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The Renormalization Group

Most quantum field theories are non-linear with infinitely many degrees of freedom, and because they describe point particles, they are rife with infinities. But short-distance effects, probably the finite sizes of the fundamental constituents of matter, mitigate these infinities so that we can cope with them by ignoring what happens at very short distances and very high energies. This procedure is called **renormalization**.

For instance, in the theory described by the Lagrange density

$$\mathcal{L} = -\frac{1}{2}\partial_\nu\phi\,\partial^\nu\phi - \frac{1}{2}m^2\phi^2 - \frac{g}{24}\phi^4 \tag{18.1}$$

we can cut-off divergent integrals at some high energy Λ. The amplitude for the elastic scattering of two bosons of initial four-momenta p_1 and p_2 and final momenta p_1' and p_2' to one-loop order (Weinberg, 1996, chap. 18) then takes the simple form (Zee, 2010, chaps. III & VI)

$$A = g - \frac{g^2}{32\pi^2}\left[\ln\left(\frac{\Lambda^6}{stu}\right) + i\pi + 3\right] \tag{18.2}$$

as long as the absolute values of the Mandelstam variables $s = -(p_1 + p_2)^2$, $t = -(p_1 - p_1')^2$, and $u = -(p_1 - p_2')^2$, which satisfy $stu > 0$ and $s + t + u = 4m^2$, are all much larger than m^2. We define the physical coupling constant g_μ, as opposed to the one that came with $\mathcal{L}$, to be the real part of the amplitude A at $s = -t = -u = \mu^2$

$$g_\mu = g - \frac{3g^2}{32\pi^2}\left[\ln\left(\frac{\Lambda^2}{\mu^2}\right) + 1\right]. \tag{18.3}$$

Thus, $g = g_\mu + 3g^2\left[\ln(\Lambda^2/\mu^2) + 1\right]$, and using this formula in our expression (18.2) for the amplitude A, we find that the amplitude no longer involves

the cut-off Λ

$$A = g_\mu - \frac{g^2}{32\pi^2}\left[\ln\left(\frac{\mu^6}{stu}\right) + i\pi\right].\qquad(18.4)$$

We further assume that with the cut-off Λ big, but finite and fixed, the coupling constants g and g_μ are so tiny that to order g_μ^2 we can replace g^2 with the square of the physical coupling constant g_μ^2

$$A = g_\mu - \frac{g_\mu^2}{32\pi^2}\left[\ln\left(\frac{\mu^6}{stu}\right) + i\pi\right].\qquad(18.5)$$

Now the amplitude depends only upon the physical coupling constant g_μ and the renormalization energy μ at which it is defined. This is the magic of renormalization.

The physical coupling "constant" g_μ is the ideal coupling at energy μ because when the Mandelstam variables are all near the renormalization point $stu = \mu^6$, the one-loop correction is tiny, and $A \approx g_\mu$.

How does the physical coupling g_μ depend upon the energy μ? The amplitude A must be independent of the renormalization energy μ, and so

$$\frac{dA}{d\mu} = \frac{dg_\mu}{d\mu}\left\{1 - \frac{g_\mu}{16\pi^2}\left[\ln\left(\frac{\mu^6}{stu}\right) + i\pi\right]\right\} - \frac{g_\mu^2}{32\pi^2}\frac{6}{\mu} = 0\qquad(18.6)$$

which is the **Callan-Symanzik equation**. To lowest order in g_μ, this is a simple differential equation

$$\mu\frac{dg_\mu}{d\mu} \equiv \beta(g_\mu) = \frac{3\,g_\mu^2}{16\pi^2}\qquad(18.7)$$

which we can integrate

$$\ln\frac{E}{M} = \int_M^E \frac{d\mu}{\mu} = \int_{g_M}^{g_E} \frac{dg_\mu}{\beta(g_\mu)} = \frac{16\pi^2}{3}\int_{g_M}^{g_E}\frac{dg_\mu}{g_\mu^2} = \frac{16\pi^2}{3}\left(\frac{1}{g_M} - \frac{1}{g_E}\right)$$
$$(18.8)$$

to find the "running" physical coupling constant at energy E

$$g_E = \frac{g_M}{1 - 3\,g_M\,\ln(E/M)/16\pi^2}.\qquad(18.9)$$

As the energy $E = \sqrt{s}$ rises above M, while staying below the singular value $E = M\exp(16\pi^2/3g_M)$, the running coupling g_E slowly increases. And so does the scattering amplitude, $A \approx g_E$.