The action of a plaquette (little square) is made from the trace of the path-ordered product along the links around the plaquette of group elements $\exp(-igaA)$ in which A is a linear combination of the $n \times n$ generators t_a of the gauge group multiplied by the fields A_i^a in the direction of the link, $A_i = t_a A_i^a$. If the center of the plaquette is x, then the action for a plaquette in the 1-2 plane is

$$\begin{aligned} S_\square = \beta \{ 1 - (1/n) \, \mathrm{Re}\, \mathrm{Tr} \, [\, & \exp(-igaA_1(x - aj/2)) \, \exp(-igaA_2(x + ai/2)) \\ & \times \exp(igaA_1(x + aj/2)) \, \exp(igaA_2(x - ai/2)) \,] \, \} \end{aligned} \tag{1}$$

where $ai/2$ adds $a/2$ to x_1, and similarly $aj/2$ adds $a/2$ to x_2.

Expand the exponentials to order a^2 and show that the product of the four of them to that order is

$$\exp\left(-i \, g \, a^2 \, F_{12}\right) \tag{2}$$

in which to order a^2

$$\begin{aligned} F_{12} = & \frac{A_2(x + ai/2) - A_2(x - ai/2)}{a} - \frac{A_1(x + aj/2) - A_1(x - aj/2)}{a} \\ & - i \, g [A_1(x), A_2(x)] \\ \approx & \; \partial_1 A_2(x) - \partial_2 A_1(x) - i \, g [A_1(x), A_2(x)]. \end{aligned} \tag{3}$$

Solution: One can expand the four exponentials to order a^2, but it is faster to use the formula

$$e^{aA} \, e^{aB} \approx e^{a(A+B) + (a^2/2)[A,B]} \tag{4}$$

which holds to order a^2 and is exact when $[A, B]$ commutes with both A and B. Thus to order a^2, the product of the first two exponentials is

$$\begin{aligned} & \exp(-igaA_1(x - aj/2)) \exp(-igaA_2(x + ai/2)) \approx \\ & \exp \{ -iga \, [A_1(x - aj/2) + A_2(x + ai/2) - iga[A_1(x), A_2(x)]/2] \} \, . \end{aligned} \tag{5}$$

Similarly, the product of the second two exponentials to order a^2 is

$$\begin{aligned} & \exp(igaA_1(x + aj/2)) \exp(igaA_2(x - ai/2)) \approx \\ & \quad \exp \{ iga \, [A_1(x + aj/2) + A_2(x - ai/2) - iga[A_1(x), A_2(x)]/2] \} \, . \end{aligned} \tag{6}$$

Thus the product of all four exponentials is to order a^2

$$\exp\left\{-iga^2\left[\frac{A_2(x+ai/2)-A_2(x-ai/2)}{a}-\frac{A_1(x+aj/2)-A_1(x-aj/2)}{a}\right]\right.$$
$$\left.-\,i\,g[A_1(x),A_2(x)]\right\} = \exp\left(-i\,g\,a^2\,F_{12}\right). \tag{7}$$

If the matrices of the representation have unit determinant, as they will for the special unitary groups $SU(N)$ and the special orthogonal groups $SO(N)$, then the generators t_a are traceless, and so to order a^4

$$\mathrm{Tr}\,\exp\left(-i\,g\,a^2\,F_{12}\right) = n - \frac{1}{2}\,g^2\,a^4\,\mathrm{Tr}\,F_{12}^2. \tag{8}$$

Thus the action of the plaquette is

$$S_\Box = \frac{\beta g^2}{2n}\,a^4\,\mathrm{Tr}\,F_{12}^2. \tag{9}$$

So replacing a^4 by d^4x and summing over all six plaquettes at each vertex of the lattice, we get in the $a \to 0$ limit

$$S = \frac{\beta g^2}{2n}\int \frac{1}{2}\,\mathrm{Tr}\,F_{\mu\nu}^2\,d^4x \tag{10}$$

in which the factor of two lets us sum over all $\mu\nu$ pairs.

In the above discussion, we used the definition

$$F_{\mu\nu}(x) \equiv \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) - i\,g[A_\mu(x), A_\nu(x)]. \tag{11}$$

Another convention is to set

$$F_{\mu\nu}(x) \equiv \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) + i\,g[A_\mu(x), A_\nu(x)]. \tag{12}$$

This convention has the advantage that the plaquette action is

$$\begin{aligned} S_\Box = \beta\,\{1-(1/n)\,\mathrm{Re}\,\mathrm{Tr}\,[\,&\exp(igaA_1(x-aj/2))\,\exp(igaA_2(x+ai/2)) \\ &\times\exp(-igaA_1(x+aj/2))\,\exp(-igaA_2(x-ai/2))\,]\,\} \end{aligned} \tag{13}$$

which is more natural.