Photon Spin Sums in Coulomb's Gauge

Because A^0 has been replaced by an integral over $\rho = j^0$, the Coulomb-gauge polarization vectors $e^\mu(p,s)$ all have

$$e^0(p,s) = 0.$$

They also have $\vec{p} \cdot \vec{e} = \vec{p} \cdot \vec{e}(p,s) = 0$ since $\vec{\nabla} \cdot \vec{A} = 0$.

They also are normalized

$$\vec{e}^{\,\dagger}(p,s) \cdot \vec{e}(p,s') = \delta_{ss'}$$

and they depend only upon $\hat{p}$. For $\hat{p} = \hat{z}$, and for right- and left-circularly polarized photons, they are

$$e(\hat{p} = \hat{z}, \pm) = e(\hat{z}, \pm) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ \pm i \\ 0 \end{pmatrix}.$$

More generally

$$e(\hat{p}, \pm) = R(\hat{p})\, \vec{e}(\hat{z}, \pm)$$

$$= e^{-i\phi L_3}\, e^{-i\theta L_2}\, e(\hat{z}, \pm).$$

The $\hat{p} = \hat{z}$ spin sum is

$$\sum_{\substack{s=-1\\ \neq 0}}^{1} e(\hat{z},s)e^{\dagger}(\hat{z},s) = \frac{1}{2}\begin{pmatrix}1\\ i\\ 0\end{pmatrix}(1\ -i\ 0) + \frac{1}{2}\begin{pmatrix}1\\ -i\\ 0\end{pmatrix}(1\ i\ 0)$$

$$= \frac{1}{2}\begin{pmatrix}1 & -i & 0\\ i & 1 & 0\\ 0 & 0 & 0\end{pmatrix} + \frac{1}{2}\begin{pmatrix}1 & i & 0\\ -i & 1 & 0\\ 0 & 0 & 0\end{pmatrix}$$

$$= \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{pmatrix} = I - \begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1\end{pmatrix}$$

$$= I - \hat{z}\hat{z}^T.$$

So the general spin sum is

$$\sum_{\substack{s=-1\\ \neq 0}}^{1} e(\hat{p},s)e^{\dagger}(\hat{p},s) = \sum_{\substack{s=-1\\ \neq 0}}^{1} R(\hat{p})e(\hat{z},s)e^{\dagger}(\hat{z},s)R^{\dagger}(\hat{p})$$

$$= R(\hat{p})\left(I - \hat{z}\hat{z}^T\right)R^{\dagger}(\hat{p})$$

$$= I - \hat{p}\hat{p}^T \quad \text{or}$$

$$\sum_{\substack{s=-1\\ \neq 0}}^{1} e^i(\hat{p},s)e^{j*}(\hat{p},s) = \delta^{ij} - \frac{p^i p^j}{|\vec{p}|^2}.$$