HW 7

11) $\langle\xi|\theta\rangle = \langle 0|(1+\bar{\xi}\psi - \frac{1}{2}\bar{\xi}\xi)(1+\psi^\dagger\theta - \frac{1}{2}\bar{\theta}\theta)|0\rangle$

in which $\bar{\xi} \equiv \xi^*$ and $\psi^\dagger \equiv \psi^* \equiv \bar{\psi}$ for ease of writing.

Now $\theta^2 = \xi^2 = \bar{\xi}^2 = \bar{\theta}^2 = 0$, and $\psi|0\rangle = \langle 0|\bar{\psi} = 0$.

So

$$\langle\xi|\theta\rangle = \langle 0|1 + \psi^\dagger\theta - \tfrac{1}{2}\bar{\theta}\theta + \bar{\xi}\psi + \bar{\xi}\psi\bar{\psi}\theta - \tfrac{1}{2}\bar{\xi}\psi\bar{\theta}\theta$$

$$- \tfrac{1}{2}\bar{\xi}\xi - \tfrac{1}{2}\bar{\xi}\xi\bar{\psi}\theta + \tfrac{1}{4}\bar{\xi}\xi\bar{\theta}\theta|0\rangle$$

$$= \langle 0|1 - \tfrac{1}{2}\bar{\theta}\theta + \bar{\xi}\psi\bar{\psi}\theta - \tfrac{1}{2}\bar{\xi}\xi + \tfrac{1}{4}\bar{\xi}\xi\bar{\theta}\theta|0\rangle$$

Now $\psi\bar{\psi} + \bar{\psi}\psi = 1$, so $\psi\bar{\psi} = -\bar{\psi}\psi + 1$, and so

$$\langle\xi|\theta\rangle = \langle 0|1 - \tfrac{1}{2}\bar{\theta}\theta + \bar{\xi}\theta - \tfrac{1}{2}\bar{\xi}\xi + \tfrac{1}{4}\bar{\xi}\xi\bar{\theta}\theta|0\rangle.$$

Also

$$e^{\bar{\xi}\theta - \frac{1}{2}(\bar{\xi}\xi + \bar{\theta}\theta)} = 1 + \bar{\xi}\theta - \tfrac{1}{2}(\bar{\xi}\xi + \bar{\theta}\theta)$$

$$+ \tfrac{1}{2}\left(\bar{\xi}\theta - \tfrac{1}{2}(\bar{\xi}\xi + \bar{\theta}\theta)\right)^2 + \text{zeros}$$

$$= 1 + \bar{\xi}\theta - \tfrac{1}{2}(\bar{\xi}\xi + \bar{\theta}\theta) + \tfrac{1}{4}\bar{\xi}\xi\bar{\theta}\theta$$

So

$$\langle\xi|\theta\rangle = \exp\left[\bar{\xi}\theta - \tfrac{1}{2}(\bar{\xi}\xi + \bar{\theta}\theta)\right].$$

12) We take it for granted that $I = |0\rangle\langle 0| + |1\rangle\langle 1|$.
Then

$$\int |\theta\rangle\langle\theta|\, d\bar{\theta}\, d\theta = \int (1 + \bar{\psi}\theta - \tfrac{1}{2}\bar{\theta}\theta)|0\rangle\langle 0|(1 + \bar{\theta}\psi - \tfrac{1}{2}\bar{\theta}\theta)\, d\bar{\theta}\, d\theta$$

$$= \int \left(-\tfrac{1}{2}\bar{\theta}\theta|0\rangle\langle 0| - \tfrac{1}{2}|0\rangle\langle 0|\bar{\theta}\theta + \bar{\psi}\theta|0\rangle\langle 0|\bar{\theta}\psi\right) d\bar{\theta}\, d\theta$$

$$= \int \left(-|0\rangle\langle 0|\bar{\theta}\theta + \bar{\psi}|0\rangle\langle 0|\psi\theta\bar{\theta}\right) d\bar{\theta}\, d\theta$$

$$= \int \left(|0\rangle\langle 0|\theta\bar{\theta} + \bar{\psi}|0\rangle\langle 0|\psi\theta\bar{\theta}\right) d\bar{\theta}\, d\theta$$

$$= \int \left(|0\rangle\langle 0| + \bar{\psi}|0\rangle\langle 0|\psi\right)\theta\bar{\theta}\, d\bar{\theta}\, d\theta$$

$$= \int \left(|0\rangle\langle 0| + \bar{\psi}|0\rangle\langle 0|\psi\right)\theta\, d\theta$$

$$= |0\rangle\langle 0| + \bar{\psi}|0\rangle\langle 0|\psi = |0\rangle\langle 0| + |1\rangle\langle 1|$$

since $|1\rangle = \bar{\psi}|0\rangle$.

13) $$|\theta\rangle = \left[\prod_{k=1}^{n} \left(1 + \bar{\psi}_k \theta_k - \tfrac{1}{2}\bar{\theta}_k \theta_k\right)\right]|0\rangle$$

Now ψ_k commutes with $\bar{\psi}_\ell \theta_\ell$ for all $\ell \neq k$ and also with $\bar{\theta}_\ell \theta_\ell$.

So

$$\psi_k|\theta\rangle = \left[\prod_{\substack{\ell \neq k \\ \ell=1}}^{n} \left(1 + \bar{\psi}_\ell \theta_\ell - \tfrac{1}{2}\bar{\theta}_\ell \theta_\ell\right)\right]\psi_k\left(1 + \bar{\psi}_k \theta_k - \tfrac{1}{2}\bar{\theta}_k \theta_k\right)|0\rangle$$

And since $\psi_k|0\rangle = 0$ while $\{\psi_k, \bar{\psi}_k\} = 1$

$$\psi_k\left(1 + \bar{\psi}_k \theta_k - \tfrac{1}{2}\bar{\theta}_k \theta_k\right)|0\rangle = \psi_k \bar{\psi}_k \theta_k|0\rangle$$

$$= \left(1 - \bar{\psi}_k \psi_k\right)\theta_k|0\rangle = \theta_k|0\rangle.$$

So

$$\psi_k|\theta\rangle = \left[\prod_{\substack{\ell=1 \\ \ell \neq k}}^{n} \left(1 + \bar{\psi}_\ell \theta_\ell - \tfrac{1}{2}\bar{\theta}_\ell \theta_\ell\right)\right]\theta_k|0\rangle \quad \text{while}$$

$$\theta_k|\theta\rangle = \left[\prod_{\substack{\ell=1 \\ \ell \neq k}}^{n} \left(1 + \bar{\psi}_\ell \theta_\ell - \tfrac{1}{2}\bar{\theta}_\ell \theta_\ell\right)\right]\theta_k\left(1 + \bar{\psi}_k \theta_k - \tfrac{1}{2}\bar{\theta}_k \theta_k\right)|0\rangle$$

$$= \left[\prod_{\substack{\ell=1 \\ \ell \neq k}}^{n} \left(1 + \bar{\psi}_\ell \theta_\ell - \tfrac{1}{2}\bar{\theta}_\ell \theta_\ell\right)\right]\theta_k|0\rangle.$$

So $\psi_k|\theta\rangle = \theta_k|\theta\rangle$.

14) One way to do this problem is to repeat the argument of problem 13 with $m, \vec{x}$ in place of the index k. One writes

$$|\chi\rangle = \left[\prod_{m,x}\left(1 + \Psi^\dagger_{mx}\chi_{mx} - \tfrac{1}{2}\chi^\dagger_{mx}\chi_{mx}\right)\right]|0\rangle$$

$$= \left[\prod_{m,x}\left(1 + \psi^\dagger_m(\vec{x})\chi_m(\vec{x}) - \tfrac{1}{2}\chi^\dagger_m(x)\chi_m(x)\right)\right]|0\rangle$$

$$= \left[\prod_x\left(1 + \psi^\dagger\chi - \tfrac{1}{2}\chi^\dagger\chi\right)\right]|0\rangle$$

in which $\psi^\dagger\chi = \psi^\dagger_m\chi_m$. Then as in (13)

$$\psi_m(\vec{x})|\chi\rangle = \left[\prod_{\substack{m',x' \\ x',m' \neq m,x}}\left(1 + \psi^\dagger\chi - \tfrac{1}{2}\chi^\dagger\chi\right)\right]\psi_m(x)\left(1 + \psi^\dagger_m(x)\chi_m(x) - \tfrac{1}{2}\chi^\dagger_m(x)\chi_m(x)\right)|0\rangle$$

$$= \left[\prod_{\substack{m'x' \\ m'x' \neq mx}}\left(1 + \psi^\dagger\chi - \tfrac{1}{2}\chi^\dagger\chi\right)\right]\chi_m(\vec{x})|0\rangle$$

while

$$\chi_m(\vec{x})|\chi\rangle = \left[\prod_{\substack{m'x' \\ m'x' \neq mx}}\left(1 + \psi^\dagger\chi - \tfrac{1}{2}\chi^\dagger\chi\right)\right]\chi_m(\vec{x})|0\rangle$$

So

$$\psi_m(\vec{x})|\chi\rangle = \chi_m(\vec{x})|\chi\rangle.$$

But a better approach is to show that

$$|\chi\rangle = D(\chi)|0\rangle = e^{\int \psi^\dagger\chi - \frac{1}{2}\chi^\dagger\chi\, d^3x}|0\rangle$$

$$= \exp\left(\int \psi^\dagger\chi - \chi^\dagger\psi\, d^3x\right)|0\rangle.$$

We use the identity

$$e^{A} e^{B} = e^{A+B+\frac{1}{2}[A,B]}$$

which holds when $[A,B]$ commutes with both A and B. Thus

$$e^{\int \psi^\dagger \chi d^3x} e^{-\int \chi^\dagger \psi d^3x} = e^{\int \psi^\dagger \chi - \chi^\dagger \psi d^3x - \frac{1}{2}[\int \psi^\dagger \chi d^3x, \int \chi^\dagger \psi d^3x]}$$

$$= D(\chi) \exp\left\{-\frac{1}{2}\int [\psi^\dagger(x)\chi(x), \chi^\dagger(y)\psi(y)] d^3x d^3y\right\}$$

$$= D(\chi) \exp\left\{-\frac{1}{2}\int \psi^\dagger(x)\chi(x)\chi^\dagger(y)\psi(y) - \chi^\dagger(y)\psi(y)\psi^\dagger(x)\chi(x) d^3x d^3y\right\}$$

$$= D(\chi) \exp\left\{+\frac{1}{2}\int \chi^\dagger(y)\chi(x)\left(\psi^\dagger_{m'}(x)\psi_m(y) + \psi_m(y)\psi^\dagger_{m'}(x)\right) d^3x d^3y\right\}$$

$$= D(\chi) \exp\left\{\frac{1}{2}\int \chi^\dagger_m(y)\chi_{m'}(x)\left(\delta_{mm'}\delta^{(3)}(\vec{x}-\vec{y})\right) d^3x d^3y\right\}$$

$$= D(\chi) \exp\left(\frac{1}{2}\int \chi^\dagger_m(x)\chi_m(x) d^3x\right)$$

So

$$D(\chi) = e^{\int \psi^\dagger \chi d^3x} e^{-\int \chi^\dagger \psi d^3x} e^{-\frac{1}{2}\int \chi^\dagger \chi d^3x}$$

whence

$$D(\chi)|0\rangle = e^{\int \psi^\dagger \chi d^3x} e^{-\int \chi^\dagger \psi d^3x} e^{-\frac{1}{2}\int \chi^\dagger \chi d^3x} |0\rangle$$

$$= \exp\left(\int \psi^\dagger \chi - \frac{1}{2}\chi^\dagger \chi\right)|0\rangle .$$

Now define

$$D(\chi,\lambda) = e^{\lambda \int \psi^\dagger \chi - \chi^\dagger \psi \, d^3x}$$

where λ is a real parameter. Next let

$$\psi(\vec{x},\lambda) = D^\dagger(\chi,\lambda)\, \psi(\vec{x})\, D(\chi,\lambda).$$

Note that $D(\chi)$ and $D(\chi,\lambda)$ are unitary operators since

$$D^\dagger(\chi)\, D(\chi) = \exp\left(\int \chi^\dagger \psi - \psi^\dagger \chi \, d^3x\right) \exp\left(\int \psi^\dagger \chi - \chi^\dagger \psi \, d^3y\right)$$

$$= I.$$

$$\frac{d\psi(\vec{x},\lambda)}{d\lambda} = D^\dagger(\chi,\lambda)\left[\int \chi^\dagger \psi - \psi^\dagger \chi \, d^3y,\, \psi(\vec{x})\right] D(\chi,\lambda)$$

$$= D^\dagger(\chi,\lambda) \int \chi_m(y)\left[\psi^\dagger_m(y)\,\psi_{m'}(x) + \psi_{m'}(x)\,\psi^\dagger_m(y)\right] d^3y\, D(\chi,\lambda)$$

$$= D^\dagger(\chi,\lambda) \int \chi_m(y)\, \delta_{mm'}\, \delta(\vec{x}-\vec{y})\, d^3y \;\; D(\chi,\lambda)$$

$$= D^\dagger(\chi,\lambda)\, \chi_{m'}(\vec{x})\, D(\chi,\lambda). \qquad (\psi')$$

(And incidentally,

$$\frac{d^2\psi(\vec{x},\lambda)}{d\lambda^2} = D^\dagger(\chi,\lambda)\left[\int \chi^\dagger \psi - \psi^\dagger \chi \, d^3y,\, \chi_{m'}(\vec{x})\right] D(\chi,\lambda).$$

But χ anti-commutes with all fermionic things.

So $\frac{d^2\psi}{d\lambda^2} = 0$. Similarly, $\frac{d^n\psi}{d\lambda^2} = 0$.)

Returning to (ψ'), we have

$$\frac{d\psi_{m'}(\vec{x}',\lambda)}{d\lambda} = D^\dagger(\chi,\lambda)\chi_{m'}(\vec{x}')\,D(\chi,\lambda)$$

$$= \chi_{m'}(\vec{x}')\;D^\dagger(\chi,\lambda)\,D(\chi,\lambda)$$

$$= \chi_{m'}(\vec{x}').$$

So

$$\psi_{m'}(\vec{x}',1) = D^\dagger(\chi)\psi_{m'}(\vec{x}')\,D(\chi),$$

$$= \psi_{m'}(x,0) + \int_0^1 \frac{d\psi_{m'}(x)}{d\lambda}\,d\lambda = \int_0^1 \chi_{m'}(x')\,d\lambda + \psi_{m'}(x)$$

$$= \psi_{m'}(\vec{x}) + \chi_{m'}(\vec{x}).$$

That is,

$$D^\dagger(\chi)\,\psi_m(\vec{x})\,D(\chi) = \psi_m(\vec{x}) + \chi_m(\vec{x}).$$

Thus

$$\phi_m(\vec{x})\,D(\chi) = D(\chi)\left(\phi_m(\vec{x}) + \chi_m(\vec{x})\right),$$

and so

$$\begin{aligned}
\phi_m(\vec{x})\,|\chi\rangle &= \phi_m(\vec{x})\,D(\chi)\,|0\rangle \\
&= D(\chi)\left(\phi_m(\vec{x}) + \chi_m(\vec{x})\right)|0\rangle \\
&= D(\chi)\,\chi_m(\vec{x})\,|0\rangle \\
&= \chi_m(\vec{x})\,D(\chi)\,|0\rangle \\
&= \chi_m(\vec{x})\,|\chi\rangle .
\end{aligned}$$