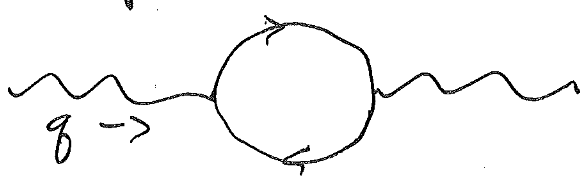
The Renormalization Group in Continuum Field Theory

Let's recall that when we computed vacuum polarization



in QED using dimensional regularization, we got

$$\pi(q^2) = \frac{e^2}{2\pi^2}\int_0^1 x(1-x)\ln\left(1+\frac{q^2x(1-x)}{m^2}\right)dx$$

in which $q^2 = \vec{q}^{\,2} - q^{0\,2}$ is the square of the photon's momentum.

Now $\pi(q^2)$ is dimensionless. So we'd expect from dimensional analysis that as $q^2 \to \infty$

$$\pi(q^2, m^2, e^2) = \pi\left(1, \frac{m^2}{q^2}, e^2\right) \to \pi(1, 0, e^2).$$

That is, π should go to a constant as $q^2 \to \infty$. This behavior is spoiled by the log term.

Such large logarithms are generic. They occur because we used a renormalization scheme at a fixed q^2, namely $q^2 = 0$.

More generally, a quantity Γ of dimension D should vary with E as

$$\Gamma(E, x, g, m) = E^D\,\Gamma(1, x, g, \tfrac{m}{E})$$

and so we'd expect

$$\Gamma(E, x, g, m) \longrightarrow E^D\,\Gamma(1, x, g, 0)$$

as $E \to \infty$. But large logs spoil this simple behavior if we renormalize at a fixed energy E.

So we try to renormalize at a sliding scale μ. Our coupling constant now is $g_\mu = g(\mu)$. Now

$$\Gamma(E, x, g_\mu, m, \mu) = E^D\,\Gamma(1, x, g_\mu, \tfrac{m}{E}, \tfrac{\mu}{E}).$$

And to compute Γ at E, we use $\mu = E$, so

$$\Gamma(E, x, g_\mu, m, \mu) = E^D\,\Gamma(1, x, g_E, \tfrac{m}{E}, 1).$$

The idea here is to have g_E independent of m for $E \gg m$. Then as $E \to \infty$ we may have

$$\Gamma(E, x, g_\mu, m, \mu) \to E^D\,\Gamma(1, x, g_E, 0, 1),$$

apart from possible non-perturbative corrections.

We expect

$$g_{\mu'} = G\,(g_\mu,\ \mu'/\mu,\ m/\mu)$$

Then

$$\frac{dg_{\mu'}}{d\mu'} = \frac{1}{\mu}\,\frac{d}{dz}G\,(g_\mu, z, m/\mu)\Big|_{z=\mu'/\mu}$$

So

$$\mu\frac{dg_{\mu'}}{d\mu'} = \frac{dG}{dz}(g_\mu, z, m/\mu)\Big|_{z=\mu'/\mu}$$

And setting $\mu'=\mu$, we have

$$\mu\,\frac{dg_\mu}{d\mu} \equiv \beta\left(g_\mu, \frac{m}{\mu}\right) = \frac{dG}{dz}\left(g_\mu, z, \frac{m}{\mu}\right)\Big|_{z=1}.$$

Thus for $\mu \gg m$, we find

$$\mu\,\frac{dg_\mu}{d\mu} = \beta(g_\mu, 0) \equiv \beta(g_\mu)$$

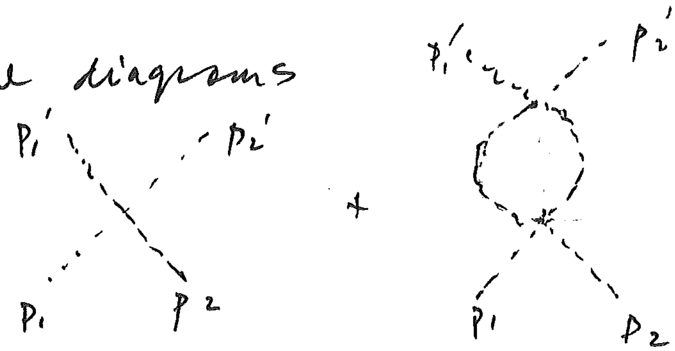
which is the Callan-Symanzik equation.
Integrating $\displaystyle\int_{g_M}^{g_E}\frac{dg}{\beta(g)} = \int_M^E\frac{d\mu}{\mu} = \ln(E/M).$

We need to choose M so that for $\mu \gtrsim M$ we can neglect m/μ and so that $\ln(M/m)$ isn't so big as to prevent us from using perturbation theory to compute g_M from g_R, the conventionally renormalized coupling constant.

Example: In the scalar theory with

$$\mathcal{L} = -\frac{1}{2}\partial_\lambda\phi\,\partial^\lambda\phi - \frac{1}{2}m^2\phi^2 - \frac{1}{24}g\phi^4$$

the diagrams



give

$$A = g - \frac{g^2}{32\pi^2}\int_0^1 dx\,\left\{\ln\frac{\Lambda^2}{m^2 - s\,x(1-x)} + \ln\left(\frac{\Lambda^2}{m^2 - t\,x(1-x)}\right) + \ln\left(\frac{\Lambda^2}{m^2 - u\,x(1-x)}\right) - 3\right\}$$

where Λ is a UV cut-off and

$$s = -(p_1+p_2)^2,\quad t = -(p_1-p_1')^2,\quad u = -(p_1-p_2')^2$$

and $s+t+u = 4m^2$ when all p's are on mass shell $p_i^2 = -m^2$.

In the conventional approach, we replace the bare g with g_R defined at some fixed scale, e.g.,

$$g_R \equiv A(s=t=u=0) = g - \frac{3g^2}{32\pi^2}\left\{\ln\frac{\Lambda^2}{m^2} - 1\right\}.$$

Setting $t = \frac{3}{32\pi^2}\left\{\ln\frac{\Lambda^2}{m^2} - 1\right\}$, we have

$$-tg^2 + g - g_R = 0 \quad \text{or} \quad g = \frac{1}{2t} \mp \sqrt{\frac{1}{4t^2} - \frac{g_R}{t}}.$$

Anyway, A then is

$$A = g_R + \frac{g_R^2}{32\pi^2}\int_0^1 dx\left\{\ln\left(1 - \frac{s\,x(1-x)}{m^2}\right)\right.$$

$$\left. + \ln\left(1 - \frac{t\,x(1-x)}{m^2}\right) + \ln\left(1 - \frac{u\,x(1-x)}{m^2}\right)\right\} + \cdots$$

and for huge s, t, and u

$$A \to g_R + \frac{g_R^2}{32\pi^2}\left\{\ln\left(-\frac{s}{m^2}\right) + \ln\left(-\frac{t}{m^2}\right) + \ln\left(-\frac{u}{m^2}\right) - 6\right\},$$

which has big logs as $-s, -t,$ or $-u \to \infty$.

Instead, we'll define

$$g_\mu \equiv A(s=t=u=-\mu^2).$$

That is,

$$g_\mu \equiv A(s=t=u=-\mu^2)$$

$$= g - \frac{3g^2}{32\pi^2}\int_0^1 dx \left\{ \ln\left(\frac{\Lambda^2}{m^2+\mu^2 x(1-x)}\right) - 1 \right\} + O(g^3).$$

In terms of $g_R = g - \frac{3g^2}{32\pi^2}\left\{ \ln \frac{\Lambda^2}{m^2} - 1 \right\}$, this is

(*) $$g_\mu = g_R + \frac{3g_R^2}{32\pi^2}\int_0^1 dx \ln\left(1 + \frac{\mu^2 x(1-x)}{m^2}\right) + O(g^3)$$

But this works only if $|g_R \ln(\mu/m)| \ll 1$.

<u>But</u> in terms of g_μ, $g_{\mu'}$ is better behaved:

$$g_{\mu'} = g - \frac{3g^2}{32\pi^2}\int_0^1 dx \left\{ \ln \Lambda^2 - \ln m^2 + \mu'^2 x(1-x) - 1 \right\}$$

$$g_\mu = g - \frac{3g^2}{32\pi^2}\int_0^1 \left\{ \ln \Lambda^2 - \ln m^2 + \mu^2 x(1-x) - 1 \right\}$$

So

$$g_{\mu'} = g_\mu - \frac{3g_\mu^2}{32\pi^2}\int_0^1 dx \ln\left(\frac{m^2+\mu^2 x(1-x)}{m^2+\mu'^2 x(1-x)}\right)$$

Here we replace g^2 with g_μ^2 or $g_{\mu'}^2$ because the difference is of order g^3.

Now

$$\beta\left(g_\mu, \frac{m}{\mu}\right) = \mu' \frac{d\, g_{\mu'}}{d\mu'}\bigg|_{\mu'=\mu}$$

$$= \frac{3 g_\mu^2}{16\pi^2} \int_0^1 dx \, \frac{\mu^2 x(1-x)}{m^2 + \mu^2 x(1-x)} + O(g_\mu^3)$$

For $\mu >> m$, this is

$$\beta(g_\mu) = \frac{3 g_\mu^2}{16\pi^2} + O(g_\mu^3).$$

To two-loop order, the beta-function is

$$\beta(g_\mu) = g_\mu \left[3\frac{g_\mu}{16\pi^2} - \frac{17}{3}\left(\frac{g_\mu}{16\pi^2}\right)^2 \right] + \cdots$$

So to one-loop order

$$\ln\frac{E}{M} = \int_M^E \frac{d\mu}{\mu} = \int_{g_M}^{g_E} \frac{dg}{\beta} = \int \frac{dg}{\frac{3g^2}{16\pi^2}} = \frac{16\pi^2}{3}\left[\frac{1}{g_M} - \frac{1}{g_E}\right]$$

M not μ

which gives us after setting $E=\mu$

$$g_E = g_\mu = \frac{g_M}{1 - \frac{3}{16\pi^2} g_M \ln\left(\frac{\mu}{M}\right)} \quad : \text{(no big logs!)}$$

We may relate g_μ to g_R by using (*) at values of $\mu^2 \gg m^2$ where

$$g_\mu \simeq g_R + \frac{3g_R^3}{32\pi^2} \int_0^1 dx \ln \frac{\mu^2}{m^2} x(1-x)$$

$$\simeq g_R + \frac{3g_R^2}{32\pi^2} \int_0^1 dx \left[2 \ln \frac{\mu}{m} + \ln x(1-x) \right]$$

$$\simeq g_R + \frac{3g_R^2}{16\pi^2} \ln \frac{\mu}{m} .$$

Although $\mu \gg m$, we want $\left| g_R \ln \frac{\mu}{m} \right| \ll 1$.

So for such a μ, say $\mu = M$, we have

$g_M = g_R$ whence for $\mu = E$

$$g_\mu = \frac{g_R}{1 - \frac{3}{16\pi^2} g_R \ln \frac{\mu}{M}} .$$

So the ϕ^4 theory exhibits asymptotic slavery.

$$g_E = \frac{g_R}{1 - \frac{3}{16\pi^2} g_R \ln \frac{E}{M}} .$$