Differential Forms and Yang-Mills Fields

$$A = A_\mu dx^\mu$$

$$A^2 = A_\mu A_\nu \, dx^\mu dx^\nu$$

$$= \tfrac{1}{2} [A_\mu, A_\nu] \, dx^\mu dx^\nu$$

This vanishes in an abelian gauge theory but not in a nonabelian one where the A_μ's are of the form $A_\mu = -i A_\mu^a T^a$, in which

$$[T_a, T_b] = i f_{abc} T_c \, .$$

Under a gauge transformation

$$A' = U A U^\dagger + U dU^\dagger$$

where $dU^\dagger = \partial_\mu U^\dagger dx^\mu$. So

$$dA' = (dU) A U^\dagger + U (dA) U^\dagger - U A \, dU^\dagger$$

$$+ (dU) dU^\dagger \qquad \text{while}$$

$$A'^2 = U A U^\dagger U A U^\dagger + U A U^\dagger U dU^\dagger + U dU^\dagger U A U^\dagger + U dU^\dagger U dU^\dagger$$

but $0 = dI = d(UU^\dagger) = (dU) U^\dagger + U dU^\dagger$, so

$$U dU^\dagger = -(dU) U^\dagger \text{ (and also } dU^\dagger U = -U^\dagger dU.)$$

So

$$dA' + A'^2 = dU\,AU^\dagger + U\,dA\,U^\dagger - UA\,dU^\dagger + dU\,dU^\dagger$$
$$+ UA^2U^\dagger + UA\,dU^\dagger - dU\,AU^\dagger - dU\,dU^\dagger$$

$$= U(A^2 + dA)U^\dagger$$

So $F = dA + A^2$ transforms as

$$F' = UFU^\dagger.$$

Also, let $D = d + A$. Then

$$D^2 = (d+A)(d+A) = dA + Ad + A^2$$

$$= (dA) - Ad + Ad + A^2 = (dA) + A^2 = F.$$

For abelian gauge theories we showed that

$$dF = d\,dA = 0.$$

Here we have

$$DF \equiv dF + [A,F] = d(dA + A^2) + [A, dA + A^2]$$

$$= dA^2 + A\,dA - (dA)A = (dA)A - A\,dA + A\,dA - (dA)A$$

$= 0$ which are the homogeneous Maxwell-Yang-Mills equations.

Now $F^2 = F_{\mu\nu}\, dx^\mu dx^\nu\, F_{\rho\sigma}\, dx^\rho dx^\sigma$
is proportional to

$$\epsilon^{\mu\nu\lambda\rho} F_{\mu\nu} F_{\lambda\rho} .$$

Consider its trace $\operatorname{tr} F^2$.

$$d \operatorname{tr} F^2 = \operatorname{tr}(dF\, F + F\, dF)$$

$$= \operatorname{tr}\left[d(dA + A^2) F + F\, d(dA + A^2) \right]$$

$$= \operatorname{tr}\left[(dA^2) F + F\, dA^2 \right]$$

$$= \operatorname{tr}\left[(dA\,A - A\,dA) F + F (dA\,A - A\,dA) \right]$$

$$= \operatorname{tr}\left[(dA\,A - A\,dA)(dA + A^2) + (dA + A^2)(dA\,A - A\,dA) \right]$$

$$= \operatorname{tr}\Big[\overset{a}{dA\,A\,dA} + \overset{b}{dA\,A^3} - \overset{d}{A(dA)^2} - \overset{c}{A\,dA\,A^2}$$
$$+ \underset{d}{(dA)^2 A} - \underset{a}{(dA)A\,dA} + \underset{c}{A^2 dA\,A} - \underset{b}{A^3 dA} \Big]$$

$= 0$ where we used $\operatorname{tr} XY = \operatorname{tr} YX$.

So $\mathrm{tr}\, F^2$ is closed because

$$d\, \mathrm{tr}\, F^2 = 0.$$

So by Poincaré's lemma, $\mathrm{tr}\, F^2$ must be locally exact. That is, locally

$$\mathrm{tr}\, F^2 = dQ.$$

What is Q. As a homework problem, show that if

$$Q = \mathrm{tr}\left(A\,dA + \tfrac{2}{3} A^3\right)$$

then $\quad dQ = \mathrm{tr}(F^2).$

Hints: Be careful, use some explicit dx_μ's etc. Also, note that $\mathrm{tr}\, A^4 = 0$.
For

$$\mathrm{tr}\, A^4 = \mathrm{tr}(A_i A_j A_k A_l\, dx^i dx^j dx^k dx^l)$$

$$= \mathrm{tr}(A_j A_k A_l A_i\, dx^i dx^j dx^k dx^l)$$

$$= -\mathrm{tr}(A_j A_k A_l A_i\, dx^j dx^k dx^l dx^i)$$

$= -\mathrm{tr}\, A^4.$ So $\mathrm{tr}\, A^4 = 0$.