Non-Relativistic Field Theory

What is the non-relativistic limit of the theory

$$\mathcal{L} = \partial\Phi^\dagger \partial\Phi - m^2 \Phi^\dagger \Phi - \lambda (\Phi^\dagger \Phi)^2 \quad ?$$

The equation of motion for $\lambda = 0$ is

$$(\partial^2 + m^2)\Phi = 0.$$

A mode with energy $E = m + \epsilon$ goes as $\Phi = e^{-iEt}$. So we let

$$\Phi(x,t) = e^{-imt}\phi(x,t)$$

Then $\partial_0 \Phi(x,t) = -im\, e^{-imt}\phi(x,t) + e^{-imt}\partial_0\phi(x,t)$

So

$$\partial_0^2 \Phi = \partial_0\, e^{-imt}(-im\phi + \partial_0\phi)$$

$$= \left[(-im)^2 - 2im\,\partial_0\phi + \partial_0^2\phi\right] e^{-imt}$$

and $(\partial^2 + m^2)\Phi = 0$ becomes

$$e^{-imt}\left[-m^2 - 2mi\,\partial_0\phi + \partial_0^2\phi\right] - \nabla^2 e^{-imt}\phi + m^2\Phi = 0$$

these cancel

we drop $\partial_0^2\phi$ compared to $-2mi\partial_0\phi$, and

So we get

$$-2mi\partial_0\varphi - \nabla^2\varphi = 0$$

$$i\partial_0\varphi = -\frac{\nabla^2}{2m}\varphi$$

which is Schrödinger's equation.

So to take the non-relativistic limit of a field theory, we replace

$$\Phi(x,t) = \frac{1}{\sqrt{2m}} e^{-imt} \varphi(x,t)$$

so that

$$\frac{\partial\Phi^*}{\partial t}\frac{\partial\Phi}{\partial t} - m^2\Phi^*\Phi \rightarrow \frac{1}{2m}\left\{\left[\left(im+\frac{\partial}{\partial t}\right)\varphi^*\right]\left[\left(-im+\frac{\partial}{\partial t}\right)\varphi\right] - m^2\varphi^*\varphi\right\}$$

$$\approx \frac{i}{2}\left(\varphi^*\frac{\partial\varphi}{\partial t} - \frac{\partial\varphi^*}{\partial t}\varphi\right).$$

Then $\mathcal{L}$ becomes

$$\mathcal{L} = \partial\Phi^*\partial\Phi - m^2\Phi^*\Phi - \lambda(\Phi^*\Phi)^2$$

$$= \frac{i}{2}(\varphi^*\partial_0\varphi - \partial_0\varphi^*\varphi) - \frac{1}{2m}\partial_i\varphi^*\partial_i\varphi - \frac{\lambda}{4m^2}(\varphi^*\varphi)^2$$

$$= i\varphi^*\partial_0\varphi - \frac{1}{2m}\nabla\varphi^*\cdot\nabla\varphi - g^2(\varphi^*\varphi)^2 \quad \text{with } g^2 = \frac{\lambda}{4m^2}.$$

The relativistic theory has a conserved current

$$J_\mu = i(\Phi^* \partial_\mu \Phi - \partial_\mu \Phi^* \Phi).$$

And $J_0 \approx \phi^* \phi$, while $J_i \approx \frac{i}{2m}(\phi^* \partial_i \phi - \partial_i \phi^* \phi)$.

In the N.R theory,

$$\Pi_\phi = \frac{\partial \mathcal{L}}{\partial\, \partial_0 \phi} = i\phi^*$$

so

$$[\phi(x,t), \Pi_\phi(x',t)] = i\,\delta(x-x')$$

gives

$$[\phi(x,t), i\phi^*(x',t)] = i\,\delta(x-x')$$

or

$$[\phi(x,t), \phi^*(x',t)] = \delta(x-x').$$

In condensed-matter physics, there's often a mean density ρ, so it's useful to write

$$\phi = \sqrt{\rho}\, e^{i\theta}$$ after which

$$\mathcal{L} = i\sqrt{\rho}\, e^{-i\theta} \partial_0(\sqrt{\rho}\, e^{i\theta}) - \frac{1}{2m}\left[\nabla(\sqrt{\rho}\, e^{-i\theta}) \cdot \nabla \sqrt{\rho}\, e^{+i\theta}\right] - g^2 \rho^2$$

$$= \frac{i}{2}\dot{\rho} - \rho\, \partial_0 \theta - \frac{1}{2m}\left[\rho(\nabla\theta)^2 + \frac{1}{4\rho}(\nabla\rho)^2\right] - g^2\rho^2$$

total divergence, which we can drop.

Now $\quad \pi_\theta = \dfrac{\partial \mathcal{L}}{\partial \dot\theta} = -\rho$ $\quad$ so

$$[\theta(x,t), -\rho(x',t)] = i\,\delta(x-x')$$

so

$$[\rho(x,t), \theta(x',t)] = i\,\delta(x-x').$$

Integrate: $\quad N = \int d^3x\, \rho(x,t)$ $\quad$ get

$$[N, \theta(x',t)] = i$$

Number is conjugate to phase.

Incidentally the physics of these two $\mathcal{L}$'s is the same:

$$\mathcal{L} = |\partial\phi|^2 - m^2|\phi|^2 - \lambda|\phi|^4 \quad \text{and}$$

$$\mathcal{L} = |\partial\phi|^2 - m^2|\phi|^2 + 2\sigma|\phi|^2 + \frac{1}{\lambda}\sigma^2$$

which some call a Hubbard-Stratonovich transformation.

The reason is that σ has no kinetic term and so its field equation is

$$0 = \frac{\partial \mathcal{L}}{\partial \sigma} = 2|\phi|^2 + \frac{2\sigma}{\lambda} = 0$$

$$\text{or} \quad \sigma = -\lambda|\phi|^2.$$

Note that the field ϕ is repulsive.

$$\mathcal{L} \sim -\lambda|\phi|^4 \quad \text{so} \quad H \sim \lambda|\phi|^4.$$

Making piles of $|\phi|^2$ costs energy.

In condensed-matter physics, one often wants a finite density $\bar{\rho}$ in the system. So the N-R $\mathcal{L}$ is

$$\mathcal{L} = i\phi^* \partial_0 \phi - \frac{1}{2m} \nabla\phi^* \cdot \nabla\phi - g^2(\phi^*\phi - \bar{\rho})^2.$$