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which is

$$I_- = -e^{2\pi a i} \int_0^\infty \frac{x^a}{(x+1)^2}\, dx = -e^{2\pi a i}\, I. \tag{5.167}$$

Now the sum of all these contour integrals is zero because it is a closed contour that encloses no singularity. So

$$0 = \left(1 - e^{2\pi a i}\right) I + 2\pi i\, a\, e^{\pi a i} \tag{5.168}$$

or

$$I = \frac{\pi a}{\sin(\pi a)}. \tag{5.169}$$

5.15 Cauchy's Principal Value

Suppose that $f(x)$ is differentiable or analytic at and near the point $x = 0$, and that we wish to evaluate the integral

$$K = \lim_{\epsilon \to 0} \int_{-a}^{b} dx\, \frac{f(x)}{x - i\epsilon} \tag{5.170}$$

for $a > 0$ and $b > 0$. First, we regularize the pole at $x = 0$ by using a method devised by Cauchy:

$$K = \lim_{\delta \to 0} \left[\lim_{\epsilon \to 0} \left(\int_{-a}^{-\delta} dx\, \frac{f(x)}{x - i\epsilon} + \int_{-\delta}^{\delta} dx\, \frac{f(x)}{x - i\epsilon} \right) + \int_{\delta}^{b} dx\, \frac{f(x)}{x - i\epsilon} \right]. \tag{5.171}$$

In the first and third integrals, since $|x| \geq \delta$, we may set $\epsilon = 0$

$$K = \lim_{\delta \to 0} \left(\int_{-a}^{-\delta} dx\, \frac{f(x)}{x} + \int_{\delta}^{b} dx\, \frac{f(x)}{x} \right) + \lim_{\delta \to 0} \lim_{\epsilon \to 0} \int_{-\delta}^{\delta} dx\, \frac{f(x)}{x - i\epsilon}. \tag{5.172}$$

We'll discuss the first two integrals before analyzing the last one.

The limit of the first two integrals is called **Cauchy's principal value**

$$P \int_{-a}^{b} dx\, \frac{f(x)}{x} \equiv \lim_{\delta \to 0} \left(\int_{-a}^{-\delta} dx\, \frac{f(x)}{x} + \int_{\delta}^{b} dx\, \frac{f(x)}{x} \right). \tag{5.173}$$

If the function $f(x)$ is nearly constant near $x = 0$, then the large negative values of $1/x$ for x slightly less than zero cancel the large positive values of $1/x$ for x slightly greater than zero.

The point $x = 0$ is not special; Cauchy's principal value is more generally defined by the limit

$$P \int_{-a}^{b} dx\, \frac{f(x)}{x - y} \equiv \lim_{\delta \to 0} \left(\int_{-a}^{y-\delta} dx\, \frac{f(x)}{x - y} + \int_{y+\delta}^{b} dx\, \frac{f(x)}{x - y} \right). \tag{5.174}$$

Using Cauchy's principal value, we may write the quantity K as

$$K = P\int_{-a}^{b} dx\,\frac{f(x)}{x} + \lim_{\delta\to 0}\lim_{\epsilon\to 0}\int_{-\delta}^{\delta} dx\,\frac{f(x)}{x-i\epsilon}. \tag{5.175}$$

To evaluate the last integral, we use differentiability of $f(x)$ near $x = 0$ to write $f(x) = f(0) + xf'(0)$ and then extract $f(0)$ from the integral:

$$\begin{aligned}\lim_{\delta\to 0}\lim_{\epsilon\to 0}\int_{-\delta}^{\delta} dx\,\frac{f(x)}{x-i\epsilon} &= \lim_{\delta\to 0}\lim_{\epsilon\to 0}\int_{-\delta}^{\delta} dx\,\frac{f(0)+x\,f'(0)}{x-i\epsilon}\\ &= f(0)\lim_{\delta\to 0}\lim_{\epsilon\to 0}\int_{-\delta}^{\delta} dx\,\frac{1}{x-i\epsilon}.\end{aligned} \tag{5.176}$$

Now since $1/(z - i\epsilon)$ is analytic, we may deform the straight contour from $x = -\delta$ to $x = \delta$ into the tiny semicircle

$$x \longrightarrow z = \delta\, e^{i\theta} \quad \text{for} \quad \pi \le \theta \le 2\pi \tag{5.177}$$

which avoids the point $x = 0$

$$K = P\int_{-a}^{b} dx\,\frac{f(x)}{x} + f(0)\lim_{\delta\to 0}\int_{-\delta}^{\delta} dz\,\frac{1}{z-i\epsilon}. \tag{5.178}$$

We now can set $\epsilon = 0$ and so write K as

$$\begin{aligned}K &= P\int_{-a}^{b} dx\,\frac{f(x)}{x} + f(0)\lim_{\delta\to 0}\int_{\pi}^{2\pi} i\delta e^{i\theta}d\theta\,\frac{1}{\delta e^{i\theta}}\\ &= P\int_{-a}^{b} dx\,\frac{f(x)}{x} + i\pi f(0).\end{aligned} \tag{5.179}$$

Recalling the definition (5.170) of K, we have

$$\lim_{\epsilon\to 0}\int_{-a}^{b} dx\,\frac{f(x)}{x-i\epsilon} = P\int_{-a}^{b}\frac{f(x)}{x} + i\pi f(0). \tag{5.180}$$

for any function $f(z)$ that is differentiable at $x = 0$.

This trick is of wide applicability. Physicists write it as

$$\frac{1}{x-i\epsilon} = P\frac{1}{x} + i\pi\delta(x). \tag{5.181}$$

It has a brother

$$\frac{1}{x+i\epsilon} = P\frac{1}{x} - i\pi\delta(x) \tag{5.182}$$

and cousins

$$\frac{1}{x-y\pm i\epsilon} = P\frac{1}{x-y} \mp i\pi\delta(x-y). \tag{5.183}$$

***Examples* of Cauchy's Trick** We may use trick (5.182) to evaluate the integral

$$I = \int_{-\infty}^{\infty} dx \, \frac{1}{x + i\epsilon} \, \frac{1}{1 + x^2} \tag{5.184}$$

as

$$I = P \int_{-\infty}^{\infty} dx \, \frac{1}{x} \, \frac{1}{1 + x^2} - i\pi \int_{-\infty}^{\infty} dx \, \frac{\delta(x)}{1 + x^2}. \tag{5.185}$$

Because the function $1/[x(1 + x^2)]$ is odd, the principal part is zero. The integral over the delta function gives unity, so we have

$$I = -i\pi. \tag{5.186}$$

***Example*:** To compute the integral

$$I = \int_0^{\infty} \frac{dk}{k} \sin k \tag{5.187}$$

which we used to derive the formula (3.126) for the Green's function of the laplacian in three dimensions, we first express it as an integral along the whole real axis

$$I = \int_0^{\infty} \frac{dk}{2ik} \left(e^{ik} - e^{-ik} \right) = \int_{-\infty}^{\infty} \frac{dk}{2ik} \, e^{ik} \tag{5.188}$$

and then add a ghost contour along the path $k = R\exp(i\theta)$ for $\theta = 0 \to \pi$ in the limit $R \to \infty$

$$I = \oint \frac{dk}{2ik} \, e^{ik} \equiv P \oint \frac{dk}{2ik} \, e^{ik} \tag{5.189}$$

in which we interpret the integral across the point $k = 0$ as Cauchy's principal value. Using Cauchy's trick (5.182), we have

$$I = P \oint \frac{dk}{2ik} \, e^{ik} = \oint \frac{dk}{2i(k + i\epsilon)} \, e^{ik} + \oint \frac{dk}{2i} \, i\pi\delta(k) \, e^{ik}. \tag{5.190}$$

The first integral vanishes because the pole is below the real axis leaving the desired result

$$I = \int_0^{\infty} \frac{dk}{k} \sin k = \frac{\pi}{2} \tag{5.191}$$

as stated in (3.125).

***Example*—The Feynman Propagator:** Adding $\pm i\epsilon$ to the denominator of a pole term of an integral formula for a function $f(x)$ can slightly shift the pole into the upper or lower half plane, causing the pole to contribute if a ghost contour goes around the UHP or the LHP. The choice of ghost contour often is influenced by the argument x of the function $f(x)$. Physicists use such $i\epsilon$'s to impose boundary conditions on Green's functions.

The Feynman propagator $\Delta_F(x)$ is a Green's function for the Klein-Gordon differential operator (Weinberg, 1995, pp. 274–280)

$$(\Box - m^2)\Delta_F(x) = -\delta^4(x) \tag{5.192}$$

in which $x = (x^0, \boldsymbol{x})$ and

$$\Box = \triangle - \frac{\partial^2}{\partial t^2} = \triangle - \frac{\partial^2}{\partial (x^0)^2} \tag{5.193}$$

is the four-dimensional version of the laplacian $\triangle \equiv \nabla \cdot \nabla$. Here $\delta^4(x)$ is the four-dimensional version of Dirac's delta function (3.27)

$$\delta^4(x) = \int \frac{d^4q}{(2\pi)^4} \exp[i(\boldsymbol{q} \cdot \boldsymbol{x} - q^0 x^0)] = \int \frac{d^4q}{(2\pi)^4} e^{iqx} \tag{5.194}$$

in which $qx = \boldsymbol{q} \cdot \boldsymbol{x} - q^0 x^0$ is the Lorentz-invariant inner product of the 4-vectors q and x. There are many Green's functions that satisfy Eq.(5.192).

Feynman's propagator $\Delta_F(x)$ is the one that satisfies certain boundary conditions which will become evident when we analyze the effect of its $i\epsilon$

$$\Delta_F(x) = \int \frac{d^4q}{(2\pi)^4} \frac{\exp(iqx)}{q^2 + m^2 - i\epsilon}. \tag{5.195}$$

The quantity $E_{\mathrm{q}} = \sqrt{\boldsymbol{q}^2 + m^2}$ is the energy of a particle of mass m and momentum $\boldsymbol{q}$ in natural units with the speed of light $c = 1$. Using this abbreviation and setting $\epsilon' = \epsilon/(2E_q)$, we may write the denominator as

$$q^2 + m^2 - i\epsilon = \boldsymbol{q} \cdot \boldsymbol{q} - \left(q^0\right)^2 + m^2 - i\epsilon = \left(E_q - i\epsilon' - q^0\right)\left(E_q - i\epsilon' + q^0\right) + \epsilon'^2 \tag{5.196}$$

in which ϵ'^2 is negligible. We now drop the prime on the ϵ and do the q^0 integral

$$I(\boldsymbol{q}) = -\int_{-\infty}^{\infty} \frac{dq^0}{2\pi} e^{-iq^0x^0} \frac{1}{[q^0 - (E_{\mathrm{q}} - i\epsilon)]\,[q^0 - (-E_{\mathrm{q}} + i\epsilon)]}. \tag{5.197}$$

The function

$$f(q^0) = e^{-iq^0x^0} \frac{1}{[q^0 - (E_{\mathrm{q}} - i\epsilon)]\,[q^0 - (-E_{\mathrm{q}} + i\epsilon)]} \tag{5.198}$$

has poles at $E_{\mathrm{q}} - i\epsilon$ and at $-E_{\mathrm{q}} + i\epsilon$, as shown in Fig. 5.9. If $x^0 > 0$, then

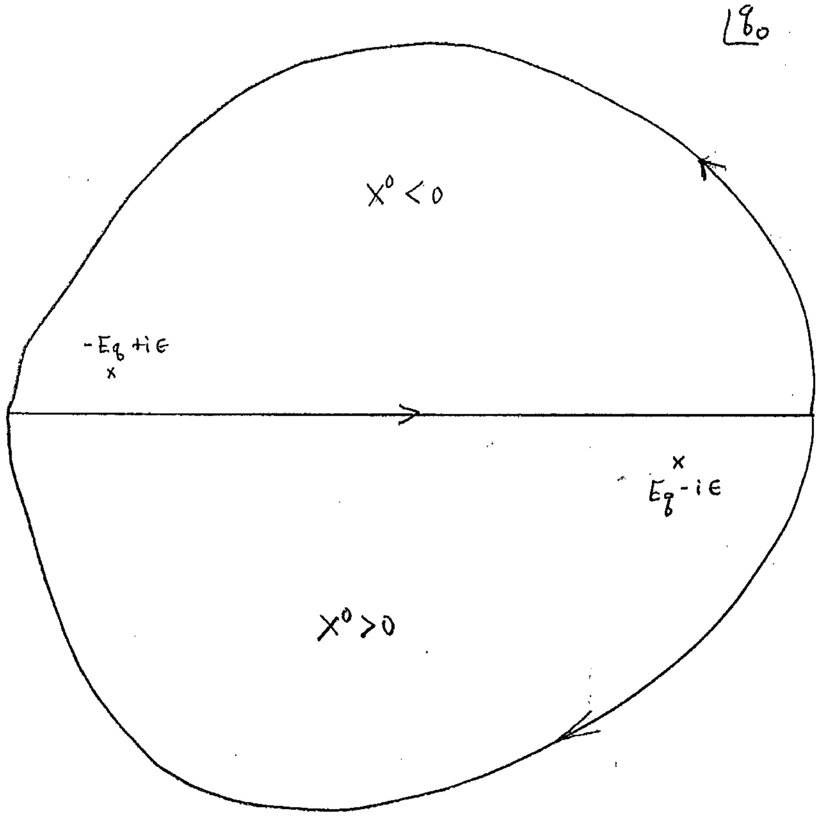

Figure 5.9 In Eq. (5.198), the function $f(q^0)$ has poles at $\pm(E_{\mathbf{q}} - i\epsilon)$, and the function $\exp(-iq^0x^0)$ is exponentially suppressed in the LHP if $x^0 > 0$ and in the UHP if $x^0 < 0$. So we can add a ghost contour in the LHP if $x^0 > 0$ and in the UHP if $x^0 < 0$.

we can add a ghost contour that goes cw around the LHP, and we get

$$I(q) = ie^{-iE_{\mathbf{q}}x^0}\frac{1}{2E_{\mathbf{q}}} \quad x^0 > 0. \tag{5.199}$$

If $x^0 < 0$, we add a ghost contour that goes ccw around the UHP, and we get

$$I(q) = ie^{iE_{\mathbf{q}}x^0}\frac{1}{2E_{\mathbf{q}}} \quad x^0 < 0. \tag{5.200}$$

Using Heaviside's step function

$$\theta(x) = \frac{x + |x|}{2}, \tag{5.201}$$

we may combine the last two equations into

$$-\,iI(\boldsymbol{q}) = \frac{1}{2E_{\mathbf{q}}}\left[\theta(x^0)\,e^{-iE_{\mathbf{q}}x^0} + \theta(-x^0)\,e^{iE_{\mathbf{q}}x^0}\right]. \tag{5.202}$$

In terms of the Lorentz-invariant function

$$\Delta_+(x) = \frac{1}{(2\pi)^3}\int \frac{d^3q}{2E_{\mathbf{q}}}\exp[i(\boldsymbol{q}\cdot\boldsymbol{x} - E_{\mathbf{q}}x^0)] \tag{5.203}$$

and with a factor of $-i$, the Feynman propagator is

$$-\,i\Delta_F(x) = \theta(x^0)\,\Delta_+(x) + \theta(-x^0)\,\Delta_+(\mathbf{x}, -x^0). \tag{5.204}$$

But the integral (5.203) defining $\Delta_+(x)$ is insensitive to the sign of $\boldsymbol{q}$, and so

$$\begin{aligned}\Delta_+(-x) &= \frac{1}{(2\pi)^3}\int \frac{d^3q}{2E_{\mathbf{q}}}\exp[i(-\boldsymbol{q}\cdot\boldsymbol{x} + E_{\mathbf{q}}x^0)] \\ &= \frac{1}{(2\pi)^3}\int \frac{d^3q}{2E_{\mathbf{q}}}\exp[i(\boldsymbol{q}\cdot\boldsymbol{x} + E_{\mathbf{q}}x^0)] = \Delta_+(\mathbf{x}, -x^0).\end{aligned} \tag{5.205}$$

Thus we arrive at the standard form of the Feynman propagator

$$-\,i\Delta_F(x) = \theta(x^0)\,\Delta_+(x) + \theta(-x^0)\,\Delta_+(-x). \tag{5.206}$$

The Lorentz-invariant function $\Delta_+(x-y)$ is the commutator of the positive-frequency part

$$\phi^+(x) = \int \frac{d^3p}{\sqrt{(2\pi)^3 2p^0}}\exp[i(\boldsymbol{p}\cdot\boldsymbol{x} - p^0x^0)]\,a(\boldsymbol{p}) \tag{5.207}$$

of a scalar field $\phi = \phi^+ + \phi^-$ with its negative-frequency part

$$\phi^-(y) = \int \frac{d^3q}{\sqrt{(2\pi)^3 2q^0}}\exp[-i(\boldsymbol{q}\cdot\boldsymbol{y} - q^0y^0)]\,a^\dagger(\boldsymbol{q}) \tag{5.208}$$

where $p^0 = E_{\mathbf{p}} = \sqrt{\mathbf{p}^2 + m^2}$ and $q^0 = E_{\mathbf{q}}$. For since the annihilation operators $a(\boldsymbol{q})$ and the creation operators $a^\dagger(\boldsymbol{p})$ satisfy the commutation relation

$$[a(\boldsymbol{q}), a^\dagger(\boldsymbol{p})] = \delta^3(\boldsymbol{q} - \boldsymbol{p}) \tag{5.209}$$

we have

$$[\phi^+(x), \phi^-(y)] = \int \frac{d^3p\, d^3q}{(2\pi)^3 2\sqrt{q^0 p^0}}\, e^{ipx-iqy}\, [a(\mathbf{p}), a^\dagger(\mathbf{q})]$$
$$= \int \frac{d^3p}{(2\pi)^3 2p^0}\, e^{ip(x-y)} = \Delta_+(x-y) \tag{5.210}$$

in which $px = \mathbf{p}\cdot\mathbf{x} - p^0 x^0$, *etc.*

Incidentally, at points x that are space-like

$$x^2 = \mathbf{x}^2 - (x^0)^2 \equiv r^2 > 0 \tag{5.211}$$

the Lorentz-invariant function $\Delta_+(x)$ depends only upon $r = +\sqrt{x^2}$ and has the value (Weinberg, 1995, p. 202)

$$\Delta_+(x) = \frac{m}{4\pi^2 r}\, K_1(mr) \tag{5.212}$$

in which the Hankel function K_1 is

$$K_1(z) = -\frac{\pi}{2}\, [J_1(iz) + iN_1(iz)] = \frac{1}{z} + \frac{z}{2j+2}\left[\ln\left(\frac{z}{2}\right) + \gamma - \frac{1}{2j+2}\right] + \ldots \tag{5.213}$$

where J_1 is the first Bessel function, N_1 is the first Neumann function, and $\gamma = 0.57721\ldots$ is the Euler-Mascheroni constant.

The Feynman propagator arises most simply as the mean value in the vacuum of the **time-ordered product** of the fields $\phi(x)$ and $\phi(y)$

$$\mathcal{T}\{\phi(x)\phi(y)\} \equiv \theta(x^0 - y^0)\phi(x)\phi(y) + \theta(y^0 - x^0)\phi(y)\phi(x). \tag{5.214}$$

Since the operators $a(\mathbf{p})$ and $a^\dagger(p)$ respectively annihilate the vacuum ket $a(p)|0\rangle = 0$ and bra $\langle 0|a^\dagger(p) = 0$, the mean value in the vacuum of the time-ordered product is $-i\Delta_F(x-y)$

$$\begin{aligned} -i\Delta_F(x-y) &= \langle 0|\mathcal{T}\{\phi(x)\phi(y)\}|0\rangle \\ &= \langle 0|\theta(x^0 - y^0)\phi^+(x)\phi^-(y) + \theta(y^0 - x^0)\phi(y)^+\phi^-(x)|0\rangle \\ &= \theta(x^0 - y^0)\langle 0|[\phi^+(x), \phi^-(y)]|0\rangle \\ &\quad +\theta(y^0 - x^0)\langle 0|[\phi^+(y), \phi^-(x)]|0\rangle \\ &= \theta(x^0 - y^0)\Delta_+(x-y) + \theta(y^0 - x^0)\Delta_+(y-x) \end{aligned} \tag{5.215}$$

which is (5.206). Feynman put $i\epsilon$ in the denominator of the Fourier transform of his propagator to get this result.