Application to QED

The hamiltonian for Coulomb-gauge QED is

$$H = H_m + \int \left[\tfrac{1}{2}\pi^2 + \tfrac{1}{2}(\nabla \times A)^2 - A \cdot J\right] d^3x + V_c \quad \text{where}$$

$$V_c = \frac{1}{2} \int \frac{j^0(x,t)\, j^0(y,t)\, d^3x\, d^3y}{4\pi |\vec{x} - \vec{y}|} \quad \text{and}$$

$$\nabla \cdot A = 0 \quad \text{and} \quad \nabla \cdot \pi = 0.$$

One can show that

$$\langle A'' | e^{-iH2T} | A' \rangle = N \int e^{i\int d^4x \left[\pi \cdot \dot{A} - \frac{1}{2}\pi^2 - \frac{1}{2}(\nabla \times A)^2 + A \cdot J + \mathcal{L}_m\right] - i\int V_c dt}$$
$$\times\, \delta(\nabla \cdot A)\, \delta(\nabla \cdot \pi)\, DA\, D\pi\, D\psi_M$$

in which $\delta(\nabla \cdot A) = \prod_x \delta(\nabla \cdot A(x))$ etc. can be expressed as

$$\delta(\nabla \cdot A) = \int e^{i\int \varphi \nabla \cdot A\, d^4x} D\varphi \quad \text{and} \quad \varphi$$

$$\delta(\nabla \cdot \pi) = \int e^{i\int \phi \nabla \cdot \pi\, d^4x} D\phi \quad \phi.$$

Then $\int \exp\left(i\int (\pi \cdot \dot{A} - \frac{\pi^2}{2}) d^4x\right) \delta(\nabla \cdot \pi)\, D\pi$

$$= \int e^{i\int \pi \cdot \dot{A} - \frac{1}{2}\pi^2 - \pi \cdot \nabla\phi\, d^4x} D\pi = \int e^{i\int \pi \cdot (\dot{A} - \nabla\phi) - \frac{1}{2}\pi^2 d^4x} D\pi$$

$$= N' e^{i\int \frac{(\dot{A} - \nabla\phi)^2}{2} d^4x}.$$

So now we have

$$\int e^{i\int \frac{\dot{A}^2}{2} + \psi \nabla\cdot A - \dot{\phi}\nabla\cdot A + \frac{\nabla\phi^2}{2}\, d^4x} D\phi D\psi$$

$$= \int e^{i\int \frac{\dot{A}^2}{2} + (\psi - \dot{\phi})\nabla\cdot A + \frac{1}{2}\nabla\phi^2 d^4x} D\phi D\psi$$

$$= \tilde{N}\int e^{i\int \frac{\dot{A}^2}{2} + \psi\nabla\cdot A \, d^4x} D\psi .$$

So now

$$\langle A''| e^{-2iTH} |A'\rangle = N'' \int e^{i\int\left[\frac{\dot{A}^2}{2} - \frac{1}{2}(\nabla\times A)^2 + A\cdot J + \mathcal{L}_m\right]d^4x - i\int V_c dt} DA\, D\psi_m\, \delta(\nabla\cdot A)$$

We now multiply by the constant factor

$$\int e^{i\int\left[-A^0 j^0 + \frac{1}{2}(\nabla A^0)^2 + \frac{1}{2}\nabla\frac{1}{\Delta}j^0\,\nabla\frac{1}{\Delta}j^0\right]d^4x} DA^0$$

$$= \int e^{i\int \frac{1}{2}(\vec{\nabla}A^0 - \vec{\nabla}\frac{1}{\Delta}j^0)^2 d^4x} DA^0$$

$$= \int e^{i\int \frac{1}{2}(\nabla A^0)^2 - A^0 j^0 - \frac{1}{2}\Delta\frac{1}{\Delta}j^0\frac{1}{\Delta}j^0\, d^4x} DA^0$$

$$= \int e^{i\int \frac{1}{2}(\nabla A^0)^2 - A^0 j^0 - \frac{1}{2}j^0\frac{1}{\Delta}j^0\, d^4x} DA^0$$

$$= \int e^{i\int\left[\frac{1}{2}(\nabla A^0)^2 - A^0 j^0\right]d^4x + i\int V_c dt} DA^0$$

So we have

$$\langle A''| e^{-2iTH} |A'\rangle = N'' \int e^{i\int\left[\frac{\dot{A}^2}{2} - \frac{1}{2}(\nabla\times A)^2 + A\cdot j - A^0 j^0 + \frac{1}{2}(\nabla A^0)^2 + \mathcal{L}_M\right] d^4x} \delta(\nabla\cdot A)\, DA\, D\psi_M$$

$$= N'' \int e^{i\int\left[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - A^\mu j_\mu + \mathcal{L}_M\right] d^4x} \delta(\nabla\cdot A)\, DA\, D\psi_M$$

Everything here is gauge invariant except for $\delta(\nabla\cdot A)$. Make a gauge transformation everywhere

$$A'_\mu = A_\mu + \partial_\mu \Lambda$$

$$\psi' = e^{iq\Lambda}\psi$$

$$(*) \qquad \langle A''| e^{-2iTH} |A'\rangle = N''' \int e^{iS} \delta(\nabla\cdot A + \nabla^2\Lambda)\, DA\, D\psi_M$$

where we assumed $DA, D\psi$ are gauge invariant.

We now play two tricks. First, we just integrate over all Λ and get

$$\langle A''| e^{-2iTH} |A'\rangle = N^{iv} \int e^{iS}\, DA\, D\psi_M$$

That is, the amplitude is an integration over all fields and all gauges. To make sense, as usual we need to focus on ratios of path integrals.

The next trick is to multiply the RHS of (*) by the constant

$$e^{-i\frac{\alpha}{2}\int d^4x\,(\dot{A}^0 - \Delta\Lambda)^2}$$ and integrate over Λ.

$$\int e^{-i\frac{\alpha}{2}\int(\dot{A}^0 - \Delta\Lambda)^2 d^4x}\,\delta(\nabla\cdot A + \Delta\Lambda)\,D\Lambda$$

$$= N' \quad e^{-i\frac{\alpha}{2}\int(\partial_\mu A^\mu)^2 d^4x}$$

So we've inserted a gauge-fixing term

$$-\frac{\alpha}{2}\int(\partial_\mu A^\mu)^2\, d^4x$$

into the gauge-invariant action S.

$$\langle A''|e^{-2iTH}|A'\rangle = N'' \int e^{iS_e}\, DA\, D\psi_m$$

where

$$S_e = \int -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - A^\mu j_\mu + \mathcal{L}_M - \frac{\alpha}{2}(\partial_\mu A^\mu)^2\; d^4x$$

For $\alpha = 1$, we get the nice propagator

$$\frac{-i\,g_{\mu\nu}}{q^2 + i\epsilon}.$$

For $\alpha = 1$, $S^0_\alpha = S^0_1$ the "free" part of S_α is

$$S^0_1 = \int -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} (\partial_\mu A^\mu)^2 \, d^4x .$$

$$= \int -\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) - \frac{1}{2}(\partial_\sigma A^\sigma)^2 \, d^4x$$

$$= \int -\frac{1}{4}\Big[\partial_\mu A_\nu \partial^\mu A^\nu + \partial_\nu A_\mu \partial^\nu A^\mu$$
$$- \partial_\nu A_\mu \partial^\mu A^\nu - \partial_\mu A_\nu \partial^\nu A^\mu$$
$$+ 2\partial_\sigma A^\sigma \partial_\lambda A^\lambda \Big] \, d^4x$$

Integrating the second term by parts, we get

$$S^0_1 = \int -\frac{1}{4}\left[2\partial_\mu A_\nu \partial^\mu A^\nu - 2\partial^\mu A_\mu \partial_\nu A^\nu + 2\partial_\sigma A^\sigma \partial_\lambda A^\lambda\right] d^4x$$

$$= \int -\frac{1}{2} \partial_\mu A_\nu \partial^\mu A^\nu \, d^4x .$$

$$A^\mu(x) = \int e^{ipx} A^\mu(p) \frac{d^4p}{(2\pi)^4}$$

or better yet

$$S^0_1 = +\frac{1}{2} \int A_\nu(x) \partial^\mu \delta^{(4)}(x-y) \partial_\mu A^\nu(y) \, d^4x \, d^4y .$$

Actually, let's just use Fourier transforms here.

$$S_1^0 = -\frac{1}{2}\int \frac{d^4x\, d^4p\, d^4q}{(2\pi)^8}\, \partial_\mu e^{ipx} A_\nu(p)\, \partial^\mu e^{iqx} A^\nu(q))$$

$$= -\frac{1}{2}\int \frac{d^4p\, d^4q\, d^4x}{(2\pi)^8}\, ip_\mu\, iq^\mu\, e^{i(p+q)x}\, A_\nu(p) A^\nu(q))$$

$$= -\frac{1}{2}\int \frac{d^4p\, d^4q}{(2\pi)^4}\, ip_\mu\, iq^\mu\, \delta^4(p+q)\, A_\nu(p) A^\nu(q))$$

$$= -\frac{1}{2}\int \frac{d^4p}{(2\pi)^4}\, A_\nu(p) A^\nu(-p)\, p^2$$

$$= -\frac{1}{2}\int \frac{d^4p}{(2\pi)^4}\, A_\nu(p) A^{*\nu}(p)\, p^2$$

the $i\epsilon$ term give us

$$S_1^0(\epsilon) = -\frac{1}{2}\int \frac{d^4p}{(2\pi)^4}\, A_\nu(p) A^{*\nu}(p)\, (p^2 - i\epsilon)$$

we now add a classical current $j^\mu A_\mu$.

$$S_1^0(\epsilon, j) = S_1^0(\epsilon) + \int j^\mu A_\mu\, d^4x$$

$$= -\frac{1}{2}\int \frac{d^4p}{(2\pi)^4}\, A_\nu(p) A^{*\nu}(p)\, (p^2 - i\epsilon) - j^{*\mu} A_\mu - j^\mu A^*_\mu$$

$$A'_\mu = A_\mu - j_\mu / p^2 - i\epsilon$$

$$S_1^0(\epsilon, j) = -\frac{1}{2}\int A'_\mu(p) A'^{*}(p)(p^2 - i\epsilon) - \frac{j^{*\mu} j_\mu}{p^2 - i\epsilon} \frac{d^4p}{(2\pi)^4}$$

$$= S_1^0(\epsilon) + \frac{1}{2}\int \frac{j^{*\mu}(p) g_{\mu\nu} j^\nu(p)}{p^2 - i\epsilon} \frac{d^4p}{(2\pi)^4}$$

So now

$$Z[j] \equiv \langle \Omega | T e^{i\int j^\mu A_\mu d^4x} | \Omega \rangle = e^{\frac{i}{2}\int \frac{j^{*\mu}(p) j^\nu(p) g_{\mu\nu}}{p^2 - i\epsilon} \frac{d^4p}{(2\pi)^4}}$$

$$= \exp\left[\frac{i}{2}\int \frac{d^4p}{(2\pi)^4} d^4x\, d^4y \frac{e^{-ipx}}{p^2 - i\epsilon} j^\nu(x) e^{ipy} j^\mu(y) g_{\mu\nu}\right]$$

$$= e^{\frac{i}{2}\int d^4x\, d^4y\; j^\nu(x) j^\mu(y) \Delta_{\mu\nu}(x-y)}$$

where $\Delta_{\mu\nu}(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{e^{-ip(x-y)} g_{\mu\nu}}{p^2 - i\epsilon}$

So

$$\langle 0 | T(A_\mu(x) A_\nu(y)) | 0 \rangle = \frac{1}{i^2} \frac{\delta^2}{\delta j^\mu(x) \delta j^\nu(y)} \cancel{\Delta_{\mu\nu}(x)} Z[j] \Big|_{j=0}$$

$$= -i\Delta_{\mu\nu}(x-y) = \int \frac{d^4p}{(2\pi)^4} e^{-ip(x-y)} \frac{-ig_{\mu\nu}}{p^2 - i\epsilon}.$$