

$$\begin{aligned} \langle x, t | x_0 \rangle &= \langle x | e^{-iH_0 t} | x_0 \rangle \\ &= \langle x | e^{-i(p^2/2m)t} | x_0 \rangle \\ &= \left(\frac{m}{2\pi i \epsilon} \right)^{3/2} e^{im(x-x_0)^2/2t} \end{aligned}$$

on with S.R.

$$\begin{aligned} \langle x, t | x_0 \rangle &= \langle x | e^{-i\sqrt{p^2+m^2}t} | x_0 \rangle \\ &= \frac{1}{2\pi^2 |x-x_0|} \int_0^\infty p \sin(px-x_0) e^{-ip\sqrt{p^2+m^2}t} dp \\ &\sim e^{-im\sqrt{x^2-t^2}} \quad \text{for } x^2 > t^2. \end{aligned}$$

Also, precise measurement requires short λ which means particle creation as $\lambda \approx 0$.

$$P-S \text{ use } p^2 = p^0{}^2 - \vec{p}^2 = m^2.$$

$$W \& S \quad \not{p}^2 = \vec{p}^2 - p^0{}^2 = -m^2.$$

$$S = \int d^4x \mathcal{L}(\phi, \partial\phi)$$

$$\delta S[\phi][h] = \frac{d}{d\epsilon} S[\phi + \epsilon h] \Big|_{\epsilon=0}$$

$$= \frac{d}{d\epsilon} \int d^4x \mathcal{L}(\phi + \epsilon h, \partial_m \phi + \epsilon \partial_m h) \Big|_{\epsilon=0}$$

$$= \frac{d}{d\epsilon} \int d^4x \mathcal{L} + \frac{\partial \mathcal{L}}{\partial \phi} \epsilon h + \frac{\partial \mathcal{L}}{\partial \partial_m \phi} \epsilon \partial_m h \Big|_{h=0}$$

$$= \int d^4x \frac{\partial \mathcal{L}}{\partial \phi} h + \frac{\partial \mathcal{L}}{\partial \partial_m \phi} \partial_m h$$

(+) (1+)

Now let $h \rightarrow \delta(x-y)$

$$\delta S[\phi][\delta h] = \int d^4x \left(\frac{\partial \mathcal{L}}{\partial \phi} - \partial_m \frac{\partial \mathcal{L}}{\partial \partial_m \phi} \right) h$$

$$\rightarrow \int d^4x \left(\frac{\partial \mathcal{L}}{\partial \phi} - \partial_m \frac{\partial \mathcal{L}}{\partial \partial_m \phi} \right) \delta(x-y)$$

(+)

$$\partial_m \left(\frac{\partial \mathcal{L}}{\partial \partial_m \phi} \right) = \frac{\partial \mathcal{L}}{\partial \phi}$$

$$\frac{\delta S[\phi]}{\delta \phi(y)} = \frac{d}{dG} S[\phi + \epsilon \delta_y] \quad \text{where } S[y] = S[x-y].$$

$[q(+), p(+)] = i$ goes to

$$[\phi(x, t), \pi(x', t')] = i \delta^{(3)}(x - x').$$

Nice, but the extension to spin $1/2$ and gauge fields is awkward and complicated.

Just as $p = \frac{\partial L}{\partial \dot{q}}$, so too

$$\begin{aligned} \pi &= \pi(\vec{x}, t) = \frac{\delta S}{\delta \dot{\phi}} = \frac{d}{d\epsilon} \int L(\phi, \nabla \phi, \dot{\phi} + \epsilon h) d^4x \\ &= \int \frac{\partial L}{\partial \dot{\phi}} h d^4x \rightarrow \left. \frac{\partial L}{\partial \dot{\phi}} \right|_{\vec{x}, t} \end{aligned}$$

or

$$\pi(\vec{x}, t) = \frac{\delta L}{\delta \dot{\phi}} = \frac{d}{d\epsilon} \int L(\phi, \nabla \phi, \dot{\phi} + \epsilon h) d^3x$$

$$= \int \frac{\partial L}{\partial \dot{\phi}} \epsilon h d^3x = \int \frac{\partial L}{\partial \dot{\phi}} h d^3x \rightarrow \frac{\partial L}{\partial \dot{\phi}}$$

Just as $H = p_i \dot{q}_i - L$, so too

$$H = \int d^3x [\pi \dot{\phi} - L]. \text{ One can use } \pi = \frac{\partial L}{\partial \dot{\phi}}.$$

$$\begin{aligned} \mathcal{L} &= \frac{1}{2} \dot{\phi}^2 - \frac{1}{2} \nabla \phi^2 - \frac{1}{2} m^2 \phi^2 \\ &= \frac{1}{2} \partial_m \phi \partial^m \phi - \frac{1}{2} m^2 \phi^2 \end{aligned}$$

we often write this as $\mathcal{L} = \frac{1}{2} (\partial_m \phi)^2 - \frac{1}{2} m^2 \phi^2$
for short.

$$\frac{\partial \mathcal{L}}{\partial \partial_m \phi} = \partial^m \phi$$

so

$$\partial_x \frac{\partial \mathcal{L}}{\partial \partial_m \phi} = \partial_m \partial^m \phi = \frac{\partial \mathcal{L}}{\partial \phi} = -m^2 \phi^2$$

so

$$\partial_m \partial^m \phi + m^2 \phi = 0$$

$$(\square + m^2) \phi = 0$$

$$\left(\frac{\partial^2}{\partial t^2} - \nabla^2 + m^2 \right) \phi = 0.$$

$$H = \int d^3x = \int d^3x \pi \dot{\phi} - \mathcal{L} = \int d^3x \pi^2 - \mathcal{L}$$

$$= \int d^3x \frac{\pi^2}{2} + \frac{\nabla \phi^2}{2} + \frac{m^2}{2} \phi^2$$

from $\sum_x \dot{g}_{xx}^2 + (\vec{g}_x \cdot \vec{g}_\phi)^2 \rightarrow \dot{g}_{xx}^2 + \cancel{\text{term}} (\nabla \phi)^2$

$$\phi' = \phi + \epsilon \Delta \phi$$

$$\mathcal{L}' = \mathcal{L} + \epsilon \partial_m T^m$$

$$\begin{aligned} \epsilon \partial_m T^m &= \frac{\partial h}{\partial \phi} \epsilon \Delta \phi + \frac{\partial L}{\partial \partial_m \phi} \partial_m (\epsilon \Delta \phi) \\ &= \frac{\partial h}{\partial \phi} \epsilon \Delta \phi + \partial_m \left(\frac{\partial h}{\partial \partial_m \phi} \epsilon \Delta \phi \right) - \cancel{\epsilon \partial_m \partial_m \frac{\partial h}{\partial \partial_m \phi}} \\ &\quad \text{[cancelation]} \\ &= \epsilon \partial_m \left(\frac{\partial h}{\partial \partial_m \phi} \Delta \phi \right) \end{aligned}$$

So the current

$$J^m = \Delta \phi \frac{\partial h}{\partial \partial_m \phi} - T^m$$

is conserved due to the symmetry

and the equations of motion

$$0 = \partial_m J^m = J^0 + \vec{\nabla} \cdot \vec{J}$$

$Q = \int d^3x J^0$ is conserved in time

$$\dot{Q} = \int d^3x \vec{J}^0 = - \int \vec{\nabla} \cdot \vec{J} d^3x = \int \vec{J} \cdot \vec{d}\sigma = 0$$

as long as \vec{J} at spatial ∞ vanishes.

Real fields ϕ_i :

Say $\phi'_i = D_{ij} \phi_j$ is a symmetry of $\mathcal{L}(x)$

$$\mathcal{L}(\phi', \partial_\mu \phi') = \mathcal{L}(\phi, \partial_\mu \phi).$$

Then $T^\mu = 0$ and

$$J^\mu = \Delta \phi_i \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_i}$$

$$= \left(\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_i} \right) D_{ij} \phi_j \text{ is conserved}$$

$$0 = \partial_\mu J^\mu.$$

Complex fields ψ_i : $\psi'_i = D_{ij} \psi_j$

$$\psi'_i = \delta_{ij}^* \psi_j^*$$

$$J^\mu = \Delta \psi_i \frac{\partial \mathcal{L}}{\partial \partial_\mu \psi_i} + \Delta \psi_i^* \frac{\partial \mathcal{L}}{\partial \partial_\mu \psi_i^*}$$

$$0 = \partial_\mu J^\mu. \quad \text{Example.}$$

$$\mathcal{L} = \partial_\mu \psi^* \partial^\mu \psi - m^2 \psi^* \psi$$

$$\psi' = e^{i\theta} \psi \quad \psi'^* = e^{-i\theta} \psi^*$$

$$J^\mu = i\theta \psi \partial^\mu \psi^* - i\theta \psi^* \partial^\mu \psi.$$

As a second example, consider

$$\phi' = \phi'(x) = \phi(x+a) = \phi(x) + a^\mu \partial_\mu \phi(x).$$

$$\mathcal{L}' \rightarrow \mathcal{L}' = \mathcal{L} + a^\mu \partial_\mu \mathcal{L}$$

$$= \mathcal{L} + a^\nu \partial_\mu (\delta^\mu_\nu \mathcal{L})$$

Then

$$a^\nu \partial_\mu (\delta^\mu_\nu \mathcal{L}) = \frac{\partial \mathcal{L}}{\partial \phi} a^\mu \partial_\mu \phi$$

$$+ \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \partial_\mu (a^\nu \partial_\nu \phi)$$

$$= \frac{\partial \mathcal{L}}{\partial \phi} a^\mu \partial_\mu \phi + \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} a^\nu \partial_\nu \phi \right] - a^\nu \partial_\nu \phi \partial_\mu \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi}$$

$$= a^\nu \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \partial_\nu \phi \right]$$

$$\text{So } T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi} \partial_\nu \phi - \delta^{\mu\nu} \cancel{\partial_\mu \mathcal{L}}$$

amounts to four conserved currents

$$0 = \partial_\mu T^{\mu\nu} \quad \text{for } \nu = 0, 1, 2, 3.$$

More generally if

$$\phi'_i = \phi_i(x+a) \text{ then}$$

$$T^{\mu}_{\nu} = \frac{\partial L}{\partial \partial_{\mu} \phi_i} \partial_{\nu} \phi_i - \delta^{\mu}_{\nu} L.$$

Note that the case $L = R\sqrt{-g}$

with $\phi_i \rightarrow g_{\mu\nu}$ is a special case

$$T^{\mu}_{\nu} = \frac{\partial (R\sqrt{-g})}{\partial \partial_{\mu} g_{ab}} \partial_{\nu} g_{ab} - \delta^{\mu}_{\nu} R\sqrt{-g}.$$

$$H = \int T^{00} d^3x = \int \rho / d^3x$$

$$P^i = \int T^{0i} d^3x = - \int \pi \partial_i \phi / d^3x$$

in simplest case.

$$[q_i, p_j] = i \delta_{ij} \quad [q_i, q_j] = [p_i, p_j] = 0$$

$$[\phi(x), \pi(y)] = i \delta^{(3)}(x-y) \quad \text{in Schrödinger picture}$$

$$[\phi(x), \phi(y)] = 0 = [\pi(x), \pi(y)]$$

Write

$$\begin{aligned} H &= \int \left(\frac{1}{2} \pi^2 + \frac{1}{2} \nabla \phi^2 + \frac{1}{2} m^2 \phi^2 \right) d^3 x \\ &= \frac{1}{2} \int \left[\sqrt{-\nabla^2 + m^2} \phi - i\pi \right] \left[\sqrt{-\nabla^2 + m^2} \phi + i\pi \right] d^3 x \\ &\quad + \underbrace{\frac{i}{2} \int [\pi, \sqrt{-\nabla^2 + m^2} \phi] d^3 x} \end{aligned}$$

We ignore this big constant. Set

$$H = \frac{1}{2} \int [\sqrt{-\nabla^2 + m^2} \phi - i\pi] [\sqrt{-\nabla^2 + m^2} \phi + i\pi] d^3 x$$

which is a non-negative hermitian operator. Use basis

$$|\phi'\rangle : \quad \phi(x) |\phi'\rangle = \phi'(x) |\phi'\rangle$$

↗ ↘

Sch. operator

function

Want

$$\langle \phi' | \sqrt{-\nabla^2 + m^2} \phi + i\pi | 0 \rangle = 0$$

Just as $p = \frac{1}{i} \frac{\partial}{\partial q}$ so $\pi =$

$$\pi = \frac{1}{i} \frac{s}{s\phi} . \quad s =$$

$$\frac{s \langle \phi' | 0 \rangle}{s \phi'(x)} = -\sqrt{-\nabla^2 + m^2} \phi'(x) \langle \phi' | 0 \rangle$$

$$-\frac{1}{2} \int \phi'(x) \sqrt{-\nabla^2 + m^2} \phi'(x) d^3x$$

$$\langle \phi' | 0 \rangle = N e$$

$$\phi'(x) = \int e^{ip \cdot x} \hat{\phi}'(p) \frac{d^3 p}{(2\pi)^3}$$

$$-\frac{1}{2} \int |\hat{\phi}'(p)|^2 \sqrt{p^2 + m^2} d^3 p$$

$$\langle \phi' | 0 \rangle = N e$$

\uparrow normalization constant.

Weinberg's view: start with particles, then make fields.

Fill in Sch. pic.

11

$$\vec{\phi}(x) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} (a_p e^{ipx} + a_p^+ e^{-ipx})$$

(i.e

$$q = \frac{1}{\sqrt{2\omega}} (a + a^+)$$

$$p = -i\sqrt{\frac{\omega}{2}} (a - a^+)$$

$$\vec{\pi}(x) = \int \frac{d^3 p}{(2\pi)^3} (-i) \sqrt{\frac{\omega_p}{2}} (a_p e^{ipx} - a_p^+ e^{-ipx})$$

$$\omega_p = \sqrt{p^2 + m^2}. \quad \text{Write } \vec{\phi}, \vec{\pi} \text{ as}$$

$$\vec{\phi}(x) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} (a_p + a_{-p}^+) e^{ipx} \rightarrow$$

$$\vec{\pi}(x) = \int \frac{d^3 p}{(2\pi)^3} (-i) \sqrt{\frac{\omega_p}{2}} (a_p - a_{-p}^+) e^{ipx}$$

$$\text{Set } [a_p, a_p^+] = (2\pi)^3 \delta^{(3)}(\vec{p} - \vec{p}')$$

$$[\phi(x), \pi(x')] = \int \frac{d^3 p d^3 p'}{(2\pi)^6} \left(-\frac{i}{2}\right) \sqrt{\frac{w p'}{w p}} e^{i(p \cdot x + p' \cdot x')} \\ \times \left([\alpha_{-p}^+, \alpha_{p'}] - [\alpha_p, \alpha_{-p'}^+] \right) e^{i(p \cdot x + p' \cdot x')}$$

$$= \int \frac{d^3 p d^3 p'}{(2\pi)^3} \left(-\frac{i}{2}\right) \sqrt{\frac{w p'}{w p}} e^{i(p \cdot x + i' p' \cdot x')} \\ \times \left(-\delta^3(p+p') - \delta^3(p+p') \right) e^{i(p \cdot x + i' p' \cdot x')}$$

$$= i \int \frac{d^3 p}{(2\pi)^3} e^{i p \cdot (x-x')} = i \delta^{(3)}(x-x').$$

$$\text{Sum. (only, } [\phi(x), \phi(x')] = 0$$

$$[\pi(x), \pi(x')] = 0$$

$$H = \frac{1}{2} \int d^3x \pi^2 + \nabla \phi^2 + m^2 \phi^2$$

$$= \frac{1}{2} \int d^3x \int \frac{d^3p d^3p'}{(2\pi)^6} e^{ipx + i p' x} \left\{ \frac{\sqrt{\omega_p \omega_{p'}}}{2} (a_{p-q_p^+}) (a_{p'-q_{p'}^+}) \right.$$

$$\left. + \left(- \frac{p \cdot p' + m^2}{2\sqrt{\omega_p \omega_{p'}}} \right) (a_{p+q_p^+}) (a_{p'+q_{p'}^+}) \right\}$$

$$= \frac{1}{4} \int \frac{d^3p d^3p'}{(2\pi)^3} \delta^3(p+p') \left\{ \dots \right\}$$

$$= \frac{1}{4} \int \frac{d^3p}{(2\pi)^3} \left\{ -\omega_p (a_{p-q_p^+}) (a_{-p}^+ a_p^+) \right. \\ \left. + \frac{p^2 + m^2}{\omega_p} (a_{p+q_p^+}) (a_{-p}^+ a_p^+) \right\}$$

$$= \frac{1}{4} \int \frac{d^3p}{(2\pi)^3} \omega_p \left\{ a_p a_{-p}^+ + a_{-p}^+ a_p^+ + a_{-p} a_{-p}^+ + a_p a_p^+ \right. \\ \left. - a_p a_{-p}^+ - a_{-p}^+ a_p^+ + a_{-p}^+ a_{-p} + a_p a_p^+ \right\}$$

$$= \frac{1}{2} \int \frac{d^3p}{(2\pi)^3} \omega_p (a_p^+ a_p + a_p a_p^+)$$

$$= \int \frac{d^3p \omega_p}{(2\pi)^3} \left\{ a_p^+ a_p + \frac{1}{2} [a_p, a_p^+] \right\}$$

$$= \int \frac{d^3p}{(2\pi)^3} \omega_p \left[a_p^+ a_p + \frac{1}{2} (2\pi)^3 \delta^{(3)}(0) \right]$$

$$[H, a_p^\dagger] = \omega_p a_p^\dagger \quad (2.32)$$

$$[H, a_p] = -\omega_p a_p$$

$H|0\rangle = 0$ after dropping \propto .

$$\vec{P} = - \int d^3x \pi \nabla \phi$$

$$= \frac{1}{(2\pi)^3} \vec{P}^\dagger a_p^\dagger a_p$$

There are two normalization conventions

$$\langle p' | p \rangle = \delta^{(3)}(\vec{p} - \vec{p}')$$

and

$$\langle p' | p \rangle = 2E_p (2\pi)^3 \delta^{(3)}(\vec{p} - \vec{p}')$$

apart from 2π and 2 factors. The second is Lorentz invariant.

Why? Well, the Lorentz group is variously called $SO(3,1)$ and $SO(1,3)$. The S means determinant 1, which follows from

$$(Lx)^\top \eta (Lx) = x^\top \eta x$$

so

$$L^\top \eta L = \eta \Rightarrow [\det(L)]^2 = 1.$$

For $O(3,1)$, $|L| = \pm 1$.

For $SO(3,1)$, $|L| = +1$.

So the Jacobian

$$J = \det\left(\frac{\partial x'}{\partial x}\right) = |L| = \pm 1$$

for $O(3,1)$ and $J = +1$ for $SO(3,1)$.

So $d^4 p \delta(p^2 - m^2)$ is invariant.

$$\begin{aligned} \text{Now } I &= \int S(g(x)) dg(x) = \int S(g(x)) g'(x) dx \\ &= \int S(x - x_0) dx \quad \text{So} \end{aligned}$$

$$S(g(x)) = \sum_{x_0} \frac{\delta(x - x_0)}{|g'(x_0)|} . \quad \left(\begin{array}{l} \text{sum} \\ \text{over} \\ \text{roots} \\ g(x_0) = 0 \end{array} \right)$$

So

$$\Theta(p) d^4 p \delta(p^2 - m^2) = d^3 p dp^0 \delta(p^0 - \vec{p}^2/m^2) \Theta(p^0)$$

$$\Theta(x) \equiv \frac{x+|x|}{2|x|}.$$

$$= \frac{d^3 p dp^0}{2p^0} \delta(p^0 - \epsilon_p) = \frac{d^3 p}{2p^0} .$$

So $\frac{d^3 p}{2p^0}$ is Lorentz invariant,

which means that $2p^0 \overset{(3)}{\delta}(p - p')$

also is Lorentz invariant. S&P give a different derivation.

So if we define $|p\rangle$ — the state of one particle of momentum \vec{p} and mass m — by

$$|p\rangle = \sqrt{2\epsilon_p} a_p^+ |0\rangle \quad (2.35)$$

then since

$$[a_p, a_{p'}^+] = (2\pi)^3 \delta^{(3)}(\vec{p} - \vec{p'})$$

we have

$$\begin{aligned} \langle p|q\rangle &= \sqrt{2\epsilon_p 2\epsilon_q} \langle 0| a_p a_q^+ |0\rangle \\ &= (2\pi)^3 2\epsilon_p \delta^3(\vec{p} - \vec{q}) . \end{aligned}$$

With these conventions, the inner product $\langle p|q\rangle$ is Lorentz invariant.

So if $U(L)$ is the unitary transformation that represents the Lorentz transformation L , then we can write

$$\langle p|q\rangle = \langle p|U^+(L)U(L)|q\rangle = \langle Lp|Lq\rangle .$$

since $\langle p|q\rangle = \langle Lp|Lq\rangle$ is Lorentz invariant

Thus $U(L)|p\rangle = |Lp\rangle$ and (2.37)

$$U(L)|p=0\rangle = |p\rangle$$

if $L \begin{pmatrix} m \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} p^0 \\ \vec{p} \end{pmatrix} .$

(Noncompact groups, like the Lorentz group, do not have finite-dimensional unitary representations, but $U(L)$ is an infinite-dimensional unitary representation.)

Using (2.35 & 2.37), we get

$$\begin{aligned} U(L)|p\rangle &= U(L)\sqrt{2\varepsilon_p} a_p^\dagger |0\rangle \\ &= U(L)\sqrt{2\varepsilon_p} a_p^\dagger U^*(L)U(L)|0\rangle \\ &= |L_p\rangle = \sqrt{2\varepsilon_{1p}} a_{1p}^\dagger |0\rangle. \end{aligned}$$

Since $U(L)|0\rangle = |0\rangle$ — the vacuum is Lorentz invariant — we get

$$U(L)\sqrt{2\varepsilon_p} a_p^\dagger U^*(L) = \sqrt{2\varepsilon_{1p}} a_{1p}^\dagger \quad \text{or}$$

$$U(L)a_p^\dagger U^*(L) = \sqrt{\frac{\varepsilon_{1p}}{\varepsilon_p}} a_{1p}^\dagger. \quad (2.38)$$

The one-particle identity operator is

$$I_1 = \int \frac{d^3 p}{(2\pi)^3} |p\rangle \frac{1}{2\varepsilon_p} \langle p|$$

$$I_2 |g\rangle = \int \frac{d^3 p}{(2\pi)^3} |p\rangle \frac{1}{2\varepsilon_p} \langle p|g\rangle = |g\rangle.$$

With P&S's conventions,

$$\phi(\vec{x}) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} (a_p e^{i\vec{p} \cdot \vec{x}} + a_p^\dagger e^{-i\vec{p} \cdot \vec{x}}).$$

$$\begin{aligned} \text{So } \phi(\vec{x}) |0\rangle &= \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} a_p^\dagger e^{-i\vec{p} \cdot \vec{x}} |0\rangle \\ &= \int \frac{d^3 p}{(2\pi)^3} \frac{e^{-i\vec{p} \cdot \vec{x}}}{2\epsilon_p} |p\rangle \propto \approx |\vec{x}\rangle \end{aligned}$$

And

$$\begin{aligned} \langle 0 | \phi(\vec{x}') | \vec{p} \rangle &= \langle 0 | \int \frac{d^3 p'}{(2\pi)^3} \frac{1}{\sqrt{2\epsilon_{p'}}} (a_{p'} e^{i\vec{p}' \cdot \vec{x}'} + a_{p'}^\dagger e^{i\vec{p}' \cdot \vec{x}'}) | \vec{p} \rangle \\ &= \langle 0 | \int \frac{d^3 p'}{(2\pi)^3} \frac{\sqrt{\epsilon_p}}{\sqrt{\epsilon_{p'}}} a_{p'} e^{i\vec{p}' \cdot \vec{x}'} a_{p'}^\dagger | 0 \rangle \\ &= e^{i\vec{p} \cdot \vec{x}'} \propto \approx \langle \vec{x}' | \vec{p} \rangle. \end{aligned}$$

So far our fields have been in the Schrödinger picture.

In Heisenberg's picture,

$$\phi(x) = \phi(\vec{x}, t) = e^{iHt} \phi(\vec{x}) e^{-iHt}, \text{ and}$$

$$\dot{\phi} = i e^{iHt} [H, \phi(\vec{x})] e^{-iHt}$$

$$= i [H, \phi(x)] \quad \text{or}$$

$$i \dot{\phi} = [\phi, H]. \quad \text{More generally}$$

$$i \dot{A} = [A, H] \text{ for any operator } A$$

that does not explicitly depend upon time.

Incidentally,

$$\begin{aligned} \langle \psi_0 | \phi(x, t) | \psi_0 \rangle &= \langle \psi_0 | e^{iHt} \phi(x, 0) e^{-iHt} | \psi_0 \rangle \\ &= \langle \psi_{-t} | \phi(x, 0) | \psi_{-t} \rangle. \end{aligned}$$

$$\text{So } e^{-iHt}$$

$e^{-iHt} |\psi_0\rangle = |\psi_{-t}\rangle$. Thus, even in non-rel Q.M.

$$|\vec{x}, t\rangle = e^{iHt} |\vec{x}, 0\rangle \quad \text{and so}$$

$$\psi(\vec{x}, t) = \langle \vec{x}, 0 | \psi_0 \rangle = \langle \vec{x}, 0 | e^{-iHt} | \psi_0 \rangle.$$

That is, $e^{-iHt} |\psi\rangle$ is not $|\psi_{-t}\rangle$; it's $|\psi_{-t}\rangle$.

Recalling (2.32), we have

$$[H, a_p^+] = \omega_p a_p^+$$

$$[H, a_p] = -\omega_p a_p \quad \text{so}$$

$$\dot{\phi} = i[H, \phi]$$

$$= \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left(i[H, a_p] e^{ip \cdot x} + i[H, a_p^+] e^{-ip \cdot x} \right)$$

$$= \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left(-i\omega_p a_p e^{ip \cdot x} + i\omega_p a_p^+ e^{-ip \cdot x} \right)$$

$$= \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} (a_p e^{ip \cdot x} + a_p^+ e^{-ip \cdot x})$$

whence (since $\omega_p \equiv E_p$) we have

$$\dot{a}_p = -iE_p a_p \quad \text{and} \quad \dot{a}_p^+ = iE_p a_p^+$$

$$\text{So} \quad a_p = e^{-iE_p t} a_p \quad a_p^+ = e^{iE_p t} a_p^+.$$

$$\text{So} \quad \phi(\vec{x}, t) = \phi(x) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left(a_p e^{ip \cdot x - iE_p t} + a_p^+ e^{-ip \cdot x + iE_p t} \right)$$

$$= \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} (a_p e^{-ipx} + a_p^+ e^{ipx})$$

$$\text{where } px = p^0 x^0 - \vec{p} \cdot \vec{x}. \quad p^0 = E_p$$

Thus

$$\frac{\partial^3}{\partial t^2} \phi = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left(-E_p^2 a_p c - E_p^2 a_p^\dagger e^{ipx} \right)$$

$$= \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left((-\vec{p}^2 - m^2) a_p e^{-ipx} + (-\vec{p}^2 - m^2) a_p^\dagger e^{ipx} \right)$$

$$= \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left((\nabla^2 - m^2) a_p e^{-ipx} + (\nabla^2 - m^2) a_p^\dagger e^{ipx} \right)$$

$$= (\nabla^2 - m^2) \phi. \quad \text{With } \square \equiv \frac{\partial^2}{\partial x^2} - \nabla^2 = \frac{\partial^2}{\partial x^2} - \Delta,$$

we have

$$(\square + m^2) \phi = 0 \quad \text{the Klein-Gordon equation.}$$

We've seen that

$$a_p(t) = e^{-iE_p t} a_p(0) e^{iHt} \quad a_p^\dagger(t) = e^{-iE_p^\dagger t} a_p^\dagger(0) e^{-iHt}$$

$$= e^{-iE_p t} a_p e^{iHt}$$

$$a_p^\dagger(t) = e^{-iE_p^\dagger t} a_p^\dagger(0) e^{-iHt}$$

$$= e^{-iE_p^\dagger t} a_p^\dagger e^{iHt}.$$

For any operator A without explicit time dependence,

$$A(t) = e^{-iHt} A(0) e^{-iHt}$$

$$= e^{-iHt} A(0) e^{iHt}.$$

Also

$$A(\vec{x}) = e^{-i\vec{P} \cdot \vec{x}} A(0) e^{i\vec{P} \cdot \vec{x}}.$$

In particular

$$a_{\vec{p}} e^{-i\vec{P} \cdot \vec{x}} = e^{-i\vec{P} \cdot \vec{x}} a_{\vec{p}} e^{i\vec{P} \cdot \vec{x}}$$

$$a_{\vec{p}}^+ e^{-i\vec{P} \cdot \vec{x}} = e^{-i\vec{P} \cdot \vec{x}} a_{\vec{p}}^+ e^{i\vec{P} \cdot \vec{x}}$$

where \vec{P} is the momentum operator

$$\vec{P} = - \int d^3x \pi(x) \vec{\nabla} \phi(x)$$

$$= \int \frac{d^3p}{(2\pi)^3} \vec{p} a_{\vec{p}}^+ a_{\vec{p}}.$$

Thus $P^\mu = (H, \vec{P})$ generates translations in space-time.

$$iHa^0 - i\vec{P} \cdot \vec{a}$$

$$U(a) = U(a^0, \vec{a}) = e^{iHa^0 - i\vec{P} \cdot \vec{a}}$$

$$U(a) \phi(x) U^\dagger(a) = \phi(x+a) = \phi(t+a^0, \vec{x} + \vec{a}).$$

$$-i\vec{\theta} \cdot \vec{\nabla}$$

Similarly $U(\vec{\theta}) = e^{-i\vec{\theta} \cdot \vec{\nabla}}$ rotates things.

$$\vec{x} \rightarrow \vec{x}' = \vec{x} + \vec{\theta} \wedge \vec{x} \quad \Rightarrow \quad \vec{x} + \hat{\theta} \times \vec{x} \quad \text{so}$$

$$e^{-i\vec{P} \cdot \vec{\theta} \wedge \vec{x}} \phi(\vec{x}, t) e^{i\vec{P} \cdot \vec{\theta} \wedge \vec{x}} = \phi(\vec{x} + \hat{\theta} \times \vec{x}, t)$$

$$\text{here } \theta^2 \ll 1.$$

Now $\vec{P} \cdot \vec{\theta} \wedge \vec{x} = P_i \epsilon_{ijk} \theta_j x_k$

$$= \theta_j \epsilon_{jki} x_k P_i = \vec{\theta} \cdot \vec{x} \wedge \vec{P} = \vec{\theta} \cdot \vec{L}$$

So

$$e^{-i\theta \cdot \vec{J}} \phi(\vec{x}, t) e^{i\theta \cdot \vec{J}} = \phi(\vec{x} + \vec{\theta} \wedge \vec{x}, t)$$

which is a right-handed active rotation
on \vec{x} about $\vec{\theta}$. More generally, for arbitrary $\vec{\theta}$

$$e^{-i\vec{\theta} \cdot \vec{J}} \phi(\vec{x}, t) e^{i\vec{\theta} \cdot \vec{J}} = \phi(R(\theta)x, t)$$

where $R(\theta)$ is a right-handed rotation about
 $\hat{\theta}$ of $| \vec{\theta} |$ radians.

The action of (H, \vec{P}) and \vec{J} on $\phi(\vec{x}, t)$
follows from their action on a_p and a_p^* :

$$e^{-i\vec{P} \cdot \vec{x}} \left(\begin{pmatrix} a_p \\ a_p^* \end{pmatrix} \right) e^{i\vec{P} \cdot \vec{x}} = \left(\begin{pmatrix} a_p e^{i\vec{p} \cdot \vec{x}} \\ a_p^* e^{-i\vec{p} \cdot \vec{x}} \end{pmatrix} \right)$$

$$e^{iHt} \left(\begin{pmatrix} a_p \\ a_p^* \end{pmatrix} \right) e^{-iHt} = \left(\begin{pmatrix} a_p e^{-ip^0 t} \\ a_p^* e^{ip^0 t} \end{pmatrix} \right)$$

$$e^{-i\vec{\theta} \cdot \vec{J}} \left(\begin{pmatrix} a_p \\ a_p^* \end{pmatrix} \right) e^{i\vec{\theta} \cdot \vec{J}} = \left(\begin{pmatrix} a_{Rp} \\ a_{Rp}^* \end{pmatrix} \right)$$

This last equation is a special case of (2.38)

$$u(L) \begin{pmatrix} a_p \\ a_p^+ \end{pmatrix} \tilde{u}'(L) = \sqrt{\frac{E_{LP}}{E_p}} \begin{pmatrix} a_{LP} \\ a_{LP}^+ \end{pmatrix}.$$

which implies that (here $x = (L, \vec{x})$)

$$\begin{aligned} u(L) \phi(x) \tilde{u}'(L) &= u(L) \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} (a_p e^{-ipx} + a_p^+ e^{ipx}) \tilde{u}'(L) \\ &= \int \frac{d^3 p}{(2\pi)^3} \frac{\sqrt{2E_{LP}}}{2E_p} (a_{LP} e^{-ipx} + a_{LP}^+ e^{ipx}). \end{aligned}$$

But $d^3 p / E_p = d^3 L_p / E_{LP}$, so

$$u(L) \phi(x) \tilde{u}'(L) = \int \frac{d^3 L_p}{(2\pi)^3} \frac{1}{\sqrt{2E_{LP}}} (a_{LP} e^{-ipx} + a_{LP}^+ e^{ipx})$$

Let $L_p = p'$. Then $px = (L'p')_x = p'(Lx)$, so

$$\begin{aligned} u(L) \phi(x) \tilde{u}'(L) &= \int \frac{d^3 p'}{(2\pi)^3} \frac{1}{\sqrt{2E_{p'}}} (a_{p'} e^{-ip' L x} + a_{p'}^+ e^{ip' L x}) \\ &= \phi(Lx). \end{aligned}$$

How fields transform follows from how a_p and a_p^+ transform.

We've seen that

$$[\phi(x, t), \phi(y, t)] = 0 \quad \text{if } (x-y)^2 > 0, \text{ so}$$

we expect that

$$U(t)[\phi(x), \phi(y)] u'(t) = [\phi(tx), \phi(ty)]$$

and so that $[\phi(x), \phi(y)] = 0 \quad \text{if } (x-y)^2 < 0$.

To check this explicitly, we write

$$\begin{aligned} [\phi(x), \phi(y)] &= \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \int d^3 q \frac{1}{(2\pi)^3 \sqrt{2E_q}} \\ &\times \left[a_p e^{-ipx} + a_p^\dagger e^{ipx}, a_q e^{-iqy} + a_q^\dagger e^{iqy} \right] \\ &= \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2E_p} \left(e^{-ip(x-y)} - e^{ip(x-y)} \right) \end{aligned} \quad (2.53a)$$

Since $(x-y)^2 < 0$, i.e. $x-y$ is space-like, we can do a Lorentz transformation on p and $x-y$ so that $x'-y' = (0, Lx - Ly)$ i.e., so that $x' = Lx$, $y' = Ly$ and $x'^0 - y'^0 = 0$.

The ratio $d^3 p / E_p$ is Lorentz invariant, so

$$\int \frac{d^3 p}{E_p} e^{ip(x-y)} = \int \frac{d^3 p'}{E_{2p}} e^{i(p \cdot L(x-y))} = \int \frac{d^3 p'}{E_{p'}} e^{i(\vec{p}' \cdot (\overrightarrow{x-y}))}$$

Now we neglect $\vec{p}' \cdot \vec{p}'(\overrightarrow{x-y}) \approx -\vec{p}' \cdot (\overrightarrow{x-y})$.

So

$$\int \frac{d^3 p}{E_p} e^{ip(x-y)} = \int \frac{d^3 p'}{E_{p'}} e^{-ip'(x'-y')} \quad \begin{matrix} \uparrow \\ 4\text{-vectors} \\ \text{now} \end{matrix}$$

but $x'^0 = y'^0$

Having reflected \vec{p}' , we can now use the invariance of $d^3 p' / E_{p'}$ and of $p'(x'-y')$ to get

$$\int \frac{d^3 p}{E_p} e^{ip(x-y)} = \int \frac{d^3 p}{E_p} e^{-ip(x-y)} \quad \begin{matrix} \uparrow \\ x-y \text{ is} \\ \underline{\text{space-like}}. \end{matrix}$$

Substituting this in (2.53a), we get

$$[\phi(x), \phi(y)] = 0 \quad \text{for } (x-y)^2 < 0.$$

This relation makes causality consistent with special relativity. It's one reason why we use fields like

$$\phi(x) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left(a_p e^{-ipx} + a_p^+ e^{ipx} \right)$$

with both a_p and a_p^+ . Were we not concerned with causality and special relativity we'd use fields like $\phi^{(+)}(x) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} a_p e^{-ipx}$ and $\phi^{(-)}(x) = [\phi^{(+)}(x)]^+$.

But using ϕ^+ and ϕ^- as fields, we'd have

$$[\phi^+(x), \phi^-(y)] = \Delta_+(x-y) = \int \frac{d^3 p}{(2\pi)^3 2p^0} e^{-ip(x-y)}$$

where for $x^2 < 0$

$$\Delta_+(x) = \frac{m}{4\pi^2 \sqrt{-x^2}} K_1(m\sqrt{-x^2})$$

which isn't zero for spacelike $x-y$. S.W.I p.202.