Local Symmetry

Suppose $\psi(x) = \begin{pmatrix} \psi_1(x) \\ \vdots \\ \psi_n(x) \end{pmatrix}$ is a vector

of matter fields, and that

$$\psi'(x) = g(x)\,\psi(x)$$

i.e.,

$$\psi'_i(x) = \sum_{j=1}^{n} g_{ij}(x)\,\psi_j(x)$$

is a transformation that we want to be a symmetry of the action density $\mathcal{L}(x)$.

Note that $g(x)$ depends upon the space-time point x. This is why the symmetry is called <u>local</u>. It also is called a <u>gauge symmetry</u>. How do we arrange that the space-time derivatives of the matter fields transform simply so as to make $\mathcal{L}(x)$ symmetric under $\psi(x) \to \psi'(x) = g(x)\,\psi(x)$?

We need a new kind of derivative $D_\mu(x)$ called a <u>covariant</u> derivative. We want

$$\left[D_\mu(x)\,\psi(x)\right]' = g(x)\,D_\mu(x)\,\psi(x).$$

Here D and g are $n \times n$ matrices, and ψ is an n-vector.

To have

$$[D_\mu(x)\psi(x)]' = D'_\mu(x)\psi'(x) = D'_\mu(x)\,g(x)\,\psi(x)$$
$$= g(x)\,D_\mu(x)\,\psi(x)$$

we need

$$D'_\mu(x)\,g(x) = g(x)\,D_\mu(x).$$

So

$$D'_\mu(x) = g(x)\,D_\mu(x)\,g^{-1}(x).$$

This is the key to both abelian and non-abelian gauge theory. To get $D_\mu(x)$ to do that, we add a matrix $A_\mu(x)$ to ∂_μ

$$D_\mu(x) = \partial_\mu + A_\mu(x).$$

We want

$$D'_\mu(x) = \partial_\mu + A'_\mu(x) = g(x)D_\mu(x)g^{-1}(x)$$
$$= g(x)\left(\partial_\mu + A_\mu(x)\right)g^{-1}(x)$$
$$= \partial_\mu + g(x)\partial_\mu g^{-1}(x) + g(x)A_\mu(x)g^{-1}(x).$$

So

$$A'_\mu(x) = g(x)A_\mu(x)g^{-1}(x) + g(x)\partial_\mu g^{-1}(x).$$

The simplest case is the group $U(1)$. The group element

$$g(x) = e^{i\theta(x)}$$

is just a phase factor and $n=1$. Then

$$A'_\mu(x) = e^{i\theta(x)} A_\mu(x) e^{-i\theta(x)} + e^{i\theta(x)} \partial_\mu e^{-i\theta(x)}$$

$$= A_\mu(x) - i\,\partial_\mu \theta(x)$$

which is the kind of gauge transformation one sees in QED.

The next simplest case is the group $SU(2)$. Now $g(x)$ is

$$g(x) = e^{i\theta^a(x)\frac{\sigma^a}{2}} = e^{i\vec{\theta}(x)\cdot\vec{\sigma}/2}.$$

Computing $\partial_\mu g^{-1}(x)$ is easiest when $\vec{\theta}(x) = \vec{\epsilon}(x)$ is an infinitesimal 3-vector. Then

$$g^{-1}(x) = e^{-i\vec{\epsilon}(x)\cdot\vec{\sigma}/2} \cong 1 - i\vec{\epsilon}(x)\cdot\frac{\vec{\sigma}}{2}$$

so

$$\partial_\mu g^{-1}(x) = -i\,\partial_\mu\vec{\epsilon}(x)\cdot\frac{\vec{\sigma}}{2}$$

and so $A_\mu(x)$ transforms to

$$A'_\mu = e^{i\epsilon\cdot\frac{\sigma}{2}} A_\mu e^{-i\epsilon\cdot\frac{\sigma}{2}} + e^{i\epsilon\cdot\frac{\sigma}{2}} \partial_\mu e^{-i\epsilon\cdot\frac{\sigma}{2}}$$

$$\simeq \left(1 + i\epsilon\cdot\frac{\sigma}{2}\right) A_\mu \left(1 - i\epsilon\cdot\frac{\sigma}{2}\right)$$

$$+ \left(1 + i\epsilon\cdot\frac{\sigma}{2}\right) \partial_\mu \left(1 - i\epsilon\cdot\frac{\sigma}{2}\right)$$

$$\simeq A_\mu + i\epsilon^a \left[\frac{\sigma^a}{2}, A_\mu\right] - i\,\partial_\mu \epsilon^a \frac{\sigma^a}{2}$$

$$A'_\mu(x) = A_\mu(x) + i\epsilon^a(x)\left[\frac{\sigma^a}{2}, A_\mu(x)\right] - i\frac{\sigma^a}{2}\partial_\mu \epsilon^a(x).$$

So A'_μ is in the Lie algebra of $SU(2)$.

$$A_\mu(x) = \sum_{a=1}^{3} \frac{\sigma^a}{2} A^a_\mu(x).$$

So

$$A'^a_\mu \frac{\sigma^a}{2} = A^a_\mu \frac{\sigma^a}{2} + i\epsilon^a\left[\frac{\sigma^a}{2}, \frac{\sigma^b}{2} A^b_\mu\right] - i\frac{\sigma^a}{2}\partial_\mu \epsilon^a$$

Now $\left[\frac{\sigma^a}{2}, \frac{\sigma^b}{2}\right] = i\epsilon_{abc}\frac{\sigma^c}{2}$

in which ϵ_{abc} gives the "structure constants" for $SU(2)$. Also

$$\sigma^a\sigma^b = \delta^{ab} + i\epsilon_{abc}\sigma^c.$$

so multiplying thru by 2

$$A'^a_\mu \sigma^a = A^a_\mu \sigma^a + i\epsilon^a i \epsilon_{abc} \sigma^c A^b_\mu - i\sigma^a \partial_\mu \epsilon^a$$

$$= A^a_\mu \sigma^a - \epsilon^a A^b_\mu \epsilon_{abc} \sigma^c - i\sigma^a \partial_\mu \epsilon^a$$

Now $\quad \mathrm{Tr}\, \sigma^a \sigma^b = 2\delta_{ab}$, so

$$\mathrm{Tr}\, A'^a_\mu \sigma^a \sigma^d = 2\delta_{ad} A'^a_\mu = 2A'^d_\mu$$

$$= \mathrm{Tr}\left[\left(A^a_\mu \sigma^a - \epsilon^a A^b_\mu \epsilon_{abc} \sigma^c - i\sigma^a \partial_\mu \epsilon^a\right)\sigma^d\right]$$

$$= 2\delta_{ad} A^a_\mu - \epsilon^a A^b_\mu \epsilon_{abc} 2\delta_{cd} - 2i\delta_{ad} \partial_\mu \epsilon^a$$

That is,

$$A'^d_\mu = A^d_\mu - \epsilon^a A^b_\mu \epsilon_{abd} - i\partial_\mu \epsilon^d$$

or

$$A'^a_\mu = A^a_\mu - \epsilon_{abc} \epsilon^b A^c_\mu - i\partial_\mu \epsilon^a .$$

In the abelian case of $U(1)$ in which the structure constants vanish, this reduces to

$$A'_\mu = A_\mu - i\partial_\mu \epsilon .$$

which is an abelian gauge transformation.

Since $D_\mu(x)$ transforms as

$$D'_\mu(x) = g(x)\, D_\mu(x)\, g^{-1}(x)$$

it's easy to find invariant objects:

$$\bar\Psi \gamma^\mu D_\mu \Psi$$

$$\mathrm{Tr}\, [D_\mu, D_\nu][D^\mu, D^\nu].$$

For

$$(\bar\Psi \gamma^\mu D_\mu \Psi)' = \bar\Psi g^{-1} \gamma^\mu g D_\mu g^{-1} g \Psi$$

$$= \bar\Psi \gamma^\mu D_\mu \Psi = \bar\Psi \not{D} \Psi.$$

And

$$\mathrm{Tr}\, [D'_\mu, D'_\nu][D^{\mu\prime}, D^{\nu\prime}]$$

$$= \mathrm{Tr}\, [g D_\mu g^{-1}, g D_\nu g^{-1}][g D^\mu g^{-1}, g D^\nu g^{-1}]$$

$$= \mathrm{Tr}\, g [D_\mu, D_\nu][D^\mu, D^\nu] g^{-1}$$

$$= \mathrm{Tr}\, [D_\mu, D_\nu][D^\mu, D^\nu].$$

So what's $[D_\mu, D_\nu]$?

$$[D_\mu, D_\nu] = [\partial_\mu + A_\mu, \partial_\nu + A_\nu]$$

$$= \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu].$$

In the case of $U(1)$, $[A_\mu, A_\nu] = 0$ and

$$[D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu = F_{\mu\nu}.$$

In the case of $SU(2)$,

$$[D_\mu, D_\nu] = \partial_\mu A_\nu^a \frac{\sigma^a}{2} - \partial_\nu A_\mu^a \frac{\sigma^a}{2} + \left[A_\mu^a \frac{\sigma^a}{2}, A_\nu^b \frac{\sigma^b}{2}\right]$$

$$= \frac{\sigma^a}{2}\left(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a\right) + i\epsilon_{abc} \frac{\sigma^c}{2} A_\mu^a A_\nu^b.$$

$$\equiv \frac{\sigma^a}{2} F_{\mu\nu}^a$$

where

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + i\epsilon_{abc} A_\mu^b A_\nu^c.$$

The generators T_a of the adjoint representation are defined by

$$(T_a)_{bc} = -i\, f_{abc}. \qquad (9.150)$$

So for $SU(2)$, $(T_a)_{bc} = -i\,\epsilon_{abc}$. So $\epsilon_{abc} = i\, T^a_{bc}$ and the gauge field of $SU(2)$ transforms as

$$A'^a_\mu(x) = A^a_\mu(x) - \epsilon_{abc}\,\epsilon^b(x) A^c_\mu(x) - i\,\partial_\mu \epsilon^a(x)$$

$$= A^a_\mu(x) - i\, T^a_{bc}\, \epsilon^b(x) A^c_\mu(x) - i\,\partial_\mu \epsilon^a(x).$$

or

$$A'^a_\mu = A^a_\mu - i\,\epsilon T^a A_\mu - i\,\partial_\mu \epsilon^a.$$

In an arbitrary compact group, the generators t^a satisfy

$$[t^a, t^b] = i\, f_{abc}\, t^c \qquad (9.62)$$

in which $t^{a\dagger} = t^a$ and

$$\mathrm{Tr}\; t^a t^b = k\, \delta^{ab} = k\, \delta_{ab}. \qquad (9.63)$$

The generators of the adjoint representation are

$$(T_a)_{bc} = T^a_{bc} = -i\, f_{abc}. \qquad (9.150)$$

For such an arbitrary compact group,

$$A'_\mu(x) = g(x)\, A_\mu(x)\, g^{-1}(x) + g(x)\, \partial_\mu g^{-1}(x)$$

and

$$g(x) = e^{i\theta^a(x) t^a} .$$

For tiny $\theta = \epsilon$, we have

$$A'_\mu = A_\mu + i\epsilon^a [t^a, A_\mu] - i t^a \partial_\mu \epsilon^a$$

Letting $A_\mu = t^a A^a_\mu$, this is

$$t^a A'^a_\mu = t^a A^a_\mu + i\epsilon^a [t^a, t^b A^b_\mu] - i t^a \partial_\mu \epsilon^a$$

$$= t^a A^a_\mu + i\epsilon^a\, i f_{abc}\, t^c A^b_\mu - i t^a \partial_\mu \epsilon^a$$

$$= t^a A^a_\mu - \epsilon^c f_{cba}\, t^a A^b_\mu - i t^a \partial_\mu \epsilon^a .$$

Since the t^a's are orthogonal, we have

$$A^{a\prime}_\mu(x) = A^a_\mu(x) - \epsilon^c f_{cba} A^b_\mu(x) - i\partial_\mu \epsilon^a(x)$$

$$= A^a_\mu(x) - \epsilon^c i T^c_{ba} A^b_\mu(x) - i\partial_\mu \epsilon^a(x)$$

or

$$A^{a\prime}_\mu = A^a_\mu - \epsilon^c f_{acb} A^b_\mu - i\partial_\mu \epsilon^a$$

$$= A^a_\mu - i\,\epsilon^c T^a_{cb} A^b_\mu - i\partial_\mu \epsilon^a .$$

Interchanging b with c, this is how the gauge fields go

$$A^{a\prime}_\mu(x) = A^a_\mu(x) - i\,\epsilon^b(x)\,T^a_{bc}\,A^c_\mu(x) - i\,\partial_\mu \epsilon^a(x)$$

under an infinitesimal gauge transformation for an arbitrary compact gauge group. Here $T^a_{bc} = -i\,f_{abc}$, where the f_{abc}'s are the structure constants of the group (not of any particular representation of it).

For this general group,

$$D_\mu = \partial_\mu + A_\mu = \partial_\mu + t^a A^a_\mu \quad \text{and so}$$

$$[D_\mu, D_\nu] = [\partial_\mu + t^a A^a_\mu, \partial_\nu + t^b A^b_\nu]$$

$$= t^b \partial_\mu A^b_\nu - \partial_\nu t^a A^a_\mu + [t^a, t^b] A^a_\mu A^b_\nu$$

$$= t^a(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu) + i\,f_{abc}\,t^c A^a_\mu A^b_\nu$$

$$= t^a(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu) + i\,f_{cba}\,t^a A^c_\mu A^b_\nu$$

$$= t^a\left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + i\,f_{abc}\,A^b_\mu A^c_\nu\right)$$

$$\equiv t^a F^a_{\mu\nu} \quad \text{where}$$

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + i\,f_{abc}\,A^b_\mu A^c_\nu$$

is the non-abelian generalization of $F_{\mu\nu}$.

Another formula for $F^a_{\mu\nu}$ is

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - A^b_\mu T^a_{bc} A^c_\nu .$$

One term in the action density is the trace

$$\mathrm{Tr}\left([D_\mu, D_\nu][D^\mu, D^\nu]\right) = \mathrm{Tr}\left(t_a F^a_{\mu\nu} t^b F^{b\mu\nu}\right)$$

$$= k\,\delta_{ab} F^a_{\mu\nu} F^{b\mu\nu} = k F^a_{\mu\nu} F_a^{\mu\nu}$$

in which the elevation of the indices of compact groups makes no difference and is done for the appearance of the formulas.

In particle physics, one replaces our A^a_μ by $-ig A^a_\mu$, where g is a coupling constant, and the i makes A hermitian. Thus with

$$^m A^a_\mu = -ig A^a_\mu$$

we have

$$D_\mu = \partial_\mu - ig\, t^a A^a_\mu$$

and

$$^m F^a_{\mu\nu} = -ig\left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu\right) - ig^2 f_{abc} A^b_\mu A^c_\nu$$

$$= -ig\left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\, f_{abc} A^b_\mu A^c_\nu\right) = -ig F^a_{\mu\nu}$$

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\, f_{abc} A^b_\mu A^c_\nu .$$

The action for a Yang-Mills theory is

$$S = \int -\frac{1}{4} F^a_{\mu\nu} F_a^{\mu\nu} + \bar{\psi}(i\not{D} - m)\psi \; d^4x$$

where

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f_{abc} A^b_\mu A^c_\nu$$

and

$$D_{\mu\alpha\beta} = \partial_\mu \delta_{\alpha\beta} - ig A^a_\mu t^a_{\alpha\beta}$$

in which $[t^a, t^b] = i f_{abc} t^c$.

For $SU(2)$, $t^a = \frac{1}{2}\sigma^a$.

For $SU(3)$, $t^a = \frac{1}{2}\lambda^a$ where the λ^a's are Gell-Mann's matrices

$$\lambda^i = \begin{pmatrix} \sigma^i & 0 \\ & 0 \\ 0\,0 & 0 \end{pmatrix} \quad \text{for } i = 1, 2, 3$$

$$\lambda^4 = \begin{pmatrix} & & 1 \\ & & \\ 1 & & \end{pmatrix}, \quad \lambda^5 = \begin{pmatrix} & & -i \\ & & \\ i & & \end{pmatrix}, \quad \lambda^6 = \begin{pmatrix} & & \\ & 0 & 1 \\ & 1 & 0 \end{pmatrix}$$

$$\lambda^7 = \begin{pmatrix} & & \\ & 0 & -i \\ & i & 0 \end{pmatrix}, \quad \lambda^8 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & & \\ & 1 & \\ & & -2 \end{pmatrix}.$$

Note that all the generators of $SU(n)$ are traceless.

The classical equations of motion that follow from

$$0 = \delta S$$

are

$$(i\not{D} - m)\psi = (i\gamma^\mu D_\mu - m)\psi$$

$$= (i\gamma^\mu(\partial_\mu - igA_\mu^a t^a) - m)\psi = 0$$

and

$$\partial^\mu F^a_{\mu\nu} + g f_{abc} A^{b\mu} F^c_{\mu\nu} = -g j^a_\nu$$

where $\quad j^a_\nu = \bar\psi \gamma_\nu t^a \psi.$

The action density $\mathcal{L}$ is

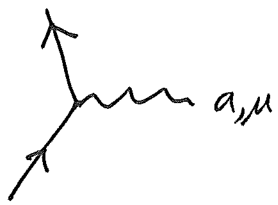
$$\mathcal{L} = \mathcal{L}_0 + g A^a_\mu \bar\psi \gamma^\mu t^a \psi - g f_{abc} (\partial_\kappa A^a_\lambda) A^{\kappa b} A^{\lambda c}$$

$$- \frac{1}{4} g^2 f_{eab} A^a_\kappa A^b_\lambda f^{ecd} A^{\kappa c} A^{\lambda d}$$

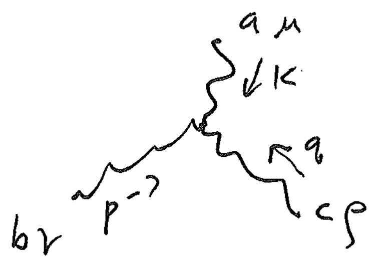
where $\quad \mathcal{L}_0 = -\frac{1}{4}(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)(\partial^\mu A^\nu_a - \partial^\nu A^\mu_a)$

$$+ \bar\psi(i\not{\partial} - m)\psi.$$

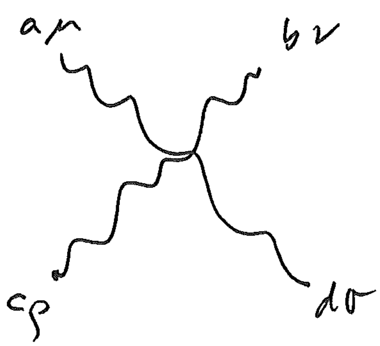
The new Feynman vertices due to $\mathcal{L}-\mathcal{L}_0$ are



$$ig\gamma^{\mu}t^{a}$$



$$gf^{abc}\left[\eta^{\mu\nu}(k-p)^{\rho} + \eta^{\nu\rho}(p-q)^{\mu} + \eta^{\rho\mu}(q-k)^{\nu}\right]$$



$$-ig^{2}\left[f^{abe}f^{cde}(\eta^{\mu\rho}\eta^{\nu\sigma}-\eta^{\mu\sigma}\eta^{\nu\rho}) + f^{ace}f^{bde}(\eta^{\mu\nu}\eta^{\rho\sigma}-\eta^{\mu\sigma}\eta^{\nu\rho}) + f^{ade}f^{bce}(\eta^{\mu\nu}\eta^{\rho\sigma}-\eta^{\mu\rho}\eta^{\nu\sigma})\right].$$

Pages 11-14 of these notes are consistent with P&S.

Another way of looking at non-abelian gauge theory is to write the vector of matter fields in terms of basis vectors $e_i(x)$:

$$\psi(x) = \sum_{i=1}^{n} e_i(x)\,\psi_i(x).$$

Now

$$\bar\psi\, i\partial\!\!\!/\,\psi = \bar\psi_i\, e_i^{+}\, i\gamma^\mu \partial_\mu e_j \psi_j$$

$$= \bar\psi_i\, i\gamma^\mu (\partial_\mu \delta_{ij} + e_i^{+}\partial_\mu e_j)\psi_j$$

$$= \bar\psi\, i\gamma^\mu D_\mu \psi$$

where

$$D_{\mu ij} = \partial_\mu \delta_{ij} + e_i^{+}\partial_\mu e_j$$

So

$$A_{\mu ij} = e_i^{+}\partial_\mu e_j.$$