

$$2.2 \quad \mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi$$

$$(a) \quad \pi_\phi = \frac{\partial h}{\partial \dot{\phi}} = \frac{\partial h}{\partial \partial_0 \phi^*} = \partial^0 \phi^* = \dot{\phi}^*$$

$$\pi_{\phi^*} = \frac{\partial h}{\partial \dot{\phi}^*} = \frac{\partial h}{\partial \partial_0 \phi^*} = \partial^0 \phi = \dot{\phi}$$

The equal-time canonical commutation relations are

$$[\phi(x, t), \pi_\phi(y, t)] = i \delta^{(3)}(\vec{x} - \vec{y})$$

i.e.,

$$[\phi(x, t), \dot{\phi}^*(y, t)] = i \delta(x - y)$$

and  $[\phi, \pi_{\phi^*}] = 0$

$$[\phi^*(x, t), \pi_{\phi^*}(y, t)] = [\phi^*(x, t), \dot{\phi}(y, t)] = i \delta(x - y).$$

$[\phi, \phi^*] = [\pi, \pi^*] = 0$  at equal times.

$$\mathcal{H} = \pi_\phi \dot{\phi} + \pi_{\phi^*} \dot{\phi}^* - \mathcal{L}$$

$$= \pi_\phi \pi_{\phi^*} + \pi_{\phi^*} \pi_\phi - \pi_\phi \pi_{\phi^*} + \nabla \phi^* \cdot \nabla \phi + m^2 \phi^* \phi$$

$$= \pi_\phi \pi_{\phi^*} + |\nabla \phi|^2 + m^2 |\phi|^2$$

$$\mathcal{H} = \int d^3x \left( |\pi_\phi|^2 + |\nabla \phi|^2 + m^2 |\phi|^2 \right)$$

$$= \int d^3x \left( \pi_\phi \pi_{\phi^*} + \nabla \phi^* \cdot \nabla \phi + m^2 \phi^* \phi \right).$$

$$i\dot{\phi} = [\phi, H]$$

$$\begin{aligned} i\dot{\phi}(y, t) &= - \int d^3x [ \bar{\Pi}_\phi \bar{\Pi}_\phi^\star + \nabla \phi \cdot \nabla \phi^\star + m^2 \phi^\star \phi ] \\ &= \int d^3x [ \phi(y, t), \bar{\Pi}_\phi(x, t) \bar{\Pi}_\phi^\star(x, t) ] \\ &= \int d^3x i \delta^3(\vec{x} - \vec{y}) \bar{\Pi}_\phi^\star(x, t) \\ &= i \bar{\Pi}_\phi^\star(y, t), \text{ which is } i\dot{\phi}(y, t). \end{aligned}$$

$$\text{So } \dot{\phi} = \bar{\Pi}_\phi^\star \text{ and}$$

$$\begin{aligned} \dot{\phi}(y, t) &= -i [ \bar{\Pi}_\phi^\star(y, t), H ] \\ &= -i \int d^3x [ \bar{\Pi}_{\phi^\star}(y, t), \bar{\Pi}_\phi \bar{\Pi}_\phi^\star + \nabla \phi \cdot \nabla \phi^\star + m^2 \phi^\star \phi ] \\ &= -i \int d^3x [ \bar{\Pi}_{\phi^\star}, -\phi^\star \Delta \phi + m^2 \phi^\star \phi ] \\ &= - \int d^3x \delta_3(x - y) (-\Delta \phi + m^2 \phi) \\ &= \Delta \phi(y, t) - m^2 \phi(y, t) \\ (\square + m^2) \phi(y, t) &= (\partial_t^2 - \Delta + m^2) \phi(y, t) = 0. \end{aligned}$$

One also gets

$$(\square + m^2) \phi^\star(x, t) = 0$$

by a similar argument.

(b) Complex notation can obscure things. Let

$$\phi_1 = \frac{1}{\sqrt{2}} (\phi + \phi^*) \quad \text{and} \quad \phi_2 = \frac{1}{i\sqrt{2}} (\phi - \phi^*)$$

$$\pi_1 = \frac{1}{\sqrt{2}} (\pi + \pi^*) \quad \pi_2 = \frac{1}{i\sqrt{2}} (\pi^* - \pi).$$

Then

$$[\phi_1, \pi_1] = \frac{1}{2} [\phi + \phi^*, \pi + \pi^*]$$

$$= \frac{1}{2} [\phi, \pi] + [\phi^*, \pi^*] = i \delta(x-y).$$

$$[\phi_1, \pi_2] = \frac{1}{i\sqrt{2}} [\phi + \phi^*, \pi - \pi^*] = \frac{1}{i\sqrt{2}} ([\phi, \pi] - [\phi^*, \pi^*]) = 0.$$

$$[\phi_2, \pi_1] = -\frac{1}{2} [\phi - \phi^*, \pi^* - \pi]$$

$$= +\frac{1}{2} [\phi, \pi] + \frac{1}{2} [\phi^*, \pi^*] = i \delta(x-y).$$

$$[\phi_2, \pi_2] = \frac{1}{2i} ([\phi - \phi^*, \pi + \pi^*]) = 0.$$

So

$$H = \int \pi^* \pi + \nabla \phi^* \cdot \nabla \phi + m^2 \phi^* \phi \, d^3x$$

$$\phi = \frac{1}{\sqrt{2}} (\phi_1 + i\phi_2) \quad \pi = \frac{1}{\sqrt{2}} (\pi_1 - i\pi_2)$$

$$\phi^* = \frac{1}{\sqrt{2}} (\phi_1 - i\phi_2) \quad \pi^* = \frac{1}{\sqrt{2}} (\pi_1 + i\pi_2)$$

S<sub>0</sub>

$$H = \int \left[ \frac{1}{2} (\pi_1 + i\pi_2)(\pi_1 - i\pi_2) + \nabla(\phi_1 - i\phi_2)\nabla(\phi_1 + i\phi_2) + \frac{m^2}{2}(\phi_1 - i\phi_2)(\phi_1 + i\phi_2) \right] d^3x$$

$$= \int \frac{1}{2} \left[ \pi_1^2 + \pi_2^2 + (\nabla\phi_1)^2 + (\nabla\phi_2)^2 + m^2(\phi_1^2 + \phi_2^2) \right] d^3x$$

and by (2.31) apart from a constant

$$H = \int \frac{\beta p}{(2\pi)^3} E_p (a_1^\dagger a_1 + a_2^\dagger a_2).$$

$$\phi_i(x) = \int (a_i(p)e^{-ipx} + a_i^\dagger(p)e^{ipx}) \frac{d^3p}{(2\pi)^3 2p^0}$$

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$$\phi(x) = \int \left[ \frac{1}{\sqrt{2}}(a_1 + ia_2)e^{-ipx} + \frac{1}{\sqrt{2}}(a_1^\dagger + ia_2^\dagger)e^{ipx} \right] \frac{d^3p}{(2\pi)^3 \sqrt{2p^0}}$$

$$\text{Let } \alpha(p) = \frac{1}{\sqrt{2}}(a_1(p) + ia_2(p))$$

$$\bar{\alpha}(p) = \frac{1}{\sqrt{2}}(a_1(p) - ia_2(p)), \text{ then}$$

$$\phi(x) = \int (\alpha(p)e^{-ipx} + \bar{\alpha}^\dagger(p)e^{ipx}) \frac{d^3p}{(2\pi)^3 \sqrt{2p^0}}$$

$$\phi^*(x) = \phi^\dagger(x) = \int (\bar{\alpha}(p)e^{-ipx} + \alpha^\dagger(p)e^{ipx}) \frac{d^3p}{(2\pi)^3 \sqrt{2p^0}}$$

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$$\text{So } a_1^+ a_1 + a_2^+ a_2 = a^+ a + \bar{a}^+ \bar{a} \quad \text{and}$$

$$H = \int E_p (a^+ a + \bar{a}^+ \bar{a}) \frac{d^3 p}{(2\pi)^3}.$$

(c) We use  $\phi$  and  $\phi^* = \phi^+$  from p. 2.2.4 and

$$\pi(x) = \int (-i p^0 \bar{a}(p) e^{-ipx} + i p^0 a^+(p) e^{ipx}) \frac{d^3 p}{(2\pi)^3 \sqrt{2p^0}}$$

$$\pi^*(x) = \pi^*(x) = \int (-i p^0 a e^{-ipx} + i p^0 \bar{a}^+ e^{ipx}) \frac{d^3 p}{(2\pi)^3 \sqrt{2p^0}}.$$

Thus

$$Q = \frac{i}{2} \int (\phi^* \pi^+ - \pi \phi) d^3 x$$

$$= \frac{1}{2} \int \frac{d^3 p d^3 q d^3 x}{(2\pi)^6 2\sqrt{p^0 q^0}} \left[ (\bar{a}(p) e^{-ipx} + a^+(p) e^{ipx}) (-i g^0 a e^{-iqx} + i g^0 \bar{a}^+ e^{iqx}) \right]$$

$$- \left( -i g^0 \bar{a}(q) e^{-iqx} + i g^0 a^+(q) e^{iqx} \right) \left( a(p) e^{-ipx} + \bar{a}(p) e^{ipx} \right)$$

$$\text{Set } t = 0 \text{ and write } \bar{a}(p) e^{ipx} + a^+(p) e^{-ipx} = [\bar{a}(p) + \bar{a}^+(-p)] e^{ipx}$$

etc.,

$$Q = \frac{i}{2} \int \frac{d^3 p d^3 q d^3 x}{(2\pi)^6 2\sqrt{p^0 q^0}} \left[ (\bar{a}(p) + \bar{a}^+(-p)) e^{ipx} (-i g^0)(a(q) - \bar{a}(-q)) e^{iqx} + i g^0 e^{ipx} \right]$$

$$+ i g^0 (\bar{a}(q) - \bar{a}^+(-q)) e^{iqx} (a(p) + \bar{a}^+(-p)) e^{-ipx} \right]$$

$$Q = \frac{e}{2} \int \frac{d^3 p d^3 q}{(2\pi)^3 2\sqrt{p_0 q_0}} \delta^3(\vec{p} + \vec{q}) \left[ (\bar{a}(p) + a^*(-p))(-ig^0)(a(q) - \bar{a}^*(-q)) \right. \\ \left. + ig^0 (\bar{a}(q) - a^*(-q))(a(p) + \bar{a}^*(-p)) \right]$$

$$= \frac{e}{2} \int \frac{d^3 p}{(2\pi)^3 2p^0} \left[ -ig^0 (\bar{a}(p) + a^*(-p))(a(-p) - \bar{a}^*(p)) \right. \\ \left. + ip^0 (\bar{a}(-p) - a^*(p))(a(p) + \bar{a}^*(-p)) \right]$$

$$= \frac{1}{4} \int \frac{d^3 p}{(2\pi)^3} \left[ (\bar{a}(p) + a^*(-p))(a(-p) - \bar{a}^*(p)) \right. \\ \left. - (\bar{a}(-p) - a^*(p))(a(p) + \bar{a}^*(-p)) \right]$$

$$= \frac{1}{4} \int \frac{d^3 p}{(2\pi)^3} -\bar{a}(p)\bar{a}^*(p) + a^*(-p)a(-p) \\ - \bar{a}(-p)\bar{a}^*(-p) + a^*(p)a(p)$$

$$= \frac{1}{2} \int \frac{d^3 p}{(2\pi)^3} a^*(p)a(p) - \bar{a}(p)\bar{a}^*(p)$$

So  $a^*(p)|0\rangle$  has charge  $\frac{1}{2}$  and  
 $\bar{a}^*(p)|0\rangle$  has charge  $-\frac{1}{2}$ .

$$(d) \quad \mathcal{L} = \sum_{a=1}^2 \partial_m \phi_a^* \partial^m \phi_a - m^2 \phi_a^* \phi_a.$$

So this is two copies of the prior theory.  
Thus one charge that is conserved is

$$Q^0 = \frac{i}{2} \left[ \left[ \sum_{a=1}^2 \phi_a^* \pi_a - \pi_a \phi_a \right] d^3x \right].$$

Let  $\psi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$  be a complex doublet.

Then

$$\chi = \partial_m \psi^+ \partial^m \psi - m^2 \psi^+ \psi$$

and  $\chi$  is invariant under

$$\psi \rightarrow \psi' = e^{-\frac{i\theta^\ell \sigma^\ell}{2}} \psi = \psi - i \frac{\theta^\ell \sigma^\ell}{2} \psi$$

so  $\delta \phi_a = -\frac{i}{2} \theta^\ell \sigma^\ell_{ab} \phi_b$  for tiny  $\vec{\theta}$ . So

$$0 = \underbrace{\frac{\partial \mathcal{L}}{\partial \phi_a} \delta \phi_a}_{\partial_m \phi_a} + \underbrace{\frac{\partial \mathcal{L}}{\partial \partial_m \phi_a} \delta \partial_m \phi_a}_{\partial_m \phi_a^*} + \underbrace{\frac{\partial \mathcal{L}}{\partial \phi_a^*} \delta \phi_a^*}_{\partial_m \phi_a^*} + \underbrace{\frac{\partial \mathcal{L}}{\partial \partial_m \phi_a^*} \delta \partial_m \phi_a^*}_{\partial_m \phi_a^*}$$

Use field equations

$$0 = \partial_m \frac{\partial \mathcal{L}}{\partial \partial_m \phi_a} - \frac{\partial \mathcal{L}}{\partial \phi_a} = \partial_m \frac{\partial \mathcal{L}}{\partial \partial_m \phi_a^*} - \frac{\partial \mathcal{L}}{\partial \phi_a^*}$$

to get

$$0 = \partial_m \frac{\partial L}{\partial \dot{\phi}_a} S\phi_a + \frac{\partial L}{\partial \phi_a} \partial_m S\phi_a$$

$$+ \partial_m \frac{\partial L}{\partial \dot{\phi}_a^*} S\phi_a^* + \frac{\partial L}{\partial \phi_a^*} \partial_m S\phi_a^*$$

$$0 = \partial_m \left[ \frac{\partial L}{\partial \dot{\phi}_a} S\phi_a + \frac{\partial L}{\partial \dot{\phi}_a^*} S\phi_a^* \right] \equiv \partial_m J^{\mu}$$

so

$$Q = \int J^0 d^3x = \int \frac{\partial L}{\partial \dot{\phi}_a} S\phi_a + \frac{\partial L}{\partial \dot{\phi}_a^*} S\phi_a^* d^3x$$

is conserved for each  $\ell$

$$S\phi_a^\ell = -\frac{i}{2} \sigma_{ab}^\ell \phi_b.$$

But  $\frac{\partial L}{\partial \dot{\phi}_a} = \pi_a$

$$\frac{\partial L}{\partial \dot{\phi}_a^*} = \pi_a^*$$

so

$$Q^\ell = \int \pi_a \left( \frac{-i}{2} \sigma_{ab}^\ell \phi_b + \pi_a^* \frac{i}{2} \sigma_{ab}^{*\ell} \phi_b^* \right) d^3x.$$

That is,

$$Q^l = \frac{i}{2} \int \phi_b^* \sigma_{ba}^l \pi_a^* - \pi_a \sigma_{ab}^l \phi_b d^3x.$$

To avoid going blind, I will focus on

$$g^l = -\frac{i}{2} \int \pi_a \sigma_{ab}^l \phi_b d^3x \text{ first.}$$

Then using the detailed work on p. 2.2.12, we have

$$[g^l, g^m] = -\frac{i}{4} \int [\pi_a \sigma_{ab}^l \phi_b, \pi_c \sigma_{cd}^m \phi_d] d^3x d^3y$$

$$= -\frac{i}{4} i \int [\pi_a \sigma_{ab}^l \delta_{bc} \delta^{(3)}(x-y) \sigma_{cd}^m \phi_d \\ - \sigma_{ab}^l \pi_c \sigma_{cd}^m \delta_{ad} \delta^{(3)}(x-y)] d^3x d^3y$$

$$= -\frac{i}{4} \int [\pi_a \sigma_{ab}^l \sigma_{bd}^m \phi_d - \pi_c \sigma_{cd}^m \sigma_{db}^l \phi_b] d^3x$$

$$= -i \int \pi_a \left[ \frac{\sigma^l}{2}, \frac{\sigma^m}{2} \right]_{ad} \phi_d d^3x$$

$$\text{But } \left[ \frac{\sigma^l}{2}, \frac{\sigma^m}{2} \right] = i \epsilon_{lmk} \frac{\sigma_{ik}}{2}, \quad \approx$$

$$[g^l, g^m] = -i \frac{i}{2} \int \pi_a \sigma_{ad}^k \phi_d d^3x \epsilon_{lmk}$$

on

$$[q^l, q^m] = i \epsilon_{lmk} \left(-\frac{i}{2}\right) \int \Pi_a \sigma_{ab}^k \phi_b dx \\ = i \epsilon_{lmk} g^k.$$

Now

$$Q^0 = q^l + q^{l+}$$

and we saw that

$$[q^l, q^m] = i \epsilon_{lmk} g^k$$

so

$$[q^{l+}, q^{l+}] = -i \epsilon_{lmk} g^{k+}$$

$$[q^{l+}, q^{l+m}] = i \epsilon_{lmk} g^{k+}$$

So since  $[q^l, q^{l+m}] = 0$ , we have

$$[Q^l, Q^m] = [q^{l+} + q^{l+}, q^m + q^{l+m}]$$

$$= [q^l, q^m] + [q^{l+}, q^{l+m}]$$

$$= i \epsilon_{lmk} g^k + i \epsilon_{lmk} g^{k+}$$

$$= i \epsilon_{lmk} Q^k.$$

2.2.12

$$\begin{aligned} [\pi_a \phi_b, \pi_c \phi_d] &= \pi_a \phi_b \pi_c \phi_d - \pi_c \phi_d \pi_a \phi_b \\ &= \pi_a (\pi_c \phi_b + i \delta_{bc}) \phi_d - \pi_c (\pi_a \phi_d + i \delta_{ad}) \phi_b \\ &= i \pi_a \phi_d \delta_{bc} - i \pi_c \phi_b \delta_{ad} \end{aligned}$$

2.2 (redux) Now let's do this problem correctly (Not required.)

$$\mathcal{L} = \sum_{a=1}^4 \frac{1}{2} \partial_\mu \phi_a \partial^\mu \phi_a - \frac{m^2}{2} \phi_a^2$$

is invariant under O(4) rotations

$\phi \rightarrow \phi' = (1 + \omega) \phi$  where  $\omega$  is real  
and anti-symmetric  $\omega^T = -\omega$  so that

$$\begin{aligned} \phi'^T \phi' &= \phi^T (1 + \omega^T) (1 + \omega) \phi = \phi^T (1 + \omega^T + \omega + \omega^T \omega) \phi \\ &= \phi^T \phi. \end{aligned}$$

The 6 conserved currents are

$$J_{ab}^\mu = \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_c} \overset{ab}{\omega_{cd}} \phi_d = \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_c} \overset{ab}{S} \phi_c$$

where  $\overset{ab}{S} \phi_c = \overset{ab}{\omega_{cd}} \phi_d$ .  $\overset{ab}{\omega_{cd}} = -\overset{ab}{\omega_{dc}}$   
for 6 pairs  $a, b$ ,  $a \neq b$ ,  $a > b$ .

$$Q_{ab} = \int \frac{1}{2} \overset{ab}{\phi_c} \overset{ab}{\omega_{cd}} \phi_d d^3x = \int \overset{ab}{\phi_c} \overset{ab}{\omega_{cd}} \phi_d dt$$

$$= \int \overset{ab}{\pi_c} \overset{ab}{\omega_{cd}} \phi_d d^3x = \int \overset{T}{\pi} \overset{ab}{\omega} \phi d^3x.$$

Extension to n complex fields.

Now

$$\mathcal{L} = \sum_{a=1}^n \partial_\mu \phi_a^* \partial^\mu \phi^a - m^2 \phi_a^* \phi_a$$

is invariant under

$$\phi_a \rightarrow \phi'_a = g_{ab} \phi_b \text{ where } g \in U(n).$$

$$\text{The tiny case is } \phi'_a = \phi_a + (i/\epsilon) T_{ab}^\alpha \phi_b$$

where  $T^\alpha$  is one of the generators of  $U(n)$ .

For each  $\alpha$ , there is a conserved current

$$J^{\mu\alpha} = -i \frac{\partial \mathcal{L}}{\partial \partial^\mu \phi_a} T_{ab}^\alpha \phi_b + \frac{\partial \mathcal{L}}{\partial \partial^\mu \phi_a^*} (+i) T_{ab}^\alpha \phi_b^*,$$

The conserved charges are

$$Q^\alpha = i \int (\phi_b^* T_{ba}^\alpha \pi_a^* - \pi_a T_{ab}^\alpha \phi_b) d^3x,$$

$$\text{They obey } [Q^\alpha, Q^\beta] = i f_{\alpha\beta\gamma} Q^\gamma$$

where the  $f_{\alpha\beta\gamma}$  are the structure constants of  $U(n)$

$$[T^\alpha, T^\beta] = i f_{\alpha\beta\gamma} T^\gamma.$$

2.2.15

However, doing this problem right would be to write

$$\mathcal{L} = \frac{1}{2} \sum_{a=1}^{2n} \partial_\mu \phi_a \partial^\mu \phi^a - m^2 \phi_a^2$$

which is invariant under  $O(2n)$

$$\phi_a \rightarrow \phi'_a = \phi_a - i\epsilon T_{ab}^\alpha \phi_b.$$

Here  $T_{ab}^\alpha = -T_{ba}^\alpha$

$$Q^\alpha = \int \pi^\mu T_{ab}^\alpha \phi^b d^3x.$$

We get more conserved charges doing things correctly. For  $U(n)$  has  $n^2$  generators and  $n^2$  conserved charges, while  $O(2n)$  has  $2n(2n-1)/2$  of each.

But Nature may prefer complex fields and unitary groups, as in the standard model,  $SU_c(3) \otimes SU_l(2) \otimes U(1)$ .