

Scattering of Fermions

Suppose the hamiltonian density of the interaction is

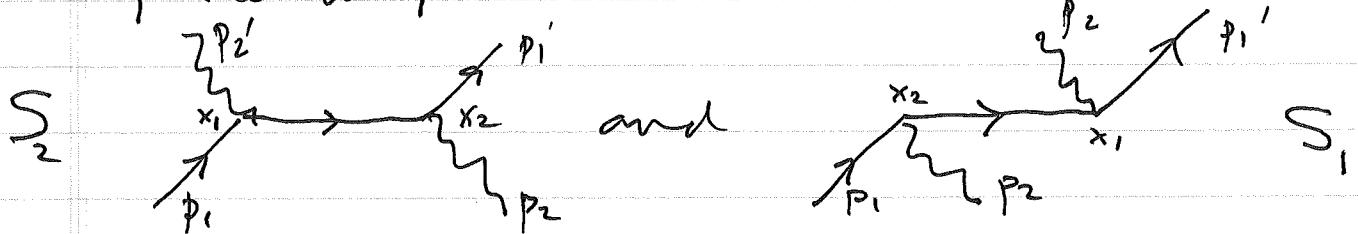
$$\mathcal{H}_I(x) = g \bar{\Psi}(x) \Psi(x) \phi(x)$$

where ϕ is a spinless neutral field but where Ψ is a Dirac field

$$\Psi = \frac{1}{\sqrt{2}} (\Psi^{(1)} + i \Psi^{(2)})$$

$$\text{Here } \bar{\Psi} \equiv \Psi^\dagger \gamma^0 = (\bar{\epsilon}, \bar{\eta}^+) (\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}) = (\bar{\eta}, \bar{\epsilon}^+).$$

Let's consider the processes represented by the diagrams



for $\phi \Psi$ scattering. The calculation is similar to the one done on pages 9-14 of "Notes on Feynman diagrams" except that now Ψ is a 4-component spin-1/2 field:

$$\Psi_\alpha(x) = \int \frac{d^3 p}{(2\pi)^3 \sqrt{2E_p}} \sum_s \alpha_{ps} u_{\alpha ps} e^{-ipx} = \int \frac{d^3 p}{(2\pi)^3 \sqrt{2E_p}} \sum_s \alpha_{ps} u_{\alpha ps}^*(p) e^{-ipx}$$

$$\Psi_\beta^{(-)}(x) = \int \frac{d^3 p}{(2\pi)^3 \sqrt{2E_p}} \sum_s b_{ps}^+ v_{\beta ps} e^{ipx} = \int \frac{d^3 p}{(2\pi)^3 \sqrt{2E_p}} \sum_s b_{ps}^*(p) v_{\beta ps}^*(p) e^{ipx} \quad (3.99)$$

$$\Psi_{\alpha}^{(+)}(x) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2\varepsilon_p}} \sum_s b_{ps} \bar{U}_{s\alpha} e^{-ipx} = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2\varepsilon_p}} \sum_s b_{ps} \bar{U}_{s\alpha} e^{-ipx} \quad (3.100)$$

$$\Psi_{\beta}^{(-)}(x) = \int \frac{d^3 p}{(2\pi)^3} \sum_s a_{ps}^+ \bar{U}_{s\beta} e^{ipx} = \int \frac{d^3 p}{(2\pi)^3} \sum_s a_{ps}^+ \bar{U}_{s\beta} e^{ipx} \quad (3.100)$$

and $U(p)$ is the 4×2 matrix

$$U(p) = \begin{pmatrix} U(p, \frac{1}{2}), U(p, -\frac{1}{2}) \end{pmatrix} = \frac{1}{\sqrt{2(m+p^0)}} \begin{pmatrix} p^0 + m - p \cdot \sigma \\ p^0 + m + p \cdot \sigma \end{pmatrix}$$

while

$$V_{as}(p) = (V_{p,\frac{1}{2}}, V_{p,-\frac{1}{2}}) = \frac{1}{\sqrt{2(m+p^0)}} \begin{pmatrix} (p^0 + m - p \cdot \sigma) \sigma_2 \\ -(p^0 + m + p \cdot \sigma) \sigma_2 \end{pmatrix}$$

and the 2×4 matrices \bar{U} and \bar{V} are

$$\bar{U}_{s\alpha}(p) = (U(p))_{s\alpha}^0 = (p^0 + m + p \cdot \sigma, p^0 + m - p \cdot \sigma) \frac{1}{\sqrt{2(m+p^0)}}$$

$$\bar{V}_{s\alpha}(p) = (V(p))_{s\alpha}^0 = (-\sigma_2(p^0 + m + p \cdot \sigma), \sigma_2(p^0 + m - p \cdot \sigma)) \frac{1}{\sqrt{2(m+p^0)}}$$

As shown on pages 36-38 of "Notes on Wigner Rotations...," the spin sums are

$$U\bar{U} = m + \cancel{\not{p}} \quad \text{and} \quad V\bar{V} = \cancel{\not{p}} - m.$$

In these notes, $b_{ps} = b(p,s)$, $a(p,s) = a_{ps}$, $U_{as} = U_{as}(p)$, $V_{as} = V_{as}(p)$, etc.

The initial state is

$$|p_1 s_1, p_2\rangle = \sqrt{2\epsilon_{p_1} 2\epsilon_{p_2}} a_{p_1 s_1}^+ c_{p_2} |0\rangle$$

and the final state is

$$|p'_1 s'_1, p'_2\rangle = \sqrt{2\epsilon_{p'_1} 2\epsilon_{p'_2}} a_{p'_1 s'_1}^+ c_{p'_2} |0\rangle$$

in which c^\dagger is the creation operator of the neutral field

$$\phi(x) = \psi^{(+)}(x) + \psi^{(-)}(x) = \frac{\int d^3 p}{(2\pi)^3 \sqrt{2\epsilon_p}} (c_p e^{-ipx} + c_p^\dagger e^{ipx}).$$

S_0

$$-i \int f_2(x) d^4 x$$

$$\langle p'_1 s'_1, p'_2 | S | p_1 s_1, p_2 \rangle = \langle p'_1 s'_1, p'_2 | \bar{e} \int T(f_1(x_1) f_2(x_2)) c_{p_2}^+ a_{p_1 s_1}^+ |0\rangle.$$

The lowest non-zero amplitude is

$$S = \sqrt{2\epsilon'_s} \frac{(-i)^3}{2} \langle 0 | a_{p'_1 s'_1}^+ c_{p'_2} \int T(f_1(x_1) f_2(x_2)) c_{p_2}^+ a_{p_1 s_1}^+ |0\rangle.$$

We stop c_{p_2} at x_2 and get a factor of 2

$$\begin{aligned} S &= -g^2 \sqrt{2\epsilon'_s} \langle 0 | a_{p'_1 s'_1}^+ c_{p'_2} \int T(\bar{\psi}_{(k_1)} \psi_{(k_1)} \phi(x_1) \bar{\psi}_{(k_2)} \psi_{(k_2)} \phi(x_2)) c_{p_2}^+ a_{p_1 s_1}^+ |0\rangle \\ &= -\frac{g^2 \sqrt{2\epsilon'_s}}{\sqrt{2\epsilon_{p_2} 2\epsilon_{p'_1}}} \langle 0 | a_{p'_1 s'_1}^+ \int T(\bar{\psi}_{(k_1)} \psi_{(k_1)} \bar{\psi}_{(k_2)} \psi_{(k_2)}) a_{p_1 s_1}^+ |0\rangle e_{d_{x_1}^4 d_{x_2}^4}^{q_1 q_2} \\ &= S_1 + S_2 \end{aligned}$$

where in S_1 , $a_{p_1 s_1}^+$ is stopped at x_2

$$S_1 = -\frac{g^2 \sqrt{2E's}}{\sqrt{2\varepsilon_{p_2} 2\varepsilon_{p_1}}} \langle 0 | a_{p_1 s_1} | T(\bar{\psi}(x_1) \psi(x_1) \bar{\psi}(x_2) \psi(x_2)) \\ i p_2' x_1 - i p_2 x_2 \\ \times e^{d^4 x_1 d^4 x_2 a_{p_1 s_1}^\dagger | 0 \rangle}$$

and S_2 is the diagram for stopping $a_{p_1 s_1}^\dagger$ at x_1

$$S_2 = -\frac{g^2 \sqrt{2E's}}{\sqrt{2\varepsilon_{p_2} 2\varepsilon_{p_1}}} \langle 0 | a_{p_1 s_1} | T(\bar{\psi}(x_1) \psi(x_1) \bar{\psi}(x_2) \psi(x_2)) \\ i p_2' x_1 - i p_2 x_2 \\ \times e^{d^4 x_1 d^4 x_2 a_{p_1 s_1}^\dagger | 0 \rangle}.$$

Now watch this in S_1 :

$$\langle 0 | a_{p_1 s_1} | T(\bar{\psi}(x_1) \psi(x_1) \bar{\psi}(x_2) \psi(x_2)) a_{p_1 s_1}^\dagger | 0 \rangle \\ = \langle 0 | a_{p_1 s_1} | \left\{ \theta(x_1^0 - x_2^0) \bar{\psi}(x_1) \psi(x_1) \bar{\psi}(x_2) \psi(x_2) \right. \\ \left. + \theta(x_2^0 - x_1^0) \bar{\psi}(x_2) \psi(x_2) \bar{\psi}(x_1) \psi(x_1) \right\} a_{p_1 s_1}^\dagger | 0 \rangle$$

$$= \frac{1}{\sqrt{2\varepsilon_{p_1}}} e^{-i p_1 x_2} \langle 0 | a_{p_1 s_1} | \left\{ \theta(x_1^0 - x_2^0) \bar{\psi}(x_1) \psi(x_1) \bar{\psi}(x_2) \right. \\ \left. + \theta(x_2^0 - x_1^0) \bar{\psi}(x_2) \psi(x_2) \bar{\psi}(x_1) \psi(x_1) \right\} | 0 \rangle u_{s_1}^{(p_1)}$$

$$= \frac{e^{i p_1 x_2}}{\sqrt{2\varepsilon_{p_1} 2\varepsilon_{p_2}}} \langle 0 | \left\{ \theta(x_1^0 - x_2^0) \psi_\beta(x_1) \bar{\psi}_\alpha(x_2) - \theta(x_2^0 - x_1^0) \bar{\psi}_\alpha(x_2) \psi_\beta(x_1) \right\} | 0 \rangle \\ \times \bar{u}_{p_1 s_1 \beta}$$

Note the minus sign

This minus sign is built into the definition of the time-ordered product of two Fermi fields:

$$(def) \quad \langle 0 | T(\bar{\psi}_\beta(x_1) \bar{\psi}_\alpha(x_2)) | 0 \rangle \equiv \langle 0 | \theta(x_1^0 - x_2^0) \bar{\psi}_\beta(x_1) \bar{\psi}_\alpha(x_2) \\ - \theta(x_2^0 - x_1^0) \bar{\psi}_\alpha(x_2) \bar{\psi}_\beta(x_1) | 0 \rangle.$$

So in S,

$$\langle 0 | a_{p'_1 s'_1} T(\bar{\psi}^{(+)}(x_1) \psi(x_1) \bar{\psi}(x_2) \psi^{(+)}(x_2)) a_{p_1 s_1}^\dagger | 0 \rangle \\ = \bar{u}_{p'_1 s'_1 \beta} \langle 0 | T(\bar{\psi}_\beta(x_1) \bar{\psi}_\alpha(x_2)) | 0 \rangle u_{p_1 s_1 \alpha} \frac{e}{\sqrt{2\epsilon_{p'_1} 2\epsilon_{p_2}}} e^{i p'_1 x_1 - i p_1 x_2}$$

whence

$$S_1 = -g^2 \int \bar{u}_{p'_1 s'_1 \beta} \langle 0 | T(\bar{\psi}_\beta(x_1) \bar{\psi}_\alpha(x_2)) | 0 \rangle u_{p_1 s_1 \alpha}^\dagger \\ \times e^{i x_1(p'_1 + p_2) - i x_2(p_1 + p_2)} d^4 x_1 d^4 x_2.$$

This need for a minus sign built in. $\langle 0 | T(\bar{\psi} \bar{\psi}) | 0 \rangle$ also shows up in S_2 :

$$\langle 0 | a_{p'_1 s'_1} T(\bar{\psi}(x_1) \psi^{(+)}(x_1) \bar{\psi}^{(-)}(x_2) \psi(x_2)) a_{p_1 s_1}^\dagger | 0 \rangle \\ = \langle 0 | a_{p'_1 s'_1} \left\{ \theta(x_1^0 - x_2^0) \bar{\psi}(x_1) \psi^{(+)}(x_1) \bar{\psi}^{(-)}(x_2) \psi(x_2) \right. \\ \left. + \theta(x_2^0 - x_1^0) \bar{\psi}^{(-)}(x_2) \psi(x_2) \bar{\psi}^{(+)}(x_1) \psi(x_1) \right\} a_{p_1 s_1}^\dagger | 0 \rangle$$

$$\begin{aligned}
 &= \frac{\bar{u}(p_1) e^{-ip_1 x_1}}{\sqrt{2\epsilon_{p_1}}} \langle 0 | \alpha(p_1, s_1) \{ \theta(x_1^0 - x_2^0) \bar{\psi}_\alpha(x_1) \bar{\psi}^{(-)}(x_2) \psi(x_2) \\
 &\quad + \theta(x_2^0 - x_1^0) \bar{\psi}^{(-)}(x_2) \psi(x_2) \bar{\psi}_\alpha(x_1) \} | 0 \rangle \\
 &= \frac{\bar{u}(p'_1) u(p_1)}{\sqrt{2\epsilon_{p_1} 2\epsilon_{p'_1}}} e^{-ip_1 x_1 + ip'_1 x_2} \\
 &\quad \times \langle 0 | -\theta(x_1^0 - x_2^0) \bar{\psi}_\alpha(x_1) \psi_\beta(x_2) + \theta(x_2^0 - x_1^0) \bar{\psi}_\beta(x_2) \bar{\psi}_\alpha(x_1) | 0 \rangle \\
 &= \frac{\bar{u}(p'_1)}{\sqrt{2\epsilon_{p'_1}}} \langle 0 | T(\psi_\beta(x_2) \bar{\psi}_\alpha(x_1)) | 0 \rangle u_{\alpha s_1}(p_1) e^{-ip_1 x_1 + ip'_1 x_2}
 \end{aligned}$$

in which we used the definition (def) with the fermionic minus sign built in. Thus

$$S_2 = -g^2 \frac{\bar{u}(p'_1)}{\sqrt{2\epsilon_{p'_1}}} \int \langle 0 | T(\psi_\beta(x_2) \bar{\psi}_\alpha(x_1)) | 0 \rangle u_{\alpha s_1}(p_1) \\
 \times e^{-i\gamma_1(p_1 - p'_1) - i\gamma_2(p_2 - p'_1)} d^4 x_1 d^4 x_2.$$

Now we evaluate the Feynman propagator for spin-one-half particles.

$$\langle 0 | T(\psi_\beta(x_1) \bar{\psi}_\alpha(x_2)) | 0 \rangle$$

$$= \langle 0 | \theta(x_1^0 - x_2^0) \psi_\beta^{(+)}(x_1) \bar{\psi}_\alpha^{(-)}(x_2) - \theta(x_2^0 - x_1^0) \bar{\psi}_\alpha^{(+)}(x_2) \psi_\beta^{(-)}(x_1) | 0 \rangle$$

$$= \langle 0 | \theta(x_1^0 - x_2^0) \psi_\beta^{(+)}(x_1) \bar{\psi}_\alpha^{(-)}(x_2) - \theta(x_2^0 - x_1^0) \bar{\psi}_\alpha^{(+)}(x_2) \psi_\beta^{(-)}(x_1) | 0 \rangle$$

$$= \langle 0 | \theta(x_1^0 - x_2^0) \{ \psi_\beta^{(+)}(x_1), \bar{\psi}_\alpha^{(-)}(x_2) \} - \theta(x_2^0 - x_1^0) \{ \bar{\psi}_\alpha^{(+)}(x_2), \psi_\beta^{(-)}(x_1) \} | 0 \rangle$$

$$= \theta(x_1^0 - x_2^0) \{ \psi_\beta^{(+)}(x_1), \bar{\psi}_\alpha^{(-)}(x_2) \} - \theta(x_2^0 - x_1^0) \{ \bar{\psi}_\alpha^{(+)}(x_2), \psi_\beta^{(-)}(x_1) \}$$

So we must evaluate the anti-commutator

$$\{ \psi_\beta^{(+)}(x_1), \bar{\psi}_\alpha^{(-)}(x_2) \} = \int \frac{d^3 p d^3 q}{(2\pi)^6} \sum_{ps} \left\{ a_{ps} u_p \beta_s e^{-ipx_1}, a_{qs}^\dagger \bar{u}_{qs}^\dagger \alpha e^{iqx_2} \right\}$$

$$= \int \frac{d^3 p}{(2\pi)^3 2\epsilon_p} \sum_s u_{ps}^{(p)} \bar{u}_{qs}^{(p)} e^{-ip(x_1 - x_2)}$$

Now

$$\sum_s u_{ps}^{(p)} \bar{u}_{qs}^{(p)} = (u(p) \bar{u}(p))_{\beta\alpha} = (\not{p} + m)_{\beta\alpha}, \text{ so}$$

$$(+) \quad \{ \psi_\beta^{(+)}(x_1), \bar{\psi}_\alpha^{(-)}(x_2) \} = \int \frac{d^3 p}{(2\pi)^3 2\epsilon_p} (\not{p} + m)_{\beta\alpha} e^{-ip(x_1 - x_2)}$$

The other term is

$$\begin{aligned}
 & \left\{ \bar{\Psi}_\alpha^{(+)}(x_2), \Psi_\beta^{(-)}(x_1) \right\} = \left\{ \Psi_\beta^{(+)}(x_1), \bar{\Psi}_\alpha^{(+)}(x_2) \right\} \\
 &= \int \frac{d^3 p d^3 q}{(2\pi)^6 \sqrt{2E_p 2E_q}} \sum_{ss'} \left\{ b_{ps}^\dagger v_p v_{ps'} e^{ipx_1} e^{-iqx_2} \right. \\
 &\quad \left. - ip(x_2 - x_1) \right\} \\
 &= \int \frac{d^3 p}{(2\pi)^3 2E_p} \sum_s v_{p\beta s} \bar{v}_{ps\alpha} e^{-ip(x_2 - x_1)} \\
 (-) \quad &= \int \frac{d^3 p}{(2\pi)^3 2E_p} (\not{p} + m)_{\beta\alpha} e^{-ip(x_1 - x_2)}
 \end{aligned}$$

Time to write these a.c.'s in terms of Δ_+ :

$$\begin{aligned}
 & \left\{ \Psi_\beta^{(+)}(x_1), \bar{\Psi}_\alpha^{(+)}(x_2) \right\} = \int \frac{d^3 p}{(2\pi)^3 2E_p} (\not{p} + m)_{\beta\alpha} e^{-ip(x_1 - x_2)} \\
 &= (i\not{x} + m)_{\beta\alpha} \int \frac{d^3 p}{(2\pi)^3 2E_p} e^{-ip(x_1 - x_2)} \\
 &= (i\not{x} + m)_{\beta\alpha} \Delta_+(x_1 - x_2)
 \end{aligned}$$

in which $\not{x} = \partial_a \gamma^a = \frac{\partial}{\partial x_1^a} \gamma^a$. Similarly,

$$\left\{ \bar{\Psi}_\alpha^{(+)}(x_2), \Psi_\beta^{(+)}(x_1) \right\} = - (i\not{x} + m)_{\beta\alpha} \Delta_+(x_2 - x_1)$$

where again $\not{x} = \frac{\partial}{\partial x_1^a} \gamma^a$.

So putting this together, we get

$$\begin{aligned}
 & \langle 0 | T(\psi_\beta(x_1) \bar{\psi}_\alpha(x_2)) | 0 \rangle \equiv S_F(x_1 - x_2)_{\beta\alpha} \\
 & = \theta(x_1^0 - x_2^0) (i\gamma^\mu \delta_{\beta\alpha} \Delta + i(x_1 - x_2)) \\
 & + \theta(x_2^0 - x_1^0) (i\gamma^\mu \delta_{\beta\alpha} \Delta + i(x_2 - x_1)) \\
 & = (i\gamma^\mu \delta_{\beta\alpha}) \left[\theta(x_1^0 - x_2^0) \Delta + i(x_1 - x_2) + \theta(x_2^0 - x_1^0) \Delta + i(x_2 - x_1) \right] \\
 & - i \left[\partial_0 \gamma^0 \theta(x_1^0 - x_2^0) \right] \Delta + i(x_1 - x_2) - i \left[\partial_0 \gamma^0 \theta(x_2^0 - x_1^0) \right] \Delta + i(x_2 - x_1) \\
 & = (i\gamma^\mu \delta_{\beta\alpha}) (-i \Delta_F(x_1 - x_2)) \\
 & - i \gamma^0 \delta(x_1^0 - x_2^0) [\Delta + i(x_1 - x_2) - \Delta + i(x_2 - x_1)] \\
 & = -i(i\gamma^\mu \delta_{\beta\alpha}) \Delta_F(x_1 - x_2) \\
 & = +i(i\gamma^\mu \delta_{\beta\alpha}) \int \frac{e^{-ik(x_1 - x_2)}}{k^2 - m^2 + i\epsilon} \frac{d^4 k}{(2\pi)^4}
 \end{aligned}$$

$$= \int e^{-ik(x_1 - x_2)} \frac{i(k + m)_{\beta\alpha}}{k^2 - m^2 + i\epsilon} \frac{d^4 k}{(2\pi)^4},$$

which is P&S's (3.121).

S_0

$$\begin{aligned}
 S_0 &= -g^2 \int \bar{u}_{p_1's_1'\beta} \frac{e^{-ik(x_1-x_2)}}{k^2 - m^2 + i\epsilon} \frac{d^4 k}{(2\pi)^4} u_{p_1s_1\alpha} e^{ikx_1} d^4 x_1 d^4 x_2 \\
 &= -g^2 \int \bar{u}_{p_1's_1'\beta} \frac{e^{-ikx_1}}{k^2 - m^2 + i\epsilon} d^4 k u_{p_1s_1\alpha} e^{ikx_1} \delta^4(k - p_1 - p_2) dx_1 \\
 &= -g^2 \int \bar{u}_{p_1's_1'} \frac{i(p_1' + p_2' + m)}{(p_1' + p_2')^2 - m^2} u_{p_1s_1} e^{ikx_1} \\
 &= -g^2 (2\pi)^4 \frac{\delta^{(4)}(p_1' + p_2' - p_1 - p_2) \bar{u}_{p_1's_1'}}{(p_1' + p_2')^2 - m^2} u_{p_1s_1}
 \end{aligned}$$

and

$$\begin{aligned}
 S_2 &= -g^2 \int \bar{u}_{p_1's_1'} \frac{e^{-ik(x_2-x_1)}}{k^2 - m^2 + i\epsilon} \frac{d^4 k}{(2\pi)^4} e^{ikx_1} d^4 x_1 d^4 x_2 \\
 &= -g^2 \int \bar{u}_{p_1's_1'} \frac{e^{-ikx_1} + x_1(p_1' - p_2')}{(p_1' - p_2')^2 - m^2} u_{p_1s_1} e^{ikx_1} \\
 &= -g^2 (2\pi)^4 \frac{\delta^{(4)}(p_1' + p_2' - p_1 - p_2) \bar{u}_{p_1's_1'}}{(p_1' - p_2')^2 - m^2} u_{p_1s_1}
 \end{aligned}$$

S_2 looks a little nicer if we use

$$p'_1 - p'_2 = p_1 - p_2 \quad \text{to write}$$

$$S_2 = -g^2 (2\pi)^4 \delta^{(4)}(p'_1 + p'_2 - p_1 - p_2) \bar{u}_{p_1 s_1} i((p_1 - p_2) + m) u_{p_2 s_2} \frac{(p_1 - p_2)^2 - m^2}{(p_1 - p_2')^2 - m^2}.$$

So the full amplitude is

$$S = S_1 + S_2 = -g^2 (2\pi)^4 \delta^{(4)}(p'_1 + p'_2 - p_1 - p_2)$$

$$\times \bar{u}_{p_1 s_1} \left[\frac{i((p_1 - p_2) + m)}{(p_1 + p_2)^2 - m^2} + \frac{i((p_1 - p_2') + m)}{(p_1 - p_2')^2 - m^2} \right] u_{p_2 s_2}$$

