

# HW 7

$$ii) \langle \bar{y}|0\rangle = \langle 0| (1 + \bar{\eta}^+ - \frac{1}{2}\bar{\eta}\eta)(1 + \psi^+\theta - \frac{1}{2}\bar{\theta}\theta) |0\rangle$$

in which  $\bar{\eta} \equiv \eta^*$  and  $\psi^+ = \psi^* = \bar{\psi}$  for ease of writing.

$$\text{Now } \theta^2 = \eta^2 = \bar{\eta}^2 = \bar{\theta}^2 = 0, \text{ and } |\psi|0\rangle = \langle 0|\bar{\psi} = 0.$$

So

$$\begin{aligned} \langle \bar{y}|0\rangle &= \langle 0| 1 + \psi^+\theta - \frac{1}{2}\bar{\theta}\theta + \bar{\eta}\psi + \bar{\eta}\psi\bar{\theta}\theta - \frac{1}{2}\bar{\eta}\psi\bar{\theta}\theta \\ &\quad - \frac{1}{2}\bar{\eta}\bar{\eta} - \frac{1}{2}\bar{\eta}\bar{\eta}\bar{\theta}\theta + \frac{1}{4}\bar{\eta}\bar{\eta}\bar{\theta}\theta |0\rangle \end{aligned}$$

$$= \langle 0| 1 - \frac{1}{2}\bar{\theta}\theta + \bar{\eta}\psi\bar{\theta}\theta - \frac{1}{2}\bar{\eta}\bar{\eta} + \frac{1}{4}\bar{\eta}\bar{\eta}\bar{\theta}\theta |0\rangle$$

Now  $\psi\bar{\psi} + \bar{\psi}\psi = 1$ , so  $\psi\bar{\psi} = -\bar{\psi}\psi + 1$ , and so

$$\langle \bar{y}|0\rangle = \langle 0| 1 - \frac{1}{2}\bar{\theta}\theta + \bar{\eta}\theta - \frac{1}{2}\bar{\eta}\eta + \frac{1}{4}\bar{\eta}\eta\bar{\theta}\theta |0\rangle.$$

$$\text{Also } \bar{\eta}\theta - \frac{1}{2}(\bar{\eta}\eta + \bar{\theta}\theta)$$

$$\Theta = 1 + \bar{\eta}\theta - \frac{1}{2}(\bar{\eta}\eta + \bar{\theta}\theta)$$

$$+ \frac{1}{2} \left( \bar{\eta}\theta - \frac{1}{2}(\bar{\eta}\eta + \bar{\theta}\theta) \right)^2 + \text{zeros}$$

$$= 1 + \bar{\eta}\theta - \frac{1}{2}(\bar{\eta}\eta + \bar{\theta}\theta) + \frac{1}{4}\bar{\eta}\eta\bar{\theta}\theta$$

So

$$\langle \bar{y}|0\rangle = \exp \left[ \bar{\eta}\theta - \frac{1}{2}(\bar{\eta}\eta + \bar{\theta}\theta) \right].$$

12) We take it for granted that  $I = 10X01 + 11X11$ .

Then

$$\begin{aligned}
 \int |0\rangle \langle 01| d\bar{\theta} d\theta &= \int (1 + \bar{\varphi}\theta - \frac{1}{2}\bar{\theta}\theta) 10X01 (1 + \bar{\theta}\bar{\varphi} - \frac{1}{2}\bar{\theta}\bar{\theta}) d\bar{\theta} d\theta \\
 &= \int \left( -\frac{1}{2}\bar{\theta}\theta 10X01 - \frac{1}{2}10X01\bar{\theta}\theta + \bar{\varphi}\theta 10X01\bar{\varphi} \right) d\bar{\theta} d\theta \\
 &= \int (-10X01\bar{\theta}\theta + \bar{\varphi}10X01\bar{\varphi}\theta) d\bar{\theta} d\theta \\
 &= \int (10X01\theta\bar{\theta} + \bar{\varphi}10X01\bar{\varphi}\bar{\theta}) d\bar{\theta} d\theta \\
 &= \int (10X01 + \bar{\varphi}10X01\bar{\varphi}) \theta\bar{\theta} d\bar{\theta} d\theta \\
 &= \int (10X01 + \bar{\varphi}10X01\bar{\varphi}) \theta d\theta \\
 &= 10X01 + \bar{\varphi}10X01\bar{\varphi} = 10X01 + 11X11
 \end{aligned}$$

since  $|11\rangle = \bar{\varphi}|0\rangle$ .

$$13) |0\rangle = \left[ \prod_{k=1}^m \left( 1 + \bar{\Psi}_k \theta_k - \frac{1}{2} \bar{\theta}_k \theta_{ik} \right) \right] |0\rangle$$

Now  $\psi_k$  commutes with  $\bar{\Psi}_l \theta_l$  for all  $l \neq k$  and also with  $\bar{\theta}_l \theta_l$ .

So

$$\psi_k |0\rangle = \left[ \prod_{\substack{l=1 \\ l \neq k}}^m \left( 1 + \bar{\Psi}_l \theta_l - \frac{1}{2} \bar{\theta}_l \theta_l \right) \right] \psi_k \left( 1 + \bar{\Psi}_k \theta_k - \frac{1}{2} \bar{\theta}_k \theta_{ik} \right) |0\rangle$$

And since  $\psi_k |0\rangle = 0$  while  $\{\psi_k, \bar{\Psi}_k\} = 1$

$$\begin{aligned} \psi_k \left( 1 + \bar{\Psi}_k \theta_k - \frac{1}{2} \bar{\theta}_k \theta_k \right) |0\rangle &= \psi_k \bar{\Psi}_k \theta_k |0\rangle \\ &= (1 - \bar{\Psi}_k \psi_k) \theta_k |0\rangle = \theta_k |0\rangle. \end{aligned}$$

So

$$\psi_k |0\rangle = \left[ \prod_{\substack{l=1 \\ l \neq k}}^m \left( 1 + \bar{\Psi}_l \theta_l - \frac{1}{2} \bar{\theta}_l \theta_l \right) \right] \theta_k |0\rangle \quad \text{while}$$

$$\begin{aligned} \theta_k |0\rangle &= \left[ \prod_{\substack{l=1 \\ l \neq k}}^m \left( 1 + \bar{\Psi}_l \theta_l - \frac{1}{2} \bar{\theta}_l \theta_l \right) \right] \theta_k \left( 1 + \bar{\Psi}_k \theta_k - \frac{1}{2} \bar{\theta}_k \theta_{ik} \right) |0\rangle \\ &= \left[ \prod_{\substack{l=1 \\ l \neq k}}^m \left( 1 + \bar{\Psi}_l \theta_l - \frac{1}{2} \bar{\theta}_l \theta_l \right) \right] \theta_k |0\rangle. \end{aligned}$$

So  $\psi_k |0\rangle = \theta_k |0\rangle$ .

14) One way to do this problem is to repeat the argument of problem 13 with  $m, \vec{x}$  in place of the index  $k$ . One writes

$$\begin{aligned} |\chi\rangle &= \left[ \prod_{m,x} \left( 1 + \bar{\psi}_{mx}^+ \chi_{mx} - \frac{1}{2} \bar{\chi}_{mx} \chi_{mx}^+ \right) \right] |0\rangle \\ &= \left[ \prod_{m,x} \left( 1 + \psi_m^+ (\vec{x}) \chi_m (\vec{x}) - \frac{1}{2} \chi_m^+ (x) \chi_m (x) \right) \right] |0\rangle \\ &= \left[ \prod_x \left( 1 + \psi_x^+ - \frac{1}{2} \chi^+ \chi \right) \right] |0\rangle \end{aligned}$$

in which  $\psi_x^+ = \psi_m^+ \chi_m$ . Then as in (13)

$$\begin{aligned} \psi_m (\vec{x}) |\chi\rangle &= \left[ \prod_{\substack{m'x' \\ m'x' \neq mx}} \left( 1 + \psi_{m'}^+ - \frac{1}{2} \chi_{m'}^+ \chi \right) \right] \psi_m^+ (x) \left( 1 + \psi_{m'}^+ \chi_{m'} (x) - \frac{1}{2} \chi_{m'}^+ \chi_{m'} (x) \right) |0\rangle \\ &= \left[ \prod_{\substack{m'x' \\ m'x' \neq mx}} \left( 1 + \psi_{m'}^+ - \frac{1}{2} \chi_{m'}^+ \chi \right) \right] \chi_m (\vec{x}) |0\rangle \end{aligned}$$

while

$$\chi_m (\vec{x}) |\chi\rangle = \left[ \prod_{\substack{m'x' \\ m'x' \neq mx}} \left( 1 + \psi_{m'}^+ - \frac{1}{2} \chi_{m'}^+ \chi \right) \right] \chi_{m'} (\vec{x}) |0\rangle$$

So

$$\psi_m (\vec{x}) |\chi\rangle = \chi_m (\vec{x}) |\chi\rangle.$$

But a better approach is to show that

$$\begin{aligned} |\chi\rangle &= D(\chi) |0\rangle = e^{\int \psi^+ \chi - \frac{1}{2} \chi^+ \chi d^3x} |0\rangle \\ &= \exp \left( \int \psi^+ \chi - \chi^+ \psi d^3x \right) |0\rangle. \end{aligned}$$

We use the identity

$$e^A e^B = e^{A+B + \frac{1}{2}[A, B]}$$

which holds when  $[A, B]$  commutes with both  $A$  and  $B$ . Thus

$$\begin{aligned} e^{\int \psi^\dagger x d^3x - \int x^\dagger \psi d^3x} &= e^{\int \psi^\dagger x - x^\dagger \psi d^3x - \frac{1}{2} [\int \psi^\dagger x d^3x, \int x^\dagger \psi d^3x]} \\ &= D(x) \exp \left\{ -\frac{1}{2} \int [\psi_{(x)}^\dagger \chi_{(x)}, \chi_{(y)}^\dagger \psi_{(y)}] d^3x d^3y \right\} \\ &= D(x) \exp \left\{ -\frac{1}{2} \int \psi_{(x)}^\dagger \chi_{(x)} \chi_{(y)}^\dagger \psi_{(y)} - \chi_{(y)}^\dagger \psi_{(y)} \psi_{(x)}^\dagger \chi_{(x)} d^3x d^3y \right\} \\ &= D(x) \exp \left\{ +\frac{1}{2} \int \chi_{(y)}^\dagger \chi_{(x)} (\psi_m^\dagger \chi_{(x)} \psi_{(y)} + \psi_{(y)}^\dagger \chi_{(x)}^\dagger \psi_{(x)}) d^3x d^3y \right\} \\ &= D(x) \exp \left\{ \frac{1}{2} \int \chi_{(y)}^\dagger \chi_{(x)} (\delta_{mm'} \delta_{nn'}^{(3)}(\vec{x} - \vec{y})) d^3x d^3y \right\} \\ &= D(x) \exp \left( \frac{1}{2} \int \chi_m^\dagger(x) \chi_m(x) d^3x \right) \end{aligned}$$

$$\text{So } D(x) = e^{\int \psi^\dagger x d^3x - \int x^\dagger \psi d^3x - \frac{1}{2} \int x^\dagger x d^3x}$$

$$\text{whence } D(x)|0\rangle = e^{\int \psi^\dagger x d^3x - \int x^\dagger \psi d^3x - \frac{1}{2} \int x^\dagger x d^3x} |0\rangle$$

$$\begin{aligned} D(x)|0\rangle &= e^{\int \psi^\dagger x d^3x - \int x^\dagger \psi d^3x - \frac{1}{2} \int x^\dagger x d^3x} |0\rangle \\ &= \exp \left( \int \psi^\dagger x - \frac{1}{2} x^\dagger x \right) |0\rangle. \end{aligned}$$

Now define  $\lambda \int 4^+ x - x^+ 4 d^3 x$

$$D(x, \lambda) = e$$

where  $\lambda$  is a real parameter. Next let

$$\psi(\vec{x}, \lambda) = D^\dagger(x, \lambda) \psi(\vec{x}) D(x, \lambda).$$

Note that  $D(x)$  and  $D(x, \lambda)$  are unitary operators since

$$D^\dagger(x) D(x) = \exp\left(\int x^+ 4 - 4^+ x d^3 x\right) \exp\left(\int 4^+ x - x^+ 4 d^3 x\right)$$

$$= I.$$

$$\begin{aligned} \frac{d\psi(\vec{x}, \lambda)}{d\lambda} &= D^\dagger(x, \lambda) \left[ \int x^+ 4 - 4^+ x d^3 y, \psi(\vec{y}) \right] D(x, \lambda) \\ &= D^\dagger(x, \lambda) \int \chi_m(y) \left[ 4^+_m \psi_m(x) + \psi_m(x) 4^+_m \right] d^3 y D(x, \lambda) \\ &= D^\dagger(x, \lambda) \int \chi_m(y) S_{mm'} \delta(\vec{x} \cdot \vec{y}) d^3 y D(x, \lambda) \\ &= D^\dagger(x, \lambda) \chi_{mm'}(\vec{x}) D(x, \lambda). \end{aligned} \quad (4')$$

And incidentally,

$$\frac{d^2\psi(\vec{x}, \lambda)}{d\lambda^2} = D^\dagger(x, \lambda) \left[ \int x^+ 4 - 4^+ x d^3 y, \chi_{mm'}(\vec{y}) \right] D(x, \lambda).$$

But  $x$  anti-commutes with all fermionic things.

$$\text{So } \frac{d^2\psi}{dx^2} = 0. \quad \text{Similarly, } \frac{d^m\psi}{dx^m} = 0.$$

Returning to (4'), we have

$$\begin{aligned} \frac{d\psi_m(\vec{x}, \lambda)}{d\lambda} &= D^\dagger(x, \lambda) \chi_{m'}(\vec{x}) D(x, \lambda) \\ &= \chi_{m'}(\vec{x}) D^\dagger(x, \lambda) D(x, \lambda) \\ &= \chi_{m'}(\vec{x}). \end{aligned}$$

$$\begin{aligned} \text{So } \psi_{m'}(\vec{x}, 1) &= D^\dagger(x) \psi_{m'}(\vec{x}) D(x) \\ &= \psi_{m'}(x, 0) + \int_0^1 \frac{d\psi_{m'}(x)}{d\lambda} d\lambda = \int_0^1 \chi_{m'}(x') d\lambda + \psi_{m'}(x) \\ &= \psi_{m'}(\vec{x}) + \chi_{m'}(\vec{x}). \end{aligned}$$

That is,

$$D^\dagger(x) \psi_m(\vec{x}) D(x) = \psi_m(\vec{x}) + \chi_m(\vec{x}).$$

Thus

$$\psi_m(\vec{x}) D(x) = D(x) \left( \psi_m(\vec{x}') + \chi_m(\vec{x}') \right),$$

and so

$$\begin{aligned} \psi_m(\vec{x}) |x\rangle &= \psi_m(\vec{x}) D(x) |0\rangle \\ &= D(x) (\psi_m(\vec{x}') + \chi_m(\vec{x}')) |0\rangle \\ &= D(x) \chi_m(\vec{x}') |0\rangle \\ &= \chi_m(\vec{x}') D(x) |0\rangle \\ &= \chi_m(\vec{x}') |x\rangle. \end{aligned}$$