

## VII.2

## Electroweak Unification

---

### The scourge of massless spin 1 particles

With the benefit of hindsight, we now know that Nature likes Yang-Mills theory. In the late 1960s and early 1970s, the electromagnetic and weak interactions were unified into an electroweak interaction, described by a nonabelian gauge theory based on the group  $SU(2) \otimes U(1)$ . Somewhat later, in the early 1970s, it was realized that the strong interaction can be described by a nonabelian gauge theory based on the group  $SU(3)$ . Nature literally consists of a web of interacting Yang-Mills fields.

But when the theory was first proposed in 1954, it seemed to be totally inconsistent with observations as they were interpreted at that time. As Yang and Mills themselves pointed out in their paper, the theory contains massless spin 1 particles, which were certainly not known experimentally. Thus, except for interest on the part of a few theorists (Schwinger, Glashow, Bludman, and others) who found the mathematical structure elegantly attractive and felt that nonabelian gauge theory must somehow be relevant for the weak interaction, the theory gradually sank into oblivion and was not part of the standard graduate curriculum in particle physics in the 1960s.

Again with the benefit of hindsight, it would seem that there are only two logical solutions to the difficulty that experimentalists do not see any massless spin 1 particles except for the photon: (1) the Yang-Mills particles somehow acquire mass, or (2) the Yang-Mills particles are in fact massless but are somehow not observed. We now know that the first possibility was realized in the electroweak interaction and the second in the strong interaction.

### Constructing the electroweak theory

We now discuss electroweak unification. It is perhaps pedagogically clearest to motivate how we would go about constructing such a theory. As I have said before, this is not a

textbook on particle physics and I necessarily will have to keep the discussion of particle physics to the bare minimum. I gave you a brief introduction to the structure of the weak interaction in chapter IV.2. The other salient fact is that weak interaction violates parity, as mentioned in chapter II.1. In particular, the left handed electron field  $e_L$  and the right handed electron field  $e_R$ , which transform into each other under parity, enter into the weak interaction quite differently.

Let us start with the weak decay of the muon,  $\mu^- \rightarrow e^- + \bar{\nu} + \nu'$ , with  $\nu$  and  $\nu'$  the electron neutrino and muon neutrino, respectively. The relevant term in the Lagrangian is  $\bar{\nu}'_L \gamma^\mu \mu_L \bar{e}_L \gamma_\mu \nu_L$ , with the left hand electron field  $e_L$ , the electron neutrino field (which is left handed)  $\nu_L$ , and so forth. The field  $\mu_L$  annihilates a muon, the field  $\bar{e}_L$  creates an electron, and so on. (Henceforth, we will suppress the word field.) As you probably know, the elementary constituents of matter form three families, with the first family consisting of  $\nu$ ,  $e$ , and the up  $u$  and down  $d$  quarks, the second of  $\nu'$ ,  $\mu$ , and the charm  $c$  and strange  $s$  quarks, and so on. For our purposes here, we will restrict our attention to the first family. Thus, we start with  $\bar{\nu}_L \gamma^\mu e_L \bar{e}_L \gamma_\mu \nu_L$ .

As I remarked in chapter III.2, a Fermi interaction of this type can be generated by the exchange of an intermediate vector boson  $W^+$  with the coupling  $W_\mu^+ \bar{\nu}_L \gamma^\mu e_L + W_\mu^- \bar{e}_L \gamma_\mu \nu_L$ .

The idea is then to consider an  $SU(2)$  gauge theory with a triplet of gauge bosons denoted by  $W_\mu^a$ , with  $a = 1, 2, 3$ . Put  $\nu_L$  and  $e_L$  into the doublet representation and the right handed electron field  $e_R$  into a singlet representation, thus

$$\psi_L \equiv \begin{pmatrix} \nu \\ e \end{pmatrix}_L, \quad e_R \quad (1)$$

(The notation is such that the upper component of  $\psi_L$  is  $\nu_L$  and the lower component is  $e_L$ .)

The fields  $\nu_L$  and  $e_L$ , but not  $e_R$ , listen to the gauge bosons  $W_\mu^a$ . Indeed, according to (IV.5.21) the Lagrangian contains

$$W_\mu^a \bar{\psi}_L \tau^a \gamma^\mu \psi_L = (W_\mu^{1-i2} \bar{\psi}_L \frac{1}{2} \tau^{1+i2} \gamma^\mu \psi_L + \text{h.c.}) + W_\mu^3 \bar{\psi}_L \tau^3 \gamma^\mu \psi_L$$

where  $W_\mu^{1-i2} \equiv W_\mu^1 - iW_\mu^2$  and so forth. We recognize  $\tau^{1+i2} \equiv \tau^1 + i\tau^2$  as the raising operator and the first two terms as  $(W_\mu^{1-i2} \bar{\nu}_L \gamma^\mu e_L + \text{h.c.})$ , precisely what we want. By design, the exchange of  $W_\mu^\pm$  generates the desired term  $\bar{\nu}_L \gamma^\mu e_L \bar{e}_L \gamma_\mu \nu_L$ .

## We need more room

We would hope that the boson  $W^3$  we were forced to introduce would turn out to be the photon so that electromagnetism is included. But alas,  $W^3$  couples to the current  $\bar{\psi}_L \tau^3 \gamma^\mu \psi_L = (\bar{\nu}_L \gamma^\mu \nu_L - \bar{e}_L \gamma_\mu e_L)$ , not the electromagnetic current  $-(\bar{e}_L \gamma_\mu e_L + \bar{e}_R \gamma_\mu e_R)$ . Oops!

Another problem lurks. To generate a mass term for the electron, we need a doublet Higgs field  $\varphi \equiv \begin{pmatrix} \varphi^+ \\ \varphi^0 \end{pmatrix}$  in order to construct the  $SU(2)$  invariant term  $f \bar{\psi}_L \varphi e_R$  in the Lagrangian so that when  $\varphi$  acquires the vacuum expectation value  $\begin{pmatrix} 0 \\ v \end{pmatrix}$  we will have

$$f\bar{\psi}_L\varphi e_R \rightarrow f(\bar{\nu}, \bar{e})_L \begin{pmatrix} 0 \\ \nu \end{pmatrix} e_R = f\nu\bar{e}_L e_R \quad (2)$$

But none of the  $SU(2)$  transformations leaves  $\begin{pmatrix} 0 \\ \nu \end{pmatrix}$  invariant: The vacuum expectation value of  $\varphi$  spontaneously breaks the entire  $SU(2)$  symmetry, leaving all three  $W$  bosons massive. There is no room for the photon in this failed theory. Aagh!

We need more room. Remarkably, we can avoid both the oops and the aagh by extending the gauge symmetry to  $SU(2) \otimes U(1)$ . Denoting the generator of  $U(1)$  by  $\frac{1}{2}Y$  (called the hypercharge) and the associated gauge potential by  $B_\mu$  [and their counterparts  $T^a$  and  $W_\mu^a$  for  $SU(2)$ ] we have the covariant derivative  $D_\mu = \partial_\mu - igW_\mu^a T^a - ig'B_\mu \frac{Y}{2}$ . With four gauge bosons, we dare to hope that one of them might turn out to be the photon.

The gauge potentials are normalized by the corresponding kinetic energy terms,  $\mathcal{L} = -\frac{1}{4}(B_{\mu\nu})^2 - \frac{1}{4}(W_{\mu\nu}^a)^2 + \dots$  with the abelian  $B_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$  and nonabelian field strength  $W_{\mu\nu}^a = \partial_\mu W_\nu^a - \partial_\nu W_\mu^a + \varepsilon^{abc}W_\mu^b W_\nu^c$ . The generators  $T^a$  are of course normalized by the commutation relations that define  $SU(2)$ . In contrast, there is no commutation relation in the abelian algebra  $U(1)$  to fix the normalization of the generator  $\frac{1}{2}Y$ . Until this is fixed, the normalization of the  $U(1)$  gauge coupling  $g'$  is not fixed.

How do we fix the normalization of the generator  $\frac{1}{2}Y$ ? By construction, we want spontaneous symmetry breaking to leave a linear combination of  $T_3$  and  $\frac{1}{2}Y$  invariant, to be identified as the generator the massless photon couples to, namely the charge operator  $Q$ . Thus, we write

$$Q = T_3 + \frac{1}{2}Y \quad (3)$$

Once we know  $T_3$  and  $\frac{1}{2}Y$  of any field, this equation tells us its charge. For example,  $Q(\nu_L) = \frac{1}{2} + \frac{1}{2}Y(\nu_L)$  and  $Q(e_L) = -\frac{1}{2} + \frac{1}{2}Y(e_L)$ . In particular, we see that the coefficient of  $T_3$  in (3) must be 1 since the charges of  $\nu_L$  and  $e_L$  differ by 1. The relation (3) fixes the normalization of  $\frac{1}{2}Y$ .

## Determining the hypercharge

The next step is to determine the hypercharge of various multiplets in the theory, which in turn determines how  $B_\mu$  couples to these multiplets. Consider  $\psi_L$ . For  $e_L$  to have charge  $-1$ , the doublet  $\psi_L$  must have  $\frac{1}{2}Y = -\frac{1}{2}$ . In contrast, the field  $e_R$  has  $\frac{1}{2}Y = -1$  since  $T_3 = 0$  on  $e_R$ .

Given the hypercharge of  $\psi_L$  and  $e_R$  we see that the invariance of the term  $f\bar{\psi}_L\varphi e_R$  under  $SU(2) \otimes U(1)$  forces the Higgs field  $\varphi$  to have  $\frac{1}{2}Y = +\frac{1}{2}$ . Thus, according to (3) the upper component of  $\varphi$  has electric charge  $Q = +\frac{1}{2} + \frac{1}{2} = +1$  and the lower component  $Q = -\frac{1}{2} + \frac{1}{2} = 0$ . Thus, we write  $\varphi = \begin{pmatrix} \varphi^+ \\ \varphi^0 \end{pmatrix}$ . Recall that  $\varphi$  has the vacuum expectation value  $\begin{pmatrix} 0 \\ \nu \end{pmatrix}$ . The fact that the electrically neutral field  $\varphi^0$  acquires a vacuum expectation value but the charged field  $\varphi^+$  does not provide a consistency check.

## The theory works itself out

Now that the couplings of the gauge bosons to the various fields, in particular, the Higgs field, are determined, we can easily work out the mass spectrum of the gauge bosons, as indeed, let me remind you, you have already done in exercise IV.6.3!

Upon spontaneous symmetry breaking  $\varphi \rightarrow (1/\sqrt{2})\begin{pmatrix} 0 \\ v \end{pmatrix}$  (the normalization is conventional): We simply plug in

$$\mathcal{L} = (D_\mu \varphi)^\dagger (D^\mu \varphi) \rightarrow \frac{g^2 v^2}{4} W_\mu^+ W^{-\mu} + \frac{v^2}{8} (g W_\mu^3 - g' B_\mu)^2 \quad (4)$$

I trust that this is what you got! Thus, the linear combination  $g W_\mu^3 - g' B_\mu$  becomes massive while the orthogonal combination remains massless and is identified with the photon. It is clearly convenient to define the angle  $\theta$  by  $\tan \theta = g'/g$ . Then,

$$Z_\mu = \cos \theta W_\mu^3 - \sin \theta B_\mu \quad (5)$$

describes a massive gauge boson known as the  $Z$  boson, while the electromagnetic potential is given by  $A_\mu = \sin \theta W_\mu^3 + \cos \theta B_\mu$ . Combine (4) and (5) and verify that the mass squared of the  $Z$  boson is  $M_Z^2 = v^2(g^2 + g'^2)/4$ , and thus by elementary trigonometry obtain the relation

$$M_W = M_Z \cos \theta \quad (6)$$

The exchange of the  $W$  boson generates the Fermi weak interaction

$$\mathcal{L} = -\frac{g^2}{2M_W^2} \bar{\nu}_L \gamma^\mu e_L \bar{e}_L \gamma_\mu \nu_L = -\frac{4G}{\sqrt{2}} \bar{\nu}_L \gamma^\mu e_L \bar{e}_L \gamma_\mu \nu_L$$

where the second equality merely gives the historical definition of the Fermi coupling  $G$ . Thus,

$$\frac{G}{\sqrt{2}} = \frac{g^2}{8M_W^2} \quad (7)$$

Next, we write the relevant piece of the covariant derivative

$$g W_\mu^3 T^3 + g' B_\mu \frac{Y}{2} = g(\cos \theta Z_\mu + \sin \theta A_\mu) T^3 + g'(-\sin \theta Z_\mu + \cos \theta A_\mu) \frac{Y}{2}$$

in terms of the physically observed  $Z$  and  $A$ . The coefficient of  $A_\mu$  works out to be  $g \sin \theta T^3 + g' \cos \theta (Y/2) = g \sin \theta (T^3 + Y/2)$ ; the fact that the combination  $Q = T^3 + Y/2$  emerges provides a nice check on the formalism. Furthermore, we obtain

$$e = g \sin \theta \quad (8)$$

Meanwhile, it is convenient to write  $g \cos \theta T^3 - g' \sin \theta (Y/2)$ , the coefficient of  $Z_\mu$  in the covariant derivative, in terms of the physically familiar electric charge  $Q$  rather than the theoretical hypercharge  $Y$ : Thus,

$$g \cos \theta T^3 - g' \sin \theta (Q - T^3) = \frac{g}{\cos \theta} (T^3 - \sin^2 \theta Q)$$

In other words, we have determined the coupling of the  $Z$  boson to an arbitrary fermion field  $\Psi$  in the theory:

$$\mathcal{L} = \frac{g}{\cos \theta} Z_\mu \bar{\Psi} \gamma^\mu (T^3 - \sin^2 \theta Q) \Psi \quad (9)$$

For example, using (9) we can immediately write the coupling of  $Z$  to leptons:

$$\mathcal{L} = \frac{g}{\cos \theta} Z_\mu \left[ \frac{1}{2} (\bar{\nu}_L \gamma^\mu \nu_L - \bar{e}_L \gamma^\mu e_L) + \sin^2 \theta \bar{e} \gamma^\mu e \right] \quad (10)$$

## Including quarks

How to include the hadrons is now almost self evident. Given that only left handed fields participate in the weak interaction, we put the quarks of the first generation into  $SU(2) \otimes U(1)$  multiplets as follows:

$$q_L^\alpha \equiv \begin{pmatrix} u^\alpha \\ d^\alpha \end{pmatrix}_L, \quad u_R^\alpha, d_R^\alpha \quad (11)$$

where  $\alpha = 1, 2, 3$  denotes the color index, which I will discuss in the next chapter. The right handed quarks  $u_R^\alpha$  and  $d_R^\alpha$  are put into singlets so that they do not hear the weak bosons  $W^a$ . Recall that the up quark  $u$  and the down quark  $d$  have electric charges  $\frac{2}{3}$  and  $-\frac{1}{3}$  respectively. Referring to (3) we see  $\frac{1}{2} Y = \frac{1}{6}, \frac{2}{3}$ , and  $-\frac{1}{3}$  for  $q_L^\alpha$ ,  $u_R^\alpha$ , and  $d_R^\alpha$ , respectively. From (9) we can immediately read off the coupling of the  $Z$  boson to the quarks:

$$\mathcal{L} = \frac{g}{\cos \theta} Z_\mu \left[ \frac{1}{2} (\bar{u}_L \gamma^\mu u_L - \bar{d}_L \gamma^\mu d_L) - \sin^2 \theta J_{em}^\mu \right] \quad (12)$$

Finally, I leave it to you to verify that of the four degrees of freedom contained in  $\varphi$  (since  $\varphi^+$  and  $\varphi^0$  are complex) three are eaten by the  $W$  and  $Z$  bosons, leaving one physical degree of freedom  $H$  corresponding to the elusive Higgs particle that experimenters are still searching for as of this writing.

## The neutral current

By virtue of its elegantly economical gauge group structure, this  $SU(2) \otimes U(1)$  electroweak theory of Glashow, Salam, and Weinberg ushered in the last great predictive era of theoretical particle physics. Writing (10) and (12) as

$$\mathcal{L} = \frac{g}{\cos \theta} Z_\mu (J_{\text{leptons}}^\mu + J_{\text{quarks}}^\mu)$$

and using (6) we see that  $Z$  boson exchange generates a hitherto unknown neutral current interaction

$$\mathcal{L}_{\text{neutral current}} = -\frac{g^2}{2M_W^2} (J_{\text{leptons}} + J_{\text{quarks}})^\mu (J_{\text{leptons}} + J_{\text{quarks}})_\mu$$

between leptons and quarks. By studying various processes described by  $\mathcal{L}_{\text{neutral current}}$  we can determine the weak angle  $\theta$ . Once  $\theta$  is determined, we can predict  $g$  from (8). Once  $g$

is determined, we can predict  $M_W$  from (7). Once  $M_W$  is determined, we can predict  $M_Z$  from (6).

## Concluding remarks

As I mentioned, there are three families of leptons and quarks in Nature, consisting of  $(\nu_e, e, u, d)$ ,  $(\nu_\mu, \mu, c, s)$ , and  $(\nu_\tau, \tau, t, b)$ . The appearance of this repetitive family structure, about which the  $SU(2) \otimes U(1)$  theory has nothing to say, represents one of the great unsolved puzzles of particle physics. The three families, with the appropriate rotation angles between them, are simply incorporated into the theory by repeating what we wrote above.

A more logical approach than the one given here would be to start with an  $SU(2) \otimes U(1)$  theory with a doublet Higgs field with some hypercharge, and to say, "Behold, upon spontaneous symmetry breaking, one linear combination of generators remains unbroken with a corresponding massless gauge field." I think that our quasi-historical approach is clearer.

As I have mentioned on several occasions, Fermi's theory of the weak interaction is nonrenormalizable. In 1999, 't Hooft and Veltman were awarded the Nobel Prize for showing that the  $SU(2) \otimes U(1)$  electroweak theory is renormalizable, thus paving the way for the triumph of nonabelian gauge theories in describing the strong, electromagnetic, and weak interactions. I cannot go into the details of their proof here, but I would like to mention that the key is to start with the nonabelian analog of the unitary gauge (recall chapter IV.6) and proceed to the  $R_\xi$  gauge. At large momenta, the massive gauge boson propagators go as  $\sim (k_\mu k_\nu/k^2)$  in the unitary gauge, but as  $\sim (1/k^2)$  in the  $R_\xi$  gauge. The theory is then renormalizable by power counting.

## Exercises

- VII.2.1** Unfortunately, the mass of the elusive Higgs particle  $H$  depends on the parameters in the double well potential  $V = -\mu^2 \phi^\dagger \phi + \lambda (\phi^\dagger \phi)^2$  responsible for the spontaneous symmetry breaking. Assuming that  $H$  is massive enough to decay into  $W^+ + W^-$  and  $Z + Z$ , determine the rates for  $H$  to decay into various modes.
- VII.2.2** Show that it is possible to stay with the  $SU(2)$  gauge group and to identify  $W^3$  as the photon  $A$ , but at the cost of inventing some experimentally unobserved lepton fields. This theory does not describe our world: For one thing, it is essentially impossible to incorporate the quarks. Show this! [Hint: We have to put the leptons into a triplet of  $SU(2)$  instead of a doublet.]