Electrodynamics: without charges

$$S = \frac{1}{2}\int E^2 - B^2 \, d^4x = -\frac{1}{4}\int F_{\mu\nu}F^{\mu\nu} \, d^4x$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = -F_{\nu\mu}$

$$E^i = +F^{i0}, \qquad \epsilon^{ijk}B^k = -F^{ij}.$$

With a charged current $j^a = (\rho, \vec{j})$,

$$S = \int\left(-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - A^\mu j_\mu\right)d^4x$$

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}\left(\partial^\mu A^\nu - \partial^\nu A^\mu\right) - j_\mu A^\mu$$

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu A_\nu} = -\partial_\mu\left(\partial^\mu A^\nu - \partial^\nu A^\mu\right) = \frac{\partial \mathcal{L}}{\partial A_\nu} = -j^\nu$$

$$\partial_\mu F^{\mu\nu} = +j^\nu$$

Set $\nu = 0$ $\qquad \partial_i F^{i0} = +j^0 = +\rho$

$$\partial_i E^i = \rho = \nabla \cdot E \quad \text{Gauss's law}$$

Set $\nu = k$ $\qquad +\partial_\mu\left(\partial^\mu A^\nu - \partial^\nu A^\mu\right) = j^k$

$$-\partial_0\left(E^k\right) - \partial_j\left(+\partial_j A^k - \partial_k A^j\right) = j^k$$

$$\vec{\nabla}\times\vec{B} - \dot{E} = \vec{j}$$

$$\vec{B} = \nabla \times A \Rightarrow \quad B^i = \epsilon^{ijk}\partial_j A^k = -\tfrac{1}{2}\epsilon^{ijk}F_{jk}$$

So

$$\partial_i B^i = \epsilon^{ijk}\partial_i\partial_j A^k = 0$$

$$\vec{\nabla}\cdot\vec{B} = 0$$

This is part of the more general identity

$$\underbrace{\epsilon^{lijk}}_{\text{a-sym}}\underbrace{\partial_i\partial_j}_{\text{sym}} A_k = 0 = \underbrace{\epsilon^{lijk}}_{\text{a-sym}}\underbrace{\partial_i\partial_k}_{\text{sym}} A_j$$

So $\quad \epsilon^{lijk}\partial_i(\partial_j A_k - \partial_k A_j) = 0$

which gives for $l=0$

$$\nabla\cdot B = 0$$

and for $l \neq 0$ gives

$$\nabla \times E + \dot{B} = 0.$$

Under $A_\mu \to A_\mu + \partial_\mu\lambda$, $F_{\mu\nu}$ is invariant

$$F'_{\mu\nu} = \partial_\mu(A_\nu + \partial_\nu\lambda) - \partial_\nu(A_\mu + \partial_\mu\lambda) = F_{\mu\nu}$$

If $\partial^\mu j_\mu = 0$, then the current j^μ is conserved. If $\partial_\mu j^\mu = 0$, then S is invariant under $A_\mu \to A_\mu + \partial_\mu\lambda$ since

$$\delta S = \int -\partial^\mu\lambda\, j_\mu\, d^4x = +\int \lambda\,\partial^\mu j_\mu\, d^4x = 0.$$

We go to the radiation gauge also known as Coulomb's gauge

$$\vec{\nabla} \cdot \vec{A} = 0 .$$

by choosing λ so that

$$0 = \nabla \cdot A' = \nabla \cdot A + \nabla \cdot \nabla \lambda$$

ie

$$\nabla^2 \lambda = - \nabla \cdot A.$$

Now drop the prime(s) and use Gauss's law

$$\nabla \cdot E = \partial_i F^{i0} = \partial_i (\partial^i A^0 - \partial^0 A^i) = \rho .$$

But $\partial_i A^i = 0$. So we get

$$- \partial_i \partial^i A^0 = \rho$$

which means we can remove A^0 from the theory, replacing it with

$$A^0(x) = \frac{1}{4\pi} \int \frac{\rho(y) \, d^3y}{|\vec{x - y}|}$$

since

$$- \Delta \frac{1}{|\vec{x - y}|} = 4\pi \, \delta(\vec{x - y}) .$$

In the integral for $A^0(x)$, $y^0 = x^0$.

Since $-j_0 A^0 = -j^0 A^0$ appears in $\mathcal{L}$, the term in $\mathcal{H}$ is

$$H_\rho = \int j^0 A^0 d^3x = \int \frac{\rho(x)\rho(y)}{4\pi|\vec{x}-\vec{y}|} d^3x\, d^3y$$

where $\rho(x) = j^0(x)$ is the charge density.

So what we have left is $\vec{A}$ subject to the constraint $\hat{\vec{\nabla}} \cdot \vec{A} = 0$. So

$$A^\mu(x) = \int \frac{d^3p}{(2\pi)^3\sqrt{2p^0}} \sum_{\substack{s=-1 \\ s\neq 0}}^{1} \left[e^{-ipx} e^\mu(p,s)\, a(p,s) + e^{ipx} e^{\mu *}(p,s)\, a^\dagger(p,s) \right]$$

where

$$\vec{p} \cdot \vec{e}(p,s) = 0 \text{ so that } \vec{\nabla} \cdot \vec{A} = 0,$$

and $e^0(p,s) = 0$ since A^0 is gone.

The index s labels the two states of circular polarization — right handed and left handed. Because the photon is massless, it has only two polarization states

$$\vec{J} \cdot \hat{p} = \pm \hbar .$$

The state $J \cdot \hat{p} = 0$ is available only to massive particles.

$$[a(p,s), a^\dagger(p',s')] = \delta_{ss'}\, \delta^{(3)}(\vec{p}-\vec{p}')(2\pi)^3$$

and $[a(p,s), a(p',s')] = 0$.

For $\vec{p} = |\vec{p}|\hat{z}$, we can set

$$\vec{e}(|\vec{p}|\hat{z}, \pm 1) = \frac{1}{\sqrt{2}}(\hat{x} \pm i\hat{y}) \text{ and}$$

$$\vec{e}(\vec{p}, \pm 1) = R(\hat{p})\,\vec{e}(|\vec{p}|\hat{z}, \pm 1)$$

$$= e^{-i\phi J_z} e^{-i\theta J_y} \vec{e}(|\vec{p}|\hat{z}, \pm 1).$$

We then have

$$\sum_{\substack{s=-1 \\ s \neq 0}}^{1} e^i(p,s)\, e^j(p,s)^* = \delta_{ij} - \frac{p_i p_j}{|\vec{p}|^2}.$$