The Pion as a Nambu-Goldstone Boson

We've seen that a conserved current

$$0 = \partial_\mu j^\mu$$

gives rise to a massless particle when its charge

$$Q = \int d^3x \, j^0(x)$$

doesn't annihilate the vacuum

$$Q|0\rangle = |s,0\rangle \neq 0.$$

The state with momentum $\vec{k}$ is

$$\int d^3x \, e^{-ik\cdot x} j^0(\vec{x},t)|0\rangle \equiv |s,\vec{k}\rangle.$$

Its limit as $\vec{k}\to 0$ is $|s,0\rangle$ which has zero energy since

$$H|s,0\rangle = HQ|0\rangle = [H,Q]|0\rangle = 0.$$

This $m=0$ particle is called a Nambu-Goldstone boson.

Back in the 1960's, people used

$$\mathcal{L} = G\, \bar{e}\, \gamma^\mu (1-\gamma_5)\nu\, (J_\mu - J_{5\mu}) \tag{1}$$

to describe weak decays of hadrons.

By translational invariance,

$$\langle p' | J_5^\mu(x) | p \rangle = e^{-i(p'-p)\cdot x} \langle p' | J_5^\mu(0) | p \rangle.$$

Lorentz invariance (and charge-conjugation and isospin symmetries) tell us that

$$\langle p' | J_5^\mu(0) | p \rangle = \bar{u}(p') \left[\gamma^\mu \gamma^5 F(q^2) + q^\mu \gamma^5 G(q^2) \right] u(p) \tag{2}$$

with $q = p' - p$. The γ^5's are there because the current J_5^μ is an <u>axial</u> vector. Similarly, for the π^-

$$\langle 0 | J_5^\mu | k \rangle = f k^\mu \tag{3}$$

where k^μ is the momentum of the π^-. So the rate $\pi^- \to e^- + \bar{\nu}$ goes as f^2.

The π is the lightest hadron — apart from the u, d quarks.

By (3) we get since the pion is so light

$$k_\mu \langle 0 | J_5^\mu | k \rangle = f k^2 = f m_\pi^2 \simeq 0 \qquad (4)$$

which says that J_5^μ is "sort of" conserved.
That is,

$$\langle 0 | J_5^\mu (x) | k \rangle = \langle 0 | J_5^\mu (0) | k \rangle e^{-ik\cdot x}$$

so

$$\langle 0 | \partial_\mu J_5^\mu (x) | k \rangle = -i k_\mu \langle 0 | J_5^\mu (0) | k \rangle e^{-ik\cdot x}$$

$$= -i f m_\pi^2 e^{-ik\cdot x} \simeq 0 .$$

So we expect that this nearly conserved current J_5^μ would lead to the existence of a nearly massless particle.
That nearly massless particle is the pion.

Multiply (2) by $(p'-p)_\mu e^{i(p'-p)x}$

$$(p'-p)_\mu \langle p' | J_5^\mu (0) | p \rangle e^{i(p'-p)x}$$

$$= (p'-p)_\mu \langle p' | J_5^\mu (x) | p \rangle$$

$$= i \langle p' | \partial_\mu J_5^\mu (x) | p \rangle \simeq 0 .$$

So $(p'-p)_\mu \langle p' | J_5^\mu(0) | p\rangle \approx 0$.

But by (2)

$(p'-p)_\mu \langle p' | J_5^\mu(0) | p\rangle$

$= (p'-p)_\mu \, \bar{u}(p') \left[\gamma^\mu \gamma^5 F(q^2) + q^\mu \gamma^5 G(q^2) \right] u(p)$

But $\{\gamma^\mu, \gamma^5\} = 0$ and

$$\not{p}\, u(p) = m\, u(p)$$

so

$$u^\dagger(p)\, p_\mu \gamma^{\mu\dagger} = m\, u^\dagger(p)$$

and so

$$\bar{u}(p)\, \not{p} = m\, \bar{u}(p).$$

So

$(p'-p)_\mu \langle p' | J_5^\mu(0) | p\rangle$

$= \bar{u}(p') \left[(\not{p}' - \not{p}) \gamma_5 F(q^2) + q^2 \gamma^5 G(q^2) \right] u(p)$

$= \bar{u}(p') \left[2m\, \gamma_5\, F(q^2) + q^2 \gamma^5 G(q^2) \right] u(p) \approx 0$

So

$$2 m_N F(q^2) + q^2 G(q^2) \approx 0.$$

This would seem to say $2 m_N F(0) \approx 0$, but because the pion is nearly massless

$$G(q^2) \sim \frac{f}{q^2} \, g_{\pi NN}$$

So

$$2 m_N F(0) + f \, g_{\pi NN} \approx 0,$$

which is the Goldberger – Treiman relation between $F(0)$ which is measured in $n \to p + e + \bar{\nu}$ and the product of the pion-nucleon coupling constant $g_{\pi NN}$ and f, which is measured in the pion decay $\pi^- \to e + \bar{\nu}$.