(2.5.47)) that takes the three-axis into the direction of $\mathbf{p}$, and

$$B(|\mathbf{p}|) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \gamma & \sqrt{\gamma^2-1} \\ 0 & 0 & \sqrt{\gamma^2-1} & \gamma \end{bmatrix} .$$

Then for an arbitrary rotation $\mathscr{R}$

$$W(\mathscr{R},p) = R(\mathscr{R}\hat{\mathbf{p}})B^{-1}(|\mathbf{p}|)R^{-1}(\mathscr{R}\hat{\mathbf{p}})\mathscr{R}\,R(\hat{\mathbf{p}})B(|\mathbf{p}|)R^{-1}(\hat{\mathbf{p}}) .$$

But the rotation $R^{-1}(\mathscr{R}\hat{\mathbf{p}})\mathscr{R}\,R(\hat{\mathbf{p}})$ takes the three-axis into the direction $\hat{\mathbf{p}}$, and then into the direction $\mathscr{R}\hat{\mathbf{p}}$, and then back to the three-axis, so it must be just a rotation by some angle θ around the three-axis

$$R^{-1}(\mathscr{R}\hat{\mathbf{p}})\mathscr{R}\,R(\hat{\mathbf{p}}) = R(\theta) \equiv \begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} .$$

Since $R(\theta)$ commutes with $B(|\mathbf{p}|)$, this now gives

$$W(\mathscr{R},p) = R(\mathscr{R}\hat{\mathbf{p}})B^{-1}(|\mathbf{p}|)R(\theta)B(|\mathbf{p}|)R^{-1}(\hat{\mathbf{p}}) = R(\mathscr{R}\hat{\mathbf{p}})R(\theta)R^{-1}(\hat{\mathbf{p}})$$

and hence

$$W(\mathscr{R},p) = \mathscr{R}$$

as was to be shown. Thus states of a moving massive particle (and, by extension, multi-particle states) have the same transformation under rotations as in non-relativistic quantum mechanics. This is another piece of good news — the whole apparatus of spherical harmonics, Clebsch–Gordan coefficients, etc. can be carried over wholesale from non-relativistic to relativistic quantum mechanics.

Mass Zero

First, we have to work out the structure of the little group. Consider an arbitrary little-group element $W^\mu{}_\nu$, with $W^\mu{}_\nu k^\nu = k^\mu$, where k^μ is the standard four-momentum for this case, $k^\mu = (0,0,1,1)$. Acting on a time-like four-vector $t^\mu = (0,0,0,1)$, such a Lorentz transformation must yield a four-vector Wt whose length and scalar product with $Wk = k$ are the same as those of t:

$$(Wt)^\mu (Wt)_\mu = t^\mu t_\mu = -1 ,$$
$$(Wt)^\mu k_\mu = t^\mu k_\mu = -1 .$$

Any four-vector that satisfies the second condition may be written

$$(Wt)^\mu = (\alpha, \beta, \zeta, 1+\zeta)$$

and the first condition then yields the relation

$$\zeta = (\alpha^2 + \beta^2)/2 . \tag{2.5.25}$$

It follows that the effect of $W^\mu{}_\nu$ on t^ν is the same as that of the Lorentz transformation

$$S^\mu{}_\nu(\alpha,\beta) = \begin{bmatrix} 1 & 0 & -\alpha & \alpha \\ 0 & 1 & -\beta & \beta \\ \alpha & \beta & 1-\zeta & \zeta \\ \alpha & \beta & -\zeta & 1+\zeta \end{bmatrix} . \tag{2.5.26}$$

This does not mean that W equals $S(\alpha,\beta)$, but it does mean that $S^{-1}(\alpha,\beta)W$ is a Lorentz transformation that leaves the time-like four-vector (0,0,0,1) invariant, and is therefore a pure rotation. Also, $S^\mu{}_\nu$ like $W^\mu{}_\nu$ leaves the light-like four-vector (0,0,1,1) invariant, so $S^{-1}(\alpha,\beta)W$ must be a rotation by some angle θ around the three-axis

$$S^{-1}(\alpha,\beta)W = R(\theta) , \tag{2.5.27}$$

where

$$R^\mu{}_\nu(\theta) \equiv \begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} .$$

The most general element of the little group is therefore of the form

$$W(\theta,\alpha,\beta) = S(\alpha,\beta)R(\theta) . \tag{2.5.28}$$

What group is this? We note that the transformations with $\theta = 0$ or with $\alpha = \beta = 0$ form subgroups:

$$S(\bar\alpha,\bar\beta)S(\alpha,\beta) = S(\bar\alpha+\alpha,\bar\beta+\beta) \tag{2.5.29}$$

$$R(\bar\theta)R(\theta) = R(\bar\theta+\theta) . \tag{2.5.30}$$

These subgroups are *Abelian* — that is, their elements all commute with each other. Furthermore, the subgroup with $\theta = 0$ is *invariant*, in the sense that its elements are transformed into other elements of the same subgroup by any member of the group

$$R(\theta)S(\alpha,\beta)R^{-1}(\theta) = S(\alpha\cos\theta + \beta\sin\theta, -\alpha\sin\theta + \beta\cos\theta) . \tag{2.5.31}$$

From Eqs. (2.5.29)–(2.5.31) we can work out the product of any group elements. The reader will recognize these multiplication rules as those of the group $ISO(2)$, consisting of translations (by a vector (α,β)) and rotations (by an angle θ) in two dimensions.

Groups that do *not* have invariant Abelian subgroups have certain simple properties, and for this reason are called *semi-simple*. As we have seen, the little group $ISO(2)$ like the inhomogeneous Lorentz group is *not* semi-simple, and this leads to interesting complications. First, let's take a look at the Lie algebra of $ISO(2)$. For θ, α, β infinitesimal, the general group element is

$$W(\theta,\alpha,\beta)^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} + \omega^{\mu}{}_{\nu} ,$$
$$\omega_{\mu\nu} = \begin{bmatrix} 0 & \theta & -\alpha & \alpha \\ -\theta & 0 & -\beta & \beta \\ \alpha & \beta & 0 & 0 \\ -\alpha & -\beta & 0 & 0 \end{bmatrix} .$$

From (2.4.3), we see then that the corresponding Hilbert space operator is

$$U(W(\theta,\alpha,\beta)) = 1 + i\alpha A + i\beta B + i\theta J_3 , \qquad (2.5.32)$$

where A and B are the Hermitian operators

$$A = -J^{13} + J^{10} = J_2 + K_1 , \qquad (2.5.33)$$
$$B = -J^{23} + J^{20} = -J_1 + K_2 , \qquad (2.5.34)$$

and, as before, $J_3 = J_{12}$. Either from (2.4.18)–(2.4.20), or directly from Eqs. (2.5.29)–(2.5.31), we see that these generators have the commutators

$$[J_3, A] = +iB , \qquad (2.5.35)$$
$$[J_3, B] = -iA , \qquad (2.5.36)$$
$$[A, B] = 0 . \qquad (2.5.37)$$

Since A and B are commuting Hermitian operators they (like the momentum generators of the inhomogeneous Lorentz group) can be simultaneously diagonalized by states $\Psi_{k,a,b}$

$$A\Psi_{k,a,b} = a\Psi_{k,a,b} ,$$
$$B\Psi_{k,a,b} = b\Psi_{k,a,b} .$$

The problem is that if we find one such set of non-zero eigenvalues of A, B, then we find a whole continuum. From Eq. (2.5.31), we have

$$U[R(\theta)]A\, U^{-1}[R(\theta)] = A\cos\theta - B\sin\theta ,$$
$$U[R(\theta)]B\, U^{-1}[R(\theta)] = A\sin\theta + B\cos\theta ,$$

and so, for arbitrary θ,

$$A\Psi^{\theta}_{k,a,b} = (a\cos\theta - b\sin\theta)\Psi^{\theta}_{k,a,b} ,$$
$$B\Psi^{\theta}_{k,a,b} = (a\sin\theta + b\cos\theta)\Psi^{\theta}_{k,a,b} ,$$

where

$$\Psi^{\theta}_{k,a,b} \equiv U^{-1}(R(\theta))\,\Psi_{k,a,b} .$$

Massless particles are not observed to have any continuous degree of freedom like θ; to avoid such a continuum of states, we must require that physical states (now called $\Psi_{k,\sigma}$) are eigenvectors of A and B with $a = b = 0$:

$$A\Psi_{k,\sigma} = B\Psi_{k,\sigma} = 0\,. \tag{2.5.38}$$

These states are then distinguished by the eigenvalue of the remaining generator

$$J_3\Psi_{k,\sigma} = \sigma\Psi_{k,\sigma}\,. \tag{2.5.39}$$

Since the momentum $\mathbf{k}$ is in the three-direction, σ gives the component of angular momentum in the direction of motion, or *helicity*.

We are now in a position to calculate the Lorentz transformation properties of general massless particle states. First note that by use of the general arguments of Section 2.2, Eq. (2.5.32) generalizes for finite α and β to

$$U(S(\alpha,\beta)) = \exp(i\alpha A + i\beta B) \tag{2.5.40}$$

and for finite θ to

$$U(R(\theta)) = \exp(iJ_3\theta)\,. \tag{2.5.41}$$

An arbitrary element W of the little group can be put in the form (2.5.28), so that

$$U(W)\Psi_{k,\sigma} = \exp(i\alpha A + i\beta B)\exp(i\theta J_3)\Psi_{k,\sigma} = \exp(i\theta\sigma)\Psi_{k,\sigma}$$

and therefore Eq. (2.5.8) gives

$$D_{\sigma'\sigma}(W) = \exp(i\theta\sigma)\delta_{\sigma'\sigma}\,,$$

where θ is the angle defined by expressing W as in Eq. (2.5.28). The Lorentz transformation rule for a massless particle of arbitrary helicity is now given by Eqs. (2.5.11) and (2.5.18) as

$$U(\Lambda)\Psi_{p,\sigma} = \sqrt{\frac{(\Lambda p)^0}{p^0}}\exp\left(i\sigma\theta(\Lambda,p)\right)\Psi_{\Lambda p,\sigma} \tag{2.5.42}$$

with $\theta(\Lambda, p)$ defined by

$$W(\Lambda,p) \equiv L^{-1}(\Lambda p)\Lambda L(p) \equiv S\left(\alpha(\Lambda,p),\beta(\Lambda,p)\right)R\left(\theta(\Lambda,p)\right)\,. \tag{2.5.43}$$

We shall see in Section 5.9 that electromagnetic gauge invariance arises from the part of the little group parameterized by α and β.

At this point we have not yet encountered any reason that would forbid the helicity σ of a massless particle from being an arbitrary real number. As we shall see in Section 2.7, there are topological considerations that restrict the allowed values of σ to integers and half-integers, just as for massive particles.

To calculate the little-group element (2.5.43) for a given Λ and p, (and also to enable us to calculate the effect of space or time inversion on these states in the next section) we need to fix a convention for the standard Lorentz transformation that takes us from $k^\mu = (0, 0, \kappa, \kappa)$ to p^μ. This may conveniently be chosen to have the form

$$L(p) = R(\hat{\mathbf{p}})B(|\mathbf{p}|/\kappa) \tag{2.5.44}$$

where $B(u)$ is a pure boost along the three-direction:

$$B(u) \equiv \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & (u^2+1)/2u & (u^2-1)/2u \\ 0 & 0 & (u^2-1)/2u & (u^2+1)/2u \end{bmatrix} \tag{2.5.45}$$

and $R(\hat{\mathbf{p}})$ is a pure rotation that carries the three-axis into the direction of the unit vector $\hat{\mathbf{p}}$. For instance, suppose we take $\hat{\mathbf{p}}$ to have polar and azimuthal angles θ and ϕ:

$$\hat{\mathbf{p}} = (\sin\theta\cos\phi,\ \sin\theta\sin\phi,\ \cos\theta)\ . \tag{2.5.46}$$

Then we can take $R(\hat{\mathbf{p}})$ as a rotation by angle θ around the two-axis, which takes $(0, 0, 1)$ into $(\sin\theta,\ 0, \cos\theta)$, followed by a rotation by angle ϕ around the three-axis:

$$U(R(\hat{\mathbf{p}})) = \exp(-i\phi J_3)\exp(-i\theta J_2)\ , \tag{2.5.47}$$

where $0 \leq \theta \leq \pi,\quad 0 \leq \phi < 2\pi$. (We give $U(R(\hat{\mathbf{p}}))$ rather than $R(\hat{\mathbf{p}})$, together with a specification of the range of ϕ and θ, because shifting θ or ϕ by 2π would give the same rotation $R(\hat{\mathbf{p}})$, but a different sign for $U(R(\hat{\mathbf{p}}))$ when acting on half-integer spin states.) Since (2.5.47) is a rotation, and does take the three-axis into the direction (2.5.46), any other choice of such an $R(\hat{\mathbf{p}})$ would differ from this one by at most an initial rotation around the three-axis, corresponding to a mere redefinition of the phase of the one-particle states.

Note that the helicity is Lorentz-invariant; a massless particle of a given helicity σ looks the same (aside from its momentum) in all inertial frames. Indeed, we would be justified in thinking of massless particles of each different helicity as different species of particles. However, as we shall see in the next section, particles of opposite helicity are related by the symmetry of space inversion. Thus, because electromagnetic and gravitational forces obey space inversion symmetry, the massless particles of helicity ± 1 associated with electromagnetic phenomena are both called *photons*, and the massless particles of helicity ± 2 that are believed to be associated with gravitation are both called *gravitons*. On the other hand, the supposedly massless particles of helicity $\pm 1/2$ that are emitted in nuclear beta decay have no interactions (apart from gravitation) that respect the symmetry of space inversion, so these particles are given different names: *neutrinos* for helicity $-1/2$, and *antineutrinos* for helicity $+1/2$.

Even though the helicity of a massless particle is Lorentz-invariant, the state itself is not. In particular, because of the helicity-dependent phase factor $\exp(i\sigma\theta)$ in Eq. (2.5.42), a state formed as a linear superposition of one-particle states with opposite helicities will be changed by a Lorentz transformation into a different superposition. For instance, a general one-photon state of four-momentum p may be written

$$\Psi_{p;\alpha} = \alpha_+ \Psi_{p,+1} + \alpha_- \Psi_{p,-1}\,,$$

where

$$|\alpha_+|^2 + |\alpha_-|^2 = 1\,.$$

The generic case is one of *elliptic polarization*, with $|\alpha_\pm|$ both non-zero and unequal. *Circular polarization* is the limiting case where either α_+ or α_- vanishes, and *linear polarization* is the opposite extreme, with $|\alpha_+| = |\alpha_-|$. The overall phase of α_+ and α_- has no physical significance, and for linear polarization may be adjusted so that $\alpha_- = \alpha_+^*$, but the relative phase is still important. Indeed, for linear polarizations with $\alpha_- = \alpha_+^*$, the phase of α_+ may be identified as the angle between the plane of polarization and some fixed reference direction perpendicular to $\mathbf{p}$. Eq. (2.5.42) shows that under a Lorentz transformation $\Lambda^\mu{}_\nu$, this angle rotates by an amount $\theta(\Lambda, p)$. Plane polarized gravitons can be defined in a similar way, and here Eq. (2.5.42) has the consequence that a Lorentz transformation Λ rotates the plane of polarization by an angle $2\theta(\Lambda, p)$.

2.6 Space Inversion and Time-Reversal

We saw in Section 2.3 that any homogeneous Lorentz transformation is either proper and orthochronous (i.e., $\mathrm{Det}\Lambda = +1$ and $\Lambda^0{}_0 \geq +1$) or else equal to a proper orthochronous transformation times either $\mathscr{P}$ or $\mathscr{T}$ or $\mathscr{P}\mathscr{T}$, where $\mathscr{P}$ and $\mathscr{T}$ are the space inversion and time-reversal transformations

$$\mathscr{P}^\mu{}_\nu = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \mathscr{T}^\mu{}_\nu = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

It used to be thought self-evident that the fundamental multiplication rule of the Poincaré group

$$U(\bar\Lambda, \bar a)\, U(\Lambda, a) = U(\bar\Lambda\Lambda, \bar\Lambda a + \bar a)$$

tensors with an *even* total number of spacetime indices, and therefore

$$\mathsf{CPT}\,\mathscr{H}(x)\,[\mathsf{CPT}]^{-1}=\mathscr{H}(-x)\,. \tag{5.8.7}$$

More generally (and somewhat more easily) we can see that the same is true for Hermitian scalars formed from the fields $\psi^{AB}_{ab}(x)$ belonging to one or more of the general irreducible representations of the homogeneous Lorentz group. Putting together our results in the previous section for the effects of inversions on such fields, we find

$$\mathsf{CPT}\,\psi^{AB}_{ab}(x)\,[\mathsf{CPT}]^{-1}=(-1)^{2B}\psi^{AB\dagger}_{ab}(-x)\,. \tag{5.8.8}$$

(For the Dirac field the factor $(-1)^{2B}$ is supplied by the matrix γ_5 in Eq. (5.8.3).) In order to couple together a product $\psi^{A_1B_1}_{a_1b_1}(x)\,\psi^{A_2B_2}_{a_2b_2}(x)\cdots$ to form a scalar $\mathscr{H}(x)$, it is necessary that both $A_1+A_2+\cdots$ and $B_1+B_2+\cdots$ be integers, so $(-1)^{2B_1+2B_2+\cdots}=1$, and so a Hermitian scalar $\mathscr{H}(x)$ will automatically satisfy Eq. (5.8.7).

From Eq. (5.8.7) it follows immediately that CPT commutes with the interaction $V\equiv\int d^3x\,\mathscr{H}(\vec{x},0)$:

$$\mathsf{CPT}\,V\,[\mathsf{CPT}]^{-1}=V\,. \tag{5.8.9}$$

Also, in any theory CPT commutes with the free-particle Hamiltonian H_0. Thus the operator CPT, which has been defined here by its operation on free-particle operators, acts on 'in' and 'out' states in the way described in Section 3.3. The physical consequences of this symmetry principle have already been discussed in Sections 3.3 and 3.6.

5.9 Massless Particle Fields

Up to this point we have dealt only with the fields of massive particles. For some of these fields, such as the scalar and Dirac fields discussed in Sections 5.2 and 5.5, there is no special problem in passing to the limit of zero mass. On the other hand, we saw in Section 5.3 that there *is* a difficulty in taking the zero-mass limit of the vector field for a particle of spin one: at least one of the polarization vectors blows up in this limit. In fact, we shall see in this section that the creation and annihilation operators for physical massless particles of spin $j\geq 1$ cannot be used to construct all of the irreducible (A,B) fields that can be constructed for finite mass. This peculiar limitation on field types will lead us naturally to the introduction of gauge invariance.

Just as we did for massive particles, let us attempt to construct a general free field for a massless particle as a linear combination of the annihilation operators $a(\mathbf{p},\sigma)$ for particles of momentum $\mathbf{p}$ and helicity σ, and the corresponding creation operators $a^{c\dagger}(\mathbf{p},\sigma)$ for the antiparticles:*

$$\psi_\ell(x) = (2\pi)^{-3/2}\int d^3p \sum_\sigma \Big[\kappa\, a(\mathbf{p},\sigma)u_\ell(\mathbf{p},\sigma)\, e^{ip\cdot x} + \lambda\, a^{c\dagger}(\mathbf{p},\sigma)v_\ell(\mathbf{p},\sigma)\, e^{-ip\cdot x}\Big] \tag{5.9.1}$$

where now $p^0 \equiv |\mathbf{p}|$. The creation operators transform just like the one-particle states in Eq. (2.5.42)

$$U(\Lambda)a^\dagger(\mathbf{p},\sigma)U^{-1}(\Lambda) = \sqrt{\frac{(\Lambda p)^0}{p^0}}\ \exp\Big(i\sigma\theta(p,\Lambda)\Big)a^\dagger\,(\mathbf{p}_\Lambda,\sigma)\ , \tag{5.9.2}$$

$$U(\Lambda)a^{c\dagger}(\mathbf{p},\sigma)U^{-1}(\Lambda) = \sqrt{\frac{(\Lambda p)^0}{p^0}}\ \exp\Big(i\sigma\theta(p,\Lambda)\Big)a^{c\dagger}\,(\mathbf{p}_\Lambda,\sigma)\ , \tag{5.9.3}$$

and hence also

$$U(\Lambda)a(\mathbf{p},\sigma)U^{-1}(\Lambda) = \sqrt{\frac{(\Lambda p)^0}{p^0}}\ \exp\Big(-i\sigma\theta(p,\Lambda)\Big)a(\mathbf{p}_\Lambda,\sigma)\,, \tag{5.9.4}$$

where $p_\Lambda \equiv \Lambda p$, and θ is the angle defined by Eqs. (2.5.43). Hence if we want the field to transform according to some representation $D(\Lambda)$ of the homogeneous Lorentz group

$$U(\Lambda)\psi_\ell(x)U^{-1}(\Lambda) = \sum_{\bar\ell} D_{\ell\bar\ell}(\Lambda^{-1})\psi_{\bar\ell}(\Lambda x)\,, \tag{5.9.5}$$

then we must take the coefficient functions u and v to satisfy the relations

$$u_{\bar\ell}(\mathbf{p}_\Lambda,\sigma)\ \exp\Big(i\sigma\theta(p,\Lambda)\Big) = \sqrt{\frac{p^0}{(\Lambda p)^0}}\sum_\ell D_{\bar\ell\ell}(\Lambda)u_\ell(\mathbf{p},\sigma)\,, \tag{5.9.6}$$

$$v_{\bar\ell}(\mathbf{p}_\Lambda,\sigma)\ \exp\Big(-i\sigma\theta(p,\Lambda)\Big) = \sqrt{\frac{p^0}{(\Lambda p)^0}}\sum_\ell D_{\bar\ell\ell}(\Lambda)v_\ell(\mathbf{p},\sigma) \tag{5.9.7}$$

in place of Eqs. (5.1.19) and (5.1.20). (Again, $p_\Lambda \equiv \Lambda p$.) As in the massive particle case, we can satisfy these requirements by setting (in place of

* We deal here with only a single species of particle, and drop the species label n. Also, κ and λ are constant coefficients to be determined by the requirement of causality with some convenient choice of normalization of the coefficient functions u_ℓ and v_ℓ.

Eqs. (5.1.21) and (5.1.22))

$$u_{\bar{\ell}}(\mathbf{p},\sigma) = \sqrt{\frac{|\mathbf{k}|}{p^0}} \sum_{\ell} D_{\bar{\ell}\ell}\left(\mathscr{L}(p)\right) u_{\ell}(\mathbf{k},\sigma) \,, \tag{5.9.8}$$

$$v_{\bar{\ell}}(\mathbf{p},\sigma) = \sqrt{\frac{|\mathbf{k}|}{p^0}} \sum_{\ell} D_{\bar{\ell}\ell}\left(\mathscr{L}(p)\right) v_{\ell}(\mathbf{k},\sigma) \,, \tag{5.9.9}$$

where $\mathbf{k}$ is a standard momentum, say $(0,0,k)$, and $\mathscr{L}(p)$ is a standard Lorentz transformation that takes a massless particle from momentum $\mathbf{k}$ to momentum $\mathbf{p}$. Also, in place of Eqs. (5.1.23) and (5.1.24), the coefficient functions at the standard momentum must satisfy

$$u_{\bar{\ell}}(\mathbf{k},\sigma)\ \exp\Big(i\sigma\theta(k,W)\Big) = \sum_{\ell} D_{\bar{\ell}\ell}(W) u_{\ell}(\mathbf{k},\sigma) \tag{5.9.10}$$

$$v_{\bar{\ell}}(\mathbf{k},\sigma)\ \exp\Big(-i\sigma\theta(k,W)\Big) = \sum_{\ell} D_{\bar{\ell}\ell}(W) v_{\ell}(\mathbf{k},\sigma) \tag{5.9.11}$$

where $W^{\mu}{}_{\nu}$ is an arbitrary element of the 'little group' for four-momentum $k = (\mathbf{k}, |\mathbf{k}|)$, i.e., an arbitrary Lorentz transformation that leaves this four-momentum invariant.

We can extract the content of Eqs. (5.9.10) and (5.9.11) by considering separately the two kinds of little-group elements in Eq. (2.5.28). For a rotation $R(\theta)$ by an angle θ around the z-axis, given by Eq. (2.5.27),

$$R^{\mu}{}_{\nu}(\theta) = \begin{bmatrix} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \,,$$

we find from Eqs. (5.9.10) and (5.9.11)

$$u_{\bar{\ell}}(\mathbf{k},\sigma) e^{i\sigma\theta} = \sum_{\ell} D_{\bar{\ell}\ell}\Big(R(\theta)\Big) u_{\ell}(\mathbf{k},\sigma) \tag{5.9.12}$$

$$v_{\bar{\ell}}(\mathbf{k},\sigma) e^{-i\sigma\theta} = \sum_{\ell} D_{\bar{\ell}\ell}\Big(R(\theta)\Big) v_{\ell}(\mathbf{k},\sigma) \ . \tag{5.9.13}$$

For combined rotations and boosts $S(\alpha,\beta)$ in the $x-y$ plane, given by (2.5.26),

$$S^{\mu}{}_{\nu}(\alpha,\beta) = \begin{bmatrix} 1 & 0 & -\alpha & \alpha \\ 0 & 1 & -\beta & \beta \\ \alpha & \beta & 1-\gamma & \gamma \\ \alpha & \beta & -\gamma & 1+\gamma \end{bmatrix} \,,$$

$$\gamma \equiv (\alpha^2 + \beta^2)/2 \,,$$

Eqs. (5.9.10) and (5.9.11) give

$$u_{\bar\ell}(\mathbf{k},\sigma) = \sum_{\ell} D_{\bar\ell\ell}\Big(S(\alpha,\beta)\Big)u_\ell(\mathbf{k},\sigma) \ , \tag{5.9.14}$$

$$v_{\bar\ell}(\mathbf{k},\sigma) = \sum_{\ell} D_{\bar\ell\ell}\Big(S(\alpha,\beta)\Big)v_\ell(\mathbf{k},\sigma) \ . \tag{5.9.15}$$

Eqs. (5.9.12)–(5.9.15) are the conditions that determine the coefficient functions u and v at the standard momentum $\mathbf{k}$; Eqs. (5.9.8) and (5.9.9) then give them at arbitrary momenta. The equations for v are just the complex conjugates of the equations for u, so with a suitable adjustment of the constants κ and λ we may normalize the coefficient functions so that

$$v_\ell(\mathbf{p},\sigma) = u_\ell(\mathbf{p},\sigma)^* \ . \tag{5.9.16}$$

The problem is that we cannot find a u_ℓ that satisfies Eq. (5.9.14) for general representations of the homogeneous Lorentz group, even for those representations for which it is possible to construct fields for particles of a given helicity in the case $m \neq 0$.

To see what goes wrong here, let's try to construct the four-vector $[(\frac{1}{2},\frac{1}{2})]$ field for a massless particle of helicity ± 1. In the four-vector representation, we have simply

$$D^\mu{}_\nu(\Lambda) = \Lambda^\mu{}_\nu \ .$$

It is conventional to write the coefficient function u_μ here in terms of a 'polarization vector' e_μ:

$$u_\mu(\mathbf{p},\sigma) \equiv (2p^0)^{-1/2}\, e_\mu(\mathbf{p},\sigma) \ , \tag{5.9.17}$$

so that Eq. (5.9.8) gives

$$e^\mu(\mathbf{p},\sigma) = \mathscr{L}(\mathbf{p})^\mu{}_\nu e^\nu(\mathbf{k},\sigma) \ . \tag{5.9.18}$$

Also, Eqs. (5.9.12) and (5.9.14) read here

$$e^\mu(\mathbf{k},\sigma)\, e^{i\sigma\theta} = R(\theta)^\mu{}_\nu e^\nu(\mathbf{k},\sigma) \ , \tag{5.9.19}$$

$$e^\mu(\mathbf{k},\sigma) = S(\alpha,\beta)^\mu{}_\nu e^\nu(\mathbf{k},\sigma) \ . \tag{5.9.20}$$

Eq. (5.9.19) requires that (up to a constant which can be absorbed into the coefficients κ and λ),

$$e^\mu(\mathbf{k},\pm 1) = (1,\pm i,0,0)/\sqrt{2} \ . \tag{5.9.21}$$

But then Eq. (5.9.20) would require also that $\alpha \pm i\beta = 0$, which is impossible for general real α,β. We therefore cannot satisfy the fundamental requirement (5.9.14) or (5.9.10); instead, we have here

$$D^\mu{}_\nu\Big(W(\theta,\alpha,\beta)\Big)e^\nu(\mathbf{k},\pm 1) = S^\mu{}_\lambda(\alpha,\beta)R^\lambda{}_\nu(\theta)e^\nu(\mathbf{k},\pm 1)$$
$$= \exp(\pm i\theta)\left\{e^\mu(\mathbf{k},\pm 1) + \frac{(\alpha \pm i\beta)}{\sqrt{2}|\mathbf{k}|}k^\mu\right\}. \quad (5.9.22)$$

We have thus come to the conclusion that no four-vector field can be constructed from the annihilation and creation operators for a particle of mass zero and helicity ± 1.

Let's temporarily close our eyes to this difficulty, and go ahead anyway, using Eqs. (5.9.18) and (5.9.21) to define a polarization vector for arbitrary momentum, and take the field as

$$a_\mu(x) = \int d^3p\,(2\pi)^{-3/2}(2p^0)^{-1/2}$$
$$\times \sum_{\sigma=\pm 1}\Big[e_\mu(\mathbf{p},\sigma)e^{ip\cdot x}a(\mathbf{p},\sigma) + e_\mu(\mathbf{p},\sigma)^* e^{-ip\cdot x}a^{c\dagger}(\mathbf{p},\sigma)\Big]. \quad (5.9.23)$$

We will come back later to consider how such a field can be used as an ingredient in a physical theory.

The field (5.9.23) of course satisfies

$$\Box a^\mu(x) = 0\,. \quad (5.9.24)$$

Other properties of the field follow from those of the polarization vector. (We shall need these properties of the polarization vector later when we come to quantum electrodynamics.). Note that the Lorentz transformation $\mathscr{L}(p)$ that takes a massless particle momentum from $\mathbf{k}$ to $\mathbf{p}$ may be written as a 'boost' $\mathscr{B}(|\mathbf{p}|)$ along the z-axis which takes the particle from energy $|\mathbf{k}|$ to energy $|\mathbf{p}|$, followed by a standardized rotation $R(\hat{\mathbf{p}})$ that takes the z-axis into the direction of $\mathbf{p}$. Since $e^\nu(\mathbf{k},\pm 1)$ is a purely spatial vector with only x and y components, it is unaffected by the boost along the z-axis, and so

$$e^\mu(\mathbf{p},\pm 1) = R(\hat{\mathbf{p}})^\mu{}_\nu\, e^\nu(\mathbf{k},\pm 1)\,. \quad (5.9.25)$$

In particular, $e^0(\mathbf{k},\pm 1) = 0$ and $\mathbf{k}\cdot\mathbf{e}(\mathbf{k},\pm 1) = 0$ so

$$e^0(\mathbf{p},\pm 1) = 0 \quad (5.9.26)$$

and

$$\mathbf{p}\cdot\mathbf{e}(\mathbf{p},\pm 1) = 0\,. \quad (5.9.27)$$

It follows that

$$a^0(x) = 0 \quad (5.9.28)$$

and

$$\nabla\cdot\mathbf{a}(x) = 0\,. \quad (5.9.29)$$

As we shall see in Chapter 9, these are the conditions satisfied by the vacuum vector potential of electrodynamics in what is called Coulomb or radiation gauge.

The fact that a^0 vanishes in all Lorentz frames shows vividly that a^μ cannot be a four-vector. Instead, Eq. (5.9.22) shows that for a general momentum $\mathbf{p}$ and a general Lorentz transformation Λ, in place of Eq. (5.9.6) we have

$$e^\mu(\mathbf{p}_\Lambda, \pm 1)\exp(\pm i\theta(\mathbf{p}, \Lambda)) = D^\mu{}_\nu(\Lambda)\, e^\nu(\mathbf{p}, \pm 1) + p^\mu \Omega_\pm(\mathbf{p}, \Lambda)\,, \qquad (5.9.30)$$

so that under a general Lorentz transformation

$$U(\Lambda) a_\mu(x) U^{-1}(\Lambda) = \Lambda^\nu{}_\mu a_\nu(\Lambda x) + \partial_\mu \Omega(x, \Lambda)\,, \qquad (5.9.31)$$

where $\Omega(x, \Lambda)$ is a linear combination of annihilation and creation operators, whose precise form will not concern us here. As we will see in more detail in Chapter 8, we will be able to use a field like $a^\mu(x)$ as an ingredient in Lorentz-invariant physical theories if the couplings of $a^\mu(x)$ are not only formally Lorentz-invariant (that is, invariant under formal Lorentz transformations under which $a^\mu \to \Lambda^\mu{}_\nu\, a^\nu$), but are also invariant under the 'gauge' transformations $a_\mu \to a_\mu + \partial_\mu \Omega$. This is accomplished by taking the couplings of a_μ to be of the form $a_\mu j^\mu$, where j^μ is a four-vector current with $\partial_\mu j^\mu = 0$.

Although there is no ordinary four-vector field for massless particles of helicity ± 1, there is no problem in constructing an antisymmetric tensor field for such particles. From Eq. (5.9.22) and the invariance of k^μ under the little group we see immediately that

$$D^\mu{}_\rho\Big(W(\theta, \alpha, \beta)\Big) D^\nu{}_\sigma\Big(W(\theta, \alpha, \beta)\Big)\Big(k^\rho e^\sigma(\mathbf{k}, \pm 1) - k^\sigma e^\rho(\mathbf{k}, \pm 1)\Big)$$
$$= e^{\pm i\theta}\Big(k^\mu e^\nu(\mathbf{k}, \pm 1) - k^\nu e^\mu(\mathbf{k}, \pm 1)\Big)\ . \qquad (5.9.32)$$

This shows that the coefficient function that satisfies Eq. (5.9.6) for the antisymmetric tensor representation of the homogeneous Lorentz group is (with an appropriate choice of normalization)

$$u^{\mu\nu}(\mathbf{p}, \pm 1) = i(2\pi)^{-3/2}(2p^0)^{-3/2}\left[p^\mu e^\nu(\mathbf{p}, \pm 1) - p^\nu e^\mu(\mathbf{p}, \pm 1)\ \right], \qquad (5.9.33)$$

where $e^\mu(\mathbf{p}, \pm 1)$ is given by Eq. (5.9.25). Using this together with Eq. (5.9.23) gives the general antisymmetric tensor field for massless particles of helicity ± 1 in the form

$$f_{\mu\nu} = \partial_\mu a_\nu - \partial_\nu a_\mu\ . \qquad (5.9.34)$$

Note that this is a tensor even though a^μ is not a four-vector, because the extra term in Eq. (5.9.31) drops out in Eq. (5.9.34). Note also that Eqs. (5.9.34), (5.9.24), (5.9.28), and (5.9.29) show that $f^{\mu\nu}$ satisfies the vacuum Maxwell equations:

$$\partial_\mu f^{\mu\nu} = 0\,, \tag{5.9.35}$$

$$\epsilon^{\rho\sigma\mu\nu}\partial_\sigma f_{\mu\nu} = 0\,. \tag{5.9.36}$$

To calculate the commutation relations for the tensor fields we need sums over helicities of the bilinears $e^\mu e^{\nu *}$. The explicit formula (5.9.21) gives

$$\sum_{\sigma=\pm 1} e^i(\mathbf{k},\sigma)e^j(\mathbf{k},\sigma)^* = \delta_{ij} - \frac{k^i k^j}{|\mathbf{k}|^2}$$

and so, using Eq. (5.9.25),

$$\sum_{\sigma=\pm 1} e^i(\mathbf{p},\sigma)e^j(\mathbf{p},\sigma)^* = \delta_{ij} - \frac{p^i p^j}{|\mathbf{p}|^2}\,. \tag{5.9.37}$$

A straightforward calculation gives then

$$[f_{\mu\nu}(x)\,,\, f_{\rho\sigma}(y)^\dagger] = (2\pi)^{-3}\left[-\eta_{\mu\rho}\partial_\nu\partial_\sigma + \eta_{\nu\rho}\partial_\mu\partial_\sigma + \eta_{\mu\sigma}\partial_\nu\partial_\rho - \eta_{\nu\sigma}\partial_\mu\partial_\rho\right] \times \int d^3p(2p^0)^{-1}\left[|\kappa|^2 e^{ip\cdot(x-y)} - |\lambda|^2 e^{-ip\cdot(x-y)}\right]\,. \tag{5.9.38}$$

This clearly vanishes for $x^0 = y^0$ if and only if

$$|\kappa|^2 = |\lambda|^2 \tag{5.9.39}$$

in which case since $f_{\mu\nu}$ is a tensor the commutator also vanishes for all space-like separations. Eq. (5.9.39) also implies that the commutator of the a^μ vanishes at equal times, and as we shall see in Chapter 8 this is enough to yield a Lorentz-invariant S-matrix. The relative phase of the creation and annihilation operators can be adjusted so that $\kappa = \lambda$; the fields are then Hermitian if the particles are their own charge-conjugates, as is the case for the photon.

Why should we want to use fields like $a^\mu(x)$ in constructing theories of massless particles of spin one, rather than being content with fields like $f^{\mu\nu}(x)$ with simple Lorentz transformation properties? The presence of the derivatives in Eq. (5.9.34) means that an interaction density constructed solely from $f_{\mu\nu}$ and its derivatives will have matrix elements that vanish more rapidly for small massless particle energy and momentum than one that uses the vector field a_μ. Interactions in such a theory will have a correspondingly rapid fall-off at large distances, faster than the usual inverse-square law. This is perfectly possible, but gauge-invariant theories that use vector fields for massless spin one particles represent a more general class of theories, including those that are actually realized in nature.

Parallel remarks apply to gravitons, massless particles of helicity ± 2. From the annihilation and creation operators for such particles we can construct a tensor $R_{\mu\nu\rho\sigma}$ with the algebraic properties of the Riemann–Christoffel curvature tensor: antisymmetric within the pairs μ,ν and ρ,σ, and symmetric between the pairs. However, in order to incorporate the usual inverse-square gravitational interactions we need to introduce a field $h_{\mu\nu}$ that transforms as a symmetric tensor, up to gauge transformations of the sort associated in general relativity with general coordinate transformations. Thus in order to construct a theory of massless particles of helicity ± 2 that incorporates long-range interactions, it is necessary for it to have a symmetry something like general covariance. As in the case of electromagnetic gauge invariance, this is achieved by coupling the field to a conserved 'current' $\theta^{\mu\nu}$, now with two spacetime indices, satisfying $\partial_\mu \theta^{\mu\nu} = 0$. The only such conserved tensor is the energy-momentum tensor, aside from possible total derivative terms that do not affect the long-range behavior of the force produced.** The fields of massless particles of spin $j \geq 3$ would have to couple to conserved tensors with three or more spacetime indices, but aside from total derivatives there are none, so *high-spin massless particles cannot produce long-range forces.*

* * *

The problems we have encountered in constructing four-vector fields for helicities ± 1 or symmetric tensor fields for helicity ± 2 are just special cases of a more general limitation. To see this, let's consider how to construct fields for massless particles belonging to arbitrary representations of the homogeneous Lorentz group. As we saw in Section 5.6, any representation $D(\Lambda)$ of the homogeneous Lorentz group can be decomposed into $(2A+1)(2B+1)$-dimensional representations (A, B), for which the generators of the homogeneous Lorentz group are represented by

$$(\mathscr{J}_{ij})_{a'b',ab} = \epsilon_{ijk}\left[(J_k^{(A)})_{a'a}\,\delta_{b'b} + (J_k^{(B)})_{b'b}\,\delta_{a'a}\right],$$
$$(\mathscr{J}_{k0})_{a'b',ab} = -i\left[(J_k^{(A)})_{a'a}\,\delta_{b'b} - (J_k^{(B)})_{b'b}\,\delta_{a'a}\right],$$

where $\mathbf{J}^{(j)}$ are the angular-momentum matrices for spin j. For θ infinites-

** If $\theta^{\mu_1\cdots\mu_N}$ is a tensor current satisfying $\partial_{\mu_1}\theta^{\mu_1\cdots\mu_N} = 0$, then $\int d^3x\,\theta^{0\mu_2\cdots\mu_N}$ is a conserved quantity that transforms like a tensor of rank $N-1$. The only such conserved tensors are the scalar 'charges' associated with various continuous symmetries, and the energy-momentum four-vector. The conservation of any other four-vector, or any tensor of higher rank, would forbid all but forward collisions.

imal, $D(R(\theta)) = 1 + i\mathscr{J}_{12}\theta$, so Eqs. (5.9.12) and (5.9.13) give

$$\sigma u_{ab}(\mathbf{k},\sigma) = (a+b)u_{ab}(\mathbf{k},\sigma)\,,$$

$$-\sigma v_{ab}(\mathbf{k},\sigma) = (a+b)v_{ab}(\mathbf{k},\sigma)\,,$$

and so $u_{ab}(\mathbf{k},\sigma)$ and $v_{ab}(\mathbf{k},\sigma)$ must vanish unless $\sigma = a+b$ and $\sigma = -a-b$, respectively. Also, letting α and β become infinitesimal in Eq. (5.9.14) gives

$$\begin{aligned}
0 &= (\mathscr{J}_{31} + \mathscr{J}_{01})_{ab,a'b'}\; u_{a'b'}(\mathbf{k},\sigma) \\
&= (J_2^{(A)} + iJ_1^{(A)})_{aa'}u_{a'b}(\mathbf{k},\sigma) + (J_2^{(B)} - iJ_1^{(B)})_{bb'}u_{ab'}(\mathbf{k},\sigma)\,, \\
0 &= (\mathscr{J}_{32} + \mathscr{J}_{02})_{ab,a'b'}\; u_{a'b'}(\mathbf{k},\sigma) \\
&= (-J_1^{(A)} + iJ_2^{(A)})_{aa'}u_{a'b}(\mathbf{k},\sigma) + (-J_1^{(B)} - iJ_2^{(B)})_{bb'}u_{ab'}(\mathbf{k},\sigma)\,,
\end{aligned}$$

or more simply

$$\left(J_1^{(A)} - i\,J_2^{(A)}\right)_{aa'}\, u_{a'b}(\mathbf{k},\sigma) = 0\,,$$

$$\left(J_1^{(B)} + i\,J_2^{(B)}\right)_{bb'}\, u_{ab'}(\mathbf{k},\sigma) = 0\quad.$$

These require that $u_{ab}(\mathbf{k},\sigma)$ vanishes unless

$$a = -A,\quad b = +B \tag{5.9.40}$$

and the same is obviously also true of $v_{ab}(\mathbf{k},\sigma)$. Putting this together, we see that a field of type (A,B) can be formed only from the annihilation operators for a massless particle of helicity σ and the creation operators for the antiparticle of helicity $-\sigma$, where

$$\sigma = B - A\,. \tag{5.9.41}$$

For instance, the $(\frac{1}{2},0)$ and $(0,\frac{1}{2})$ parts of the Dirac field for a massless particle can only destroy particles of helicity $-\frac{1}{2}$ and $+\frac{1}{2}$ respectively, and create antiparticles of helicity $+\frac{1}{2}$ and $-\frac{1}{2}$, respectively. In the 'two-component' theory of the neutrino, there is only a $(\frac{1}{2},0)$ field and its adjoint, so neutrinos have helicity $-\frac{1}{2}$ and antineutrinos helicity $+\frac{1}{2}$ in this theory.

By the same methods as in Section 5.7, it can be shown that the $(j,0)$ and $(0,j)$ fields for massless particles of spin j (i.e., helicity $\mp j$) commute with each other and their adjoints at space-like separations if the coefficients of the annihilation and creation terms in Eq. (5.9.1) satisfy Eq. (5.9.39). The relative phase of the annihilation and creation operators may then be adjusted so that these coefficients are equal. It is easy to see that the fields for a massless particle of spin j of type $(A, A+j)$ or $(B+j, B)$ are just the $2A$th or $2B$th derivatives of fields of type $(0,j)$ or $(j,0)$, respectively, so these more general fields do not need to be considered separately here.

We can now see why it was impossible to construct a vector field for massless particles of helicity ± 1. A vector field transforms according to the $(\frac{1}{2}, \frac{1}{2})$ representation, and hence according to Eq. (5.9.41) can only describe helicity zero. (It *is*, of course, possible to construct a vector field for helicity zero — just take the derivative $\partial_\mu \phi$ of a massless scalar field ϕ.) The simplest covariant massless field for helicity ± 1 has the Lorentz transformation type $(1,0) \oplus (0,1)$; that is, it is an antisymmetric tensor $f_{\mu\nu}$. Similarly, the simplest covariant massless field for helicity ± 2 has the Lorentz transformation type $(2,0) \oplus (0,2)$: a fourth rank tensor which like the Riemann–Christoffel curvature tensor is antisymmetric within each pair of indices and symmetric between the two pairs.

The discussion of the inversions P, C, T given in the previous section can be carried over to the case of zero mass with only obvious modifications.

Problems

1. Show that if the zero-momentum coefficient functions satisfy the conditions (5.1.23) and (5.1.24), then the coefficient functions (5.1.21) and (5.1.22) for arbitrary momentum satisfy the defining conditions Eqs. (5.1.19) and (5.1.20).

2. Consider a free field $\psi^\mu_\ell(x)$ which annihilates and creates a self-charge-conjugate particle of spin $\frac{3}{2}$ and mass $m \neq 0$. Show how to calculate the coefficient functions $u^\mu_\ell(\mathbf{p}, \sigma)$, which multiply the annihilation operators $a(\mathbf{p}, \sigma)$ in this field, in such a way that the field transforms under Lorentz transformations like a Dirac field ψ_ℓ with an extra four-vector index μ. What field equations and algebraic and reality conditions does this field satisfy? Evaluate the matrix $P^{\mu\nu}(p)$, defined (for $p^2 = -m^2$) by

$$\sum_\sigma u^\mu_\ell(\mathbf{p}, \sigma) u^{\nu\,*}_m(\mathbf{p}, \sigma) \equiv (2p^0)^{-1} P^{\mu\nu}_{\ell m}(p).$$

   What are the commutation relations of this field? How does the field transform under the inversions P, C, T?

3. Consider a free field $h^{\mu\nu}(x)$ satisfying $h^{\mu\nu}(x) = h^{\nu\mu}(x)$ and $h^\mu{}_\mu(x) = 0$, which annihilates and creates a particle of spin two and mass $m \neq 0$. Show how to calculate the coefficient functions $u^{\mu\nu}(\mathbf{p}, \sigma)$, which multiply the annihilation operators $a(\mathbf{p}, \sigma)$ in this field, in such a way that the field transforms under Lorentz transformations like a tensor. What field equations does this field satisfy? Evaluate the function $P^{\mu\nu,\kappa\lambda}(p)$, defined by

$$\sum_\sigma u^{\mu\nu}(\mathbf{p}, \sigma) u^{\kappa\lambda *}(\mathbf{p}, \sigma) \equiv (2p^0)^{-1} P^{\mu\nu,\kappa\lambda}(p).$$