The action of a plaquette (little square) is made from the trace of the path-ordered product along the links around the plaquette of group elements $\exp(-igaA)$ in which A is a linear combination of the $n \times n$ generators t_a of the gauge group multiplied by the fields A_i^a in the direction of the link, $A_i = t_a A_i^a$. If the center of the plaquette is x, then the action for a plaquette in the 1-2 plane is

$$\begin{aligned} S_\square = \beta \{ 1 - (1/n) \, \mathrm{Re}\, \mathrm{Tr} \, [\exp(-igaA_1(x - aj/2)) \, \exp(-igaA_2(x + ai/2)) \\ \times \exp(igaA_1(x + aj/2)) \, \exp(igaA_2(x - ai/2)) \,] \, \} \end{aligned} \tag{1}$$

where $ai/2$ adds $a/2$ to x_1, and similarly $aj/2$ adds $a/2$ to x_2.

Expand the exponentials to order a^2 and show that the product of the four of them to that order is

$$\exp\left(-i \, g \, a^2 \, F_{12}\right) \tag{2}$$

in which to order a^2

$$\begin{aligned} F_{12} = & \frac{A_2(x + ai/2) - A_2(x - ai/2)}{a} - \frac{A_1(x + aj/2) - A_1(x - aj/2)}{a} \\ & - i \, g [A_1(x), A_2(x)] \\ \approx & \, \partial_1 A_2(x) - \partial_2 A_1(x) - i \, g [A_1(x), A_2(x)]. \end{aligned} \tag{3}$$

If the matrices of the representation have unit determinant, as they will for the special unitary groups $SU(N)$ and the special orthogonal groups $SO(N)$, then the generators t_a are traceless, and so to order a^4

$$\mathrm{Tr} \, \exp\left(-i \, g \, a^2 \, F_{12}\right) = n - \frac{1}{2} \, g^2 \, a^4 \, \mathrm{Tr} \, F_{12}^2. \tag{4}$$

Thus the action of the plaquette is

$$S_\square = \frac{\beta g^2}{2n} \, a^4 \, \mathrm{Tr} \, F_{12}^2. \tag{5}$$

So replacing a^4 by $d^4 x$ and summing over all six plaquettes at each vertex of the lattice, we get in the $a \to 0$ limit

$$S = \frac{\beta g^2}{2n} \int \frac{1}{2} \, \mathrm{Tr} \, F_{\mu\nu}^2 \, d^4 x \tag{6}$$

in which the factor of two lets us sum over all $\mu\nu$ pairs.

In the above discussion, we used the definition

$$F_{\mu\nu}(x) \equiv \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) - i\, g[A_\mu(x), A_\nu(x)]. \tag{7}$$

Another convention is to set

$$F_{\mu\nu}(x) \equiv \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) + i\, g[A_\mu(x), A_\nu(x)]. \tag{8}$$

An advantage of this convention is that we can define the plaquette action as

$$\begin{aligned} S_\Box = \beta\, \{1 - (1/n)\, \mathrm{Re}\,\mathrm{Tr}\, [\, &\exp(igaA_1(x - aj/2))\, \exp(igaA_2(x + ai/2)) \\ &\times \exp(-igaA_1(x + aj/2))\, \exp(-igaA_2(x - ai/2))\,]\, \} \end{aligned} \tag{9}$$

which is more natural.