiments are done by observing the precession of the muon spin in a storage ring.
8. W. Pauli and F. Villars, *Rev. Mod. Phys.* **21**, 434 (1949). Also see J. Rayski, *Phys. Rev.* **75**, 1961 (1949).
9. See, e.g., A. I. Miller, *Theory of Relativity — Emergence (1905) and Early Interpretation (1905–1911)* (Addison-Wesley, Reading, MA, 1981): Chapter 1.