Instantons

Consider a "pure" gauge theory, that is, one with just gauge fields, no scalars or fermions

$$S = -\int d^4x \, \frac{1}{4g^2} F^a_{\mu\nu} F_a^{\mu\nu} . \qquad (1)$$

The euclidean action is

$$S_E = -\int d^4x \, \frac{1}{4g^2} (F^a_{\mu\nu})^2 . \qquad (2)$$

As $|x| \to \infty$, $|F^a_{\mu\nu}| \to 0$, so A must be a pure gauge there if S_E is to be finite.

In (1 & 2), the coupling constant g appears as $-1/g^2$ because the 1-form

$$A^a = A^a_\mu dx^\mu \qquad (3)$$

has a g in it, as well as an i. Under a gauge transformation by U

$$A' = U A U^\dagger + U d U^\dagger \qquad (4)$$

so a pure gauge is

$$A = U d U^\dagger . \qquad (5)$$

Consider the group $SU(2)$ to be the gauge group. Let

$$U = x_4 + i\vec{x}\cdot\vec{\sigma}$$

where $\vec{\sigma}$ is the triplet of Pauli matrices. Then

$$U^\dagger U = (x_4 - i\vec{x\cdot\sigma})(x_4 + i\vec{x\cdot\sigma})$$

$$= x_4^2 + (x\cdot\sigma)^2 = x_4^2 + \vec{x}^2 = x_\mu^2.$$

So if $x_\mu^2 = 1$, then $U = x_4 + i\vec{x\cdot\sigma}$ is unitary. Also, $\det U = 1$. So $U \in SU(2)$.

Also, we see that we can label the elements of $SU(2)$ by the points of the sphere S^3.

Now we learned that

$$\mathrm{tr}\, F^2 = \mathrm{tr}(dA + A^2)^2 = d\,\mathrm{tr}(AdA + \tfrac{2}{3}A^3).$$

So the euclidean action is

$$S_E = -\frac{1}{4g^2}\int d^4x\, \mathrm{tr}\, F^2 = -\frac{1}{4g^2}\int d^4x\, d\,\mathrm{tr}(AdA + \tfrac{2}{3}A^3).$$

But our A is $A = U dU^\dagger$, so

since $dA = F - A^2$

$$S_E = -\frac{1}{4g^2}\int d^4x\, d\, \mathrm{tr}\left[A(F - A^2) + \frac{2}{3}A^3\right]$$

$$= -\frac{1}{4g^2}\int d^4x\, d\, \mathrm{tr}\left[AF - \frac{1}{3}A^3\right].$$

$$= -\frac{1}{4g^2}\int_{S^3} \mathrm{tr}\left[AF - \frac{1}{3}A^3\right]$$

in which the integral now is over the boundary of 4-space, that is, over the 3-sphere S^3 at spatial infinity.

But as $|x|^2 = x_\mu^2 \to \infty$, $|F| \to 0$, so

$$S_E = +\frac{1}{12g^2}\int_{S^3} \mathrm{tr}\, A^3$$

$$= +\frac{1}{12g^2}\int_{S^3} \mathrm{tr}\,(u\, du^\dagger)^3.$$

This last quantity is Pontryagin's index which labels how many times the map from spatial S^3 to the S^3 of $SU(2)$ wraps around $SU(2)$. So

$$S_E = + \frac{n}{12 g^2}$$

where n is the Pontryagin index of the instanton.