

Magnetic Monopoles

Differential forms:

Let $A \equiv A_\mu dx^\mu$. This is a 1-form

If x and x' are two coordinate systems, then

$$dx^\mu = \frac{\partial x^\mu}{\partial x'^\nu} dx'^\nu, \quad \text{so}$$

$$A = A_\mu dx^\mu = A_\mu \frac{\partial x^\mu}{\partial x'^\nu} dx'^\nu \equiv A'_\nu dx'^\nu$$

So $A'_\nu = \frac{\partial x^\mu}{\partial x'^\nu} A_\mu$. Note that the

1-form A is invariant under coordinate transformations.

Consider $A = \cos\theta d\phi$. Then $A_0 = 0$ and $A_\phi = \cos\theta$.

A p -form is $H = \frac{1}{p!} H_{\mu_1 \dots \mu_p} dx^{\mu_1} \dots dx^{\mu_p}$

A function $f(x)$ is a 0-form.

$F \equiv \frac{1}{2} F_{\mu\nu} dx^\mu dx^\nu$ is a 2-form.

We want the differentials dx and dy , etc., to anti-commute. To see why,

consider the element of area $dx dy$
under a change of variables

$$dx dy = \left(\frac{\partial x}{\partial x'} dx' + \frac{\partial x}{\partial y'} dy' \right) \left(\frac{\partial y}{\partial x'} dx' + \frac{\partial y}{\partial y'} dy' \right).$$

$$\text{If } dx' dy' \equiv dx' \wedge dy' = -dy' \wedge dx' \equiv -dy' dx'$$

$$\text{and } dx' dx' = dx' \wedge dx' = 0$$

$$\text{and } dy' dy' = dy' \wedge dy' = 0, \text{ then}$$

$$dx dy = \left(\frac{\partial x}{\partial x'} \frac{\partial y}{\partial y'} - \frac{\partial x}{\partial y'} \frac{\partial y}{\partial x'} \right) dx' dy'$$

$$= J(x, y; x', y') dx' dy'$$

That is, if the differentials anti-commute,
then $dx dy$ automatically transforms
as an element of area.

Since the dx^{μ} 's are Grassmann (anticommuting) variables, it follows that the tensors that multiply them in p -forms must be anti-symmetric. Thus $dx^{\nu} dx^{\mu} = -dx^{\mu} dx^{\nu}$ implies that in

$$F = \frac{1}{2} F_{\mu\nu} dx^{\mu} dx^{\nu} = -\frac{1}{2} F_{\nu\mu} dx^{\nu} dx^{\mu}$$

$$= -\frac{1}{2} F_{\nu\mu} dx^{\mu} dx^{\nu}. \text{ So } F_{\mu\nu} = -F_{\nu\mu}.$$

P-forms are invariant.

$$F = \frac{1}{2} F_{\mu\nu} dx^\mu dx^\nu$$

$$= \frac{1}{2} F_{\mu\nu} \left(\frac{\partial x^\mu}{\partial y^\lambda} dy^\lambda + \frac{\partial x^\mu}{\partial y^\sigma} dy^\sigma \right) \left(\frac{\partial x^\nu}{\partial y^\lambda} dy^\lambda + \frac{\partial x^\nu}{\partial y^\sigma} dy^\sigma \right)$$

$$= \frac{1}{2} F_{\mu\nu} \left(\frac{\partial x^\mu}{\partial y^\lambda} \frac{\partial x^\nu}{\partial y^\sigma} - \frac{\partial x^\mu}{\partial y^\sigma} \frac{\partial x^\nu}{\partial y^\lambda} \right) dy^\lambda dy^\sigma$$

$$= \frac{1}{2} F_{\lambda\sigma} dy^\lambda dy^\sigma = \frac{1}{2} F(y) dy^\lambda dy^\sigma.$$

And $F_{\mu\nu}$ transforms as a rank-2 antisymmetric tensor

$$F'_{\lambda\sigma} = (x^\mu_{,\lambda} x^\nu_{,\sigma'} - x^\mu_{,\sigma'} x^\nu_{,\lambda}) F_{\mu\nu},$$

where $A_{,\mu'} \equiv \frac{\partial A}{\partial x'^{\mu'}}$.

Now we define d :

$$df = \frac{\partial f}{\partial x^\mu} dx^\mu$$

$$\begin{aligned} dA &= d(A_\mu dx^\mu) = \frac{\partial A_\mu}{\partial x^\nu} dx^\nu dx^\mu \\ &= \frac{1}{2} (\partial_\mu A_\nu - \partial_\nu A_\mu) dx^\mu dx^\nu \end{aligned}$$

Note that $dA = F$. Very slick.

$$dH = d \frac{1}{p!} H_{\mu_1 \mu_2 \dots \mu_p} dx^{\mu_1} dx^{\mu_2} \dots dx^{\mu_p}$$

$$= \frac{1}{p!} \frac{\partial H_{\mu_1 \dots \mu_p}}{\partial x^\nu} dx^\nu dx^{\mu_1} \dots dx^{\mu_p}$$

dH is a $(p+1)$ -form if H is a p -form.

Note that $dd = d^2 = 0$.

Thus $d(dA) = dd(A_{\mu} dx^{\mu})$

$$= dF = \frac{1}{2} \frac{\partial F_{\mu\nu}}{\partial x^\sigma} dx^\sigma dx^\mu dx^\nu = 0$$

Why? Because

$$dF = d(dA) = dd(A_{\mu} dx^{\mu})$$

$$= d(A_{\mu\nu} dx^\nu dx^\mu)$$

$$= A_{\mu\nu,\sigma} dx^\sigma dx^\nu dx^\mu = 0$$

because $A_{\mu\nu,\sigma}$ is symmetric in ν & σ while $dx^\sigma dx^\nu$ is anti-symmetric in ν & σ .

More simply

$$d d = d \quad dx^m \partial_m = \underbrace{dx^\nu dx^\mu}_{A.S.} \underbrace{\partial_\nu \partial_\mu}_S = 0.$$

We usually write $dF = 0$ as

$$0 = \epsilon^{\mu\nu\lambda\sigma} F_{\nu\lambda,\sigma}$$

in which $\epsilon^{\mu\nu\lambda\sigma}$ is totally antisymmetric. These are the two homogeneous Maxwell equations.

A p -form α is closed if $d\alpha = 0$.

A p -form α is exact if there is a $(p-1)$ -form β such that $\alpha = d\beta$.

Every exact form α is closed because $d\alpha = dd\beta = 0$.

Poincaré's lemma says that a closed form ($d\alpha = 0$) is locally exact, that is, locally $\alpha = d\beta$.

E.g., if $\nabla \times F = 0$, then locally $F = -\nabla V$. To see this in form language, we go to

3 dimensions and consider the force 1-form

$$F = F_i dx^i$$

Then $\nabla \times F = 0$ is $dF = F_{ij} dx^j dx^i = 0$

That is, $F_{1,2} - F_{2,1} = (\nabla \times F)_3 = 0$, etc.

So $dF = 0$ and at least locally there's a (-) potential $-V$ such that

$$F = d(-V) = -V_{,i} dx^i, \text{ or } \vec{F} = -\vec{\nabla} V.$$

Forms are even slicker. Since $dx^u dx^v$ is an element of area, the integral of F over a 2-manifold ∂M (a surface) is

$$\int_{\partial M} F = \int \frac{1}{2} F_{uv} dx^u dx^v$$

Thus, the form F contains its surface element $dx^u dx^v$. One may show that

$$\int_M dA = \int_{\partial M} A$$

for any p -form A and $(p-1)$ -form dA .
Here M is a $(p+1)$ -dimensional manifold.

Then for the 1-form $A = A_m dx^m$

$$\int_{\partial M} A = \int A_m dx^m = \int_M dA = \int_M F$$

In 3-D, this is simply

$$\int \vec{A} \cdot d\vec{x} = \int \vec{B} \cdot d\vec{\sigma}$$

Dirac, Yang, and Wu

Let $F = \frac{g}{4\pi} d\cos\theta d\phi$. Integrate over the 2-sphere S^2 and get

$$\int_{S^2} F = \int \frac{g}{4\pi} d\cos\theta d\phi = g.$$

$$\text{But } \int_{S^2} F = \int_{\partial S^3} F = \int_{S^3} dF = 0,$$

so what's going on here? Well, what A would give us this F ? Try

$$A = \frac{g}{4\pi} \cos\theta d\phi \text{ which gives } F = \frac{g}{4\pi} d\cos\theta d\phi.$$

since $d^2\phi = 0$.

But $d\phi$ is not defined at the north and south poles.

$$\text{Try } A_N = \frac{g}{4\pi} (\cos\theta - 1) d\phi$$

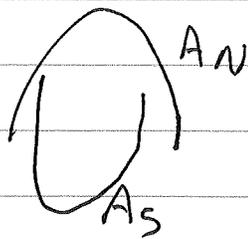
Then $F = dA_N$ and $A_N = 0$ at north pole
But A_N is not defined at the south pole.

So use

$$A_S = \frac{g}{4\pi} (\cos\theta + 1) d\phi$$

which vanishes at the south pole but is undefined at the north pole.

Yang and Wu say use both, that is, use A_N
from AK to Australia



and A_S from Antarctica
to the US.

But what about $A_S - A_N = 2 \frac{g}{4\pi} d\phi$
where they overlap?

Recall that a $U(1)$ gauge transformation is

$$\psi'(x) = e^{i\Lambda(x)} \psi(x)$$

$$A'_m(x) = A_m(x) + \frac{1}{ie} \partial_\mu e^{-i\Lambda(x)} e^{i\Lambda(x)}$$

or

$$A' = A'_m dx^m = A + \frac{1}{ie} e^{-i\Lambda} d e^{i\Lambda}$$

$$= A_m dx^m + \frac{1}{e} \Lambda_{,m} dx^m$$

The difference $A_S - A_N = \frac{g}{2\pi} d\phi$

won't cause problems if it's just a gauge transformation, that is, if

$$A_S - A_N = \frac{g}{2\pi} d\phi = \frac{1}{ie} e^{-i\Lambda} d e^{i\Lambda}$$

i.e.

$$\frac{g d\phi}{2\pi} = \frac{1}{e} \Lambda, \phi d\phi$$

So let $\Lambda = \frac{eg}{2\pi} \phi$.

We need $e^{i\Lambda} = e^{i \frac{eg}{2\pi} \phi}$ to be single valued. That is,

$$e^{\frac{ieg}{2\pi} (\phi + 2\pi)} = e^{\frac{ieg}{2\pi} \phi} = e^{\frac{ieg}{2\pi} \phi + ieg}$$

So we need $eg = 2\pi n$ where n is an integer. So the charge of a magnetic monopole must be

$$g = \frac{2\pi}{e} n.$$

Here F is locally but not globally exact, which is why $g = \int_{S^2} F \neq 0$.

With magnetic monopoles, Maxwell's equations are invariant under

$$(\vec{E} + i\vec{B})' = e^{i\theta} (\vec{E} + i\vec{B}).$$

\vec{E} and \vec{B} are dual, and turning e means huge g and vice versa.

Currents. The current of a world line is

$$J^M(x) = \int d\tau \frac{dx^M}{d\tau} \delta^{(D)}[x - X(\tau)]$$

is invariant under a change $\tau \rightarrow \tau'(\tau)$.

The current of a string is

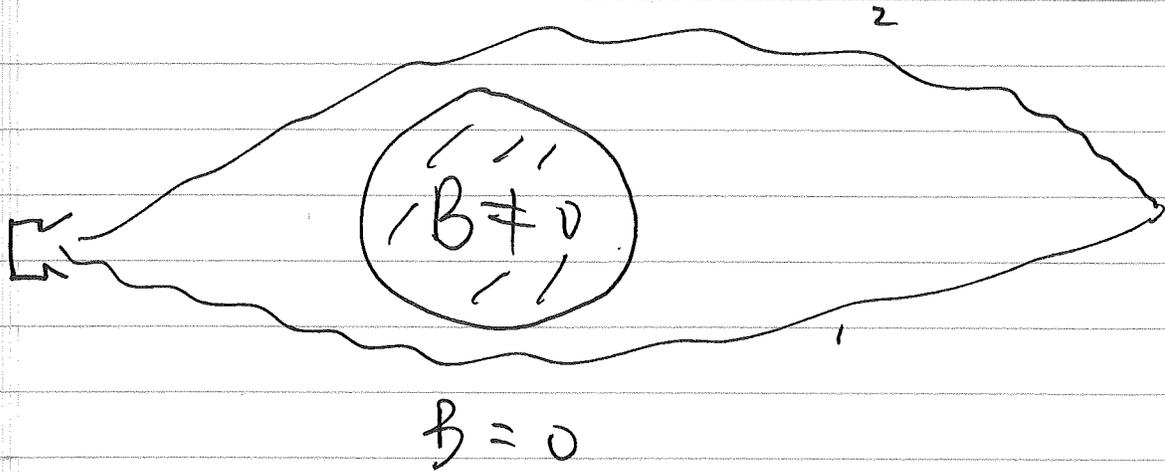
$$J^{M\nu}(x) = \int d\tau d\sigma \det \begin{pmatrix} X^M_{,\tau} & X^{\nu}_{,\tau} \\ X^M_{,\sigma} & X^{\nu}_{,\sigma} \end{pmatrix} \delta^{(D)}[x - X(\tau, \sigma)].$$

$$= \int dx^M dx^{\nu} \delta^{(D)}[x - X(\tau, \sigma)].$$

is invariant under $\tau \rightarrow \tau'(\tau, \sigma)$ & $\sigma \rightarrow \sigma'(\tau, \sigma)$.

$J^{M\nu}$ is an anti-symmetric tensor. So just as $A_{,\mu} J^{\mu}$, we have $B_{,\mu\nu} J^{M\nu}$. String theory has a 2-form potential $B = \frac{1}{2} B_{\mu\nu} dx^{\mu} dx^{\nu}$ and a 3-form field $H = dB$.

The Aharonov-Bohm effect.



phases are e^{iS_1} and e^{iS_2}

S contains $e \int A_n dx^n = e \int A$

The phase difference is

$$e^{i(S_1 - S_2)} = e^{ie \int A_n dx^n}$$

$$\text{In static case, } e^{i(S_1 - S_2)} = e^{ie \int \vec{A} \cdot d\vec{x}}$$

$$= e^{ie \int \vec{B} \cdot \vec{d}\vec{v}} = e^{ie \Phi}$$

where Φ is the flux thru the surface bounded by the curves 1 and 2.