

$$\langle p, s \rangle = \underbrace{U(L(p))}_{\text{some spin index}} |p_0, s\rangle \quad p_0 = (m, \vec{p}^*), \quad m > 0.$$

$$U(1) |p, s\rangle = U(1) U(L(p)) |p_0, s\rangle$$

$$= -i\partial J_3 - i\partial J_2 + i\partial J_2 i\partial J_3$$

$$L(p) = R(p) B(p^0) R(p)^\dagger = e^{-i\partial J_3} B_2(p^0) e^{i\partial J_3}$$

is a standard Lorentz transformation from p_0 to p .

$$U(1) |p, s\rangle = U(L(1p)) U^{-1}(L(1p)) U(1) U(L(p)) |p_0, s\rangle$$

$$= U(L(1p)) U(L^{-1}(1p) \wedge L(p)) |p_0, s\rangle$$

But $L^{-1}(1p) \wedge L(p)p_0 = L^{-1}(1p)1p = p_0$.

So $W(1, p) \equiv L^{-1}(1p) \wedge L(p)$ is a notation,

the Wigner notation. So

$$U(1) |p, s\rangle = U(L(1p)) U(W(1, p)) |p_0, s\rangle.$$

But $U(w) |p_0, s\rangle = \sum_{s'=-j}^j D_{s's}^{(j)}(w) |p_0, s'\rangle$.

So $U(1) |p, s\rangle = U(L(1p)) \sum_{s'} D_{s's}^{(j)}(w) |p_0, s'\rangle$

$$= \sum_{s'=-j}^j D_{s's}^{(j)}(w) U(L(1p)) |p_0, s'\rangle$$

$$U(\lambda) |p,s\rangle = \sum_{s'=-j}^j D_{s's}^{(j)}(w(\lambda, p)) |\lambda p, s'\rangle.$$

$$U(\lambda) \sqrt{2p^0} a^+(p,s) |0\rangle = \sum_{s'=-j}^j D_{s's}^{(j)}(w) \sqrt{2(\lambda p)^0} a^+(\lambda p, s') |0\rangle$$

$$\tilde{U}'(\lambda) |0\rangle = |0\rangle.$$

$$U(\lambda) a^+(p,s) \tilde{U}'(\lambda) = \sqrt{\frac{(\lambda p)^0}{p^0}} \sum_{s'=-j}^j D_{s's}^{(j)}(w(\lambda, p)) a^+(\lambda p, s')$$

generalized the spin-zero case (2.38)

$$U(\lambda) a^+(p) \tilde{U}'(\lambda) = \sqrt{\frac{(\lambda p)^0}{p^0}} a^+(\lambda p).$$

$$(T) \quad U(a) a^+(p,s) \tilde{U}(a) = e^{+i\vec{P} \cdot \vec{a}} a^+(p,s) e^{-i\vec{P} \cdot \vec{a}} = e^{i\vec{p} \cdot \vec{a}} a^+(p,s).$$

$$\vec{P} \cdot \vec{a} = P^0 a^0 - \vec{p} \cdot \vec{a}.$$

Since $D_{s's}^{(j)}(w) = D_{s's}^{(j)*}(w^{-1}) = D_{ss'}^{*(j)}(w^{-1})$, we get

$$(a^+) \quad U(\lambda) a^+(p,s) \tilde{U}'(\lambda) = \sqrt{\frac{(\lambda p)^0}{p^0}} \sum_{s'=-j}^j D_{ss'}^{*(j)}(w^{-1}(\lambda p)) a^+(\lambda p, s').$$

Taking the adjoint of the last equation, we have

$$(a) \quad U(\lambda) a(p, s) U^\dagger(\lambda) = \sqrt{\frac{(1p)^0}{p^0}} \sum_{s'=-j}^j D_{ss'}^{(i)} (w'(\lambda, p)) a(1p, s').$$

So now we know how massive fields transform under Lorentz transformations, since fields are linear combinations of a and a^\dagger .

$$\psi_\ell(x) = \psi_\ell^{(+)}(x) + \psi_\ell^{(-)}(x)$$

$$\psi_\ell^{(+)}(x) = \sum_s \int d^3p u_\ell(x; p, s) a(p, s)$$

$$\psi_\ell^{(-)}(x) = \sum_s \int d^3p v_\ell(x; p, s) a^\dagger(p, s).$$

We already know the translation part

$$iP.a \quad i(Ha^0 - \vec{P} \cdot \vec{a})$$

$$U(a) = e^{iP.a} = e^{i(Ha^0 - \vec{P} \cdot \vec{a})}$$

and its effect on a^\dagger (and a) as given by (T).

$$\text{We want } \psi_\ell(x+a) = U(a) \psi_\ell^{(\pm)}(x) U^\dagger(a).$$

So using (T), and setting $x=0$ and $a=x$ ($\neq 0$), we get

$$\psi_\ell^{(\pm)}(x) = U(x) \psi_\ell^{(\pm)}(0) U^\dagger(x) \quad \text{, that is ,}$$

$$\begin{aligned} \psi_\ell^{(+)}(x) &= \sum_s \int d^3p u_\ell(x; p, s) a(p, s) = \sum_s \int d^3p u_\ell(0; p, s) U(x) a(p, s) U^\dagger(x) \\ &= \sum_s \int d^3p u_\ell(0; p, s) e^{ipx} a(p, s). \end{aligned}$$

$$\text{So } u_e(x; ps) = e^{-ipx} u_e(0; ps).$$

Again using (7), we have

$$\begin{aligned}\psi_e^{(-)}(x) &= \sum_s \int d^3 p v_e(x; ps) a^\dagger(p, s) = \sum_s \int d^3 p v_e(0; ps) U(x) a^\dagger(p, s) U(x) \\ &= \sum_s \int d^3 p v_e(0; ps) e^{ipx} a^\dagger(p, s).\end{aligned}$$

$$v_e(x; ps) = e^{ipx} v_e(0; ps).$$

So now

$$\psi_e^{(+)}(x) = \sum_s \int d^3 p u_e(p, s) e^{-ipx} a(p, s)$$

$$\psi_e^{(-)}(x) = \sum_s \int d^3 p v_e(ps) e^{ipx} a^\dagger(ps),$$

which we already knew, but this S.W. derivation shows how particles transform implies how fields transform.

Now under Lorentz transformations, we want

$$(w^3) \quad u(\Lambda) \psi_e^{(\pm)}(x) u^{-1}(\Lambda) = \sum_{g'} D_{ee'}(\Lambda^{-1}) \psi_{g'}^{(\pm)}(\Lambda x),$$

where D is some $D^{(i,j)}$ representation of the Lorentz group.

So from (a) and (a⁺), we get (w³) if

$$\begin{aligned}
 u(\lambda) \psi_e^{(+)}(x) u'(\lambda) &= \sum_s \int d^3 p u_e(p s) u(\lambda) \alpha(p s) u'(\lambda) e^{-ipx} \\
 &= \sum_{ss'} \int d^3 p u_e(p s) \sqrt{\frac{(\lambda p)^0}{p^0}} D_{ss'}^{(j)}(w'(1, p)) \alpha(\lambda p, s') e^{-ipx} \\
 &= \sum_{\ell'} D_{\ell \ell'}^{ee'}(\lambda^{-1}) \psi_{\ell'}^{(+)}(\lambda x) \\
 &= \sum_{\ell' s} \int d^3 p u_{\ell'}(p s) \alpha(p, s) e^{-ip\lambda x} D_{\ell \ell'}^{ee'}(\lambda^{-1})
 \end{aligned}$$

with a similar equation for $\psi_{\ell'}^{(-)}(x)$:

$$\begin{aligned}
 (b) \quad \sum_{ss'} \int d^3 p v_e(p, s) \sqrt{\frac{(\lambda p)^0}{p^0}} D_{ss'}^{(j)}(w'(1, p)) \alpha^+(\lambda p, s) e^{ipx} \\
 &= \sum_{\ell' s} \int d^3 p D_{\ell \ell'}(\lambda^{-1}) v_{\ell'}(p s) \alpha^+(p s) e^{ip\lambda x}.
 \end{aligned}$$

So we need to replace $d^3 p$ by $d^3 \lambda p (\rho / (\lambda p)^0)$ in the integrals with the phase factors $e^{\mp ipx}$; noting that

$$e^{ip\lambda x} = e^{i\lambda p\lambda x}, \quad e^{-ip\lambda x} = e^{-i\lambda p\lambda x}, \quad e^{+ip\lambda x} = e^{-i\lambda p\lambda x}.$$

So we need

(t)

$$\sum_{ss'} \int d^3 \Lambda p \frac{p^0}{(\Lambda p)^0} u_e(p, s) \sqrt{\frac{(\Lambda p)^0}{p^0}} D_{ss'}^{(j)}(w^-) a(\Lambda p, s') e^{-i \Lambda p \Lambda x}$$

$$= \sum_{e's} \int d^3 p D_{ee'}(\Lambda^-) u_{e'}(p, s) a(p, s) e^{-i p \Lambda x}$$

$$= \sum_{e's} \int d^3 \Lambda p D_{ee'}(\Lambda^-) u_{e'}(\Lambda p, s) a(\Lambda p, s) e^{-i \Lambda p \Lambda x}$$

where we just replaced p by Λp everywhere.

This is an example of $\int f(q) dy = \int f(y(x)) dy(x) = \int f(x) dx$.

So we need (switching s and s' in the top equations (t) & (b))

(1)

$$\sum_{e'} D_{ee'}(\Lambda^-) u_{e'}(\Lambda p, s) = \sqrt{\frac{p^0}{(\Lambda p)^0}} \sum_{s'}^{(j)} u_{e'}(p, s') D_{ss'}^{(j)}(w^-)$$

and

(2)

$$\sum_{e'} D_{ee'}(\Lambda^-) v_{e'}(\Lambda p, s) = \sqrt{\frac{p^0}{(\Lambda p)^0}} \sum_{s'}^{(j)*} D_{ss'}^{(j)*}(w^-) v_{e'}(p, s')$$

We now multiply (1) by $D_{\ell\ell'}^{(j)}(\Lambda)$
to get

$$(3) \quad u_{\ell''}(\Lambda p, s) = \sqrt{\frac{p^0}{(\Lambda p)^0}} \sum_{s'\ell} D_{\ell''\ell}^{(j)}(\Lambda) D_{ss'}^{(j)}(w^-(\Lambda p)) u_{\ell}(p, s')$$

Similarly, multiplying eq (2) by $D(\Lambda)$, we get

$$(4) \quad v_{\ell''}(\Lambda p, s) = \sqrt{\frac{p_0}{(1p)}} \sum_{s' \in \ell} D_{ss'}^{(\ell)}(w') D_{\ell''}(\Lambda) v_{\ell'}(p, s').$$

Equations (3) and (4) tell us what the spinors u and v must be. In them, we set $p = (\vec{m}, 0)$ and $\Lambda = L(g)$ the standard Lorentz transformation from $(\vec{m}, 0)$ to g .

Then

$$\begin{aligned} w(\Lambda, p) &= L^{-1}(\Lambda p) \wedge L(p) \quad \text{but } L(p) = I \\ &= L^{-1}(L(g)p) L(g) \quad p_0 = (\vec{m}, 0) \\ &= L^{-1}(g) L(g) = I. \end{aligned}$$

Thus w and w' are just the identity matrix. So (3 & 4) give

$$(5) \quad u_{\ell'}(g, s) = \sqrt{\frac{m}{g_0}} D_{\ell'}(L(g)) u_0(p_0, s)$$

$$(6) \quad v_{\ell'}(g, s) = \sqrt{\frac{m}{g_0}} D_{\ell'}(L(g)) v_0(p_0, s).$$

So the spinors $u(g)$ and $v(g)$ are related to the spinors for a particle at rest by the Lorentz matrix $D(L(g))$ that represents the standard boost from $p_0^2(\vec{m}, 0)$ to g .

The standard $L(p)$ has the form

$$(sb) \quad L(p) = R(\vec{p}) B(p^0) R'(\vec{p}).$$

And in the $D^{1/2} \oplus D^{0/2}$ representation —
the Dirac representation — it is

$$(f) \quad D(L(p)) = \frac{m + \not{p}^0}{\sqrt{2m(p^0 + m)}}.$$

$$= \frac{1}{\sqrt{2m(p^0 + m)}} \begin{pmatrix} p^0 + m - \vec{p} \cdot \vec{\sigma} & 0 \\ 0 & p^0 + m + \vec{p} \cdot \vec{\sigma} \end{pmatrix}.$$

The spinors for $p = p_0 = (m, \vec{0})$ are

$$u(0, \frac{1}{2}) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad u(0, -\frac{1}{2}) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

$$v(0, \frac{1}{2}) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} \quad v(0, -\frac{1}{2}) = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

which satisfy the Majorana conditions

$$u(\vec{0}, s) = -i \gamma^2 v^*(\vec{0}, s) = -i \gamma^2 v(0, s) \quad \text{and}$$

$$v(\vec{0}, s) = -i \gamma^2 u^*(0, s) = -i \gamma^2 u(0, s).$$

The standard boost $L(p)$ is a boost in the \hat{p} direction.

$$L(p) = R(\hat{p}) B(p^0) \tilde{R}'(\hat{p}) = B(p)$$

$$= e^{\alpha \hat{p} \cdot \vec{B}}$$

where you may show that

$$\cosh \alpha = \cosh \alpha = p^0/m \quad \text{and}$$

$$\sinh \alpha = \sinh \alpha = i\hat{p}^0/m.$$

The $D^{(1/2, 0)}_{(L(p))}$ matrix then is

$$D^{(1/2, 0)}_{(L(p))} = e^{-i \hat{p} \cdot \vec{K}} = e^{-\alpha \hat{p} \cdot \vec{\sigma}/2}$$

since $-\vec{K} = -i\vec{\sigma}/2$ in this basis.

You may show that

$$e^{-\alpha \hat{p} \cdot \vec{\sigma}/2} = \cosh(\alpha/2) + \hat{p} \cdot \vec{\sigma} \sinh(\alpha/2)$$

where $\cosh \alpha/2 = \sqrt{\frac{p^0 + m}{2m}}$ and

$$\sinh \alpha/2 = \sqrt{\frac{p^0 - m}{2m}}.$$

Thus

$$e^{-\alpha \vec{p} \cdot \frac{\vec{\sigma}}{2}} = \sqrt{\frac{p^0 + m}{2m}} - \vec{p} \cdot \vec{\sigma} \sqrt{\frac{p^0 - m}{2m}}$$

so that

$$D^{(1/2, 0)}(L(p)) = \frac{p^0 + m - \vec{p} \cdot \vec{\sigma}}{\sqrt{2m(p^0 + m)}}.$$

Similarly, the $D^{(0, 1/2)}(L(p))$ matrix is

$$D^{(0, 1/2)}(L(p)) = e^{i \alpha \vec{p} \cdot \vec{k}} = e^{\alpha \vec{p} \cdot \vec{\sigma}/2}$$

$$\begin{aligned} D^{(0, 1/2)} &= \cosh \alpha/2 + \vec{p} \cdot \vec{\sigma} \sinh \alpha/2 \\ &= \frac{p^0 + m + \vec{p} \cdot \vec{\sigma}}{\sqrt{2m(p^0 + m)}}. \end{aligned}$$

The formula (+) for the Dirac matrix

$$D(L(p)) = \frac{m + \vec{p} \cdot \vec{\tau}^0}{\sqrt{2m(p^0 + m)}}$$

now follows from $(1/2, 0)$ and $(0, 1/2)$.

To figure out what the $\hat{p}' = 0$ operators $u_{\ell'}(0, s)$ and $v_{\ell'}(0, s)$ are, we go back to (3) and (4) but now set $A = R$ a notation and also set $\hat{p}' = 0$. Then

$$(7) \quad W(A, p) = W(A, 0) = L^{-1}(R_0) R L(0)$$

$$= L^{-1}(0) R L(0) = R.$$

Then we get

$$(8) \quad u_{\ell'}(0, s) = \sum_{s' \ell} D_{\ell' \ell}(R) D_{s's}^{(j)}(R^{-1}) u_{\ell}(0, s')$$

$$(9) \quad v_{\ell'}(0, s) = \sum_{s' \ell} D_{\ell' \ell}(R) D_{s's}^{(j)*}(R^{-1}) v_{\ell}(0, s').$$

Multiply (8) by $D_{ss''}^{(j)}(R)$:

$$u_{\ell'}(0, s) D_{ss''}^{(j)}(R) = \sum_{s' \ell} D_{\ell' \ell}(R) D_{s's}^{(j)}(R^{-1}) D_{ss''}^{(j)}(R) u_{\ell}(0, s')$$

$$= \sum_{s' \ell} D_{\ell' \ell}(R) \delta_{s's''} u_{\ell}(0, s')$$

$$= \sum_{\ell} D_{\ell' \ell}(R) u_{\ell}(0, s'')$$

or

$$\sum_s u_{\ell}(0, s) D_{ss''}^{(j)}(R) = \sum_{\ell} D_{\ell' \ell}(R) u_{\ell}(0, s'')$$

Multiply (9) by $D_{S-S''}^{(j)*}(R)$

$$\sum_s D_{S-S''}^{(j)*}(R) V_{\ell'}(0, s) = \sum_{ss'e} D_{e'e}(R) D_{S-S''}^{(j)*}(R) D_{ss'}^{(j)*}(R') V_{\ell'}(0, s')$$

$$(11) \quad = \sum_{\ell'} D_{e'e}(R) V_{\ell'}(0, s'')$$

In matrix notation (10) & (11) are

$$(12) \quad u(0) D^{(j)}(R) = D(R) u(0) \quad \text{and}$$

$$(13) \quad v(0) D^{(j)*}(R) = D(R) v(0)$$

where now u and v are matrices

$$u_{\ell s}(0) = u_{\ell}(0, s) \quad \text{and} \quad v_{\ell s}(0) = v_{\ell}(0, s).$$

With all indices explicit, they are

$$\sum_s u_{\ell}(0, s) D_{SS'}^{(j)}(R) = \sum_{\ell'} D_{\ell\ell'}(R) u_{\ell'}(0, s')$$

$$\sum_s v_{\ell}(0, s) D_{S-S'}^{*(j)}(R) = \sum_{\ell'} D_{\ell\ell'}(R) v_{\ell'}(0, s').$$

$$\text{So if } D^{(j)}(R) = e^{-i\theta \cdot J} \quad \text{and} \quad D(R) = e^{-i\phi \cdot J},$$

then

$$u_e(0, s') J_{s's}^{i'} = J_{e'e}^i u_e(0, s) \quad \text{and}$$

$$v_e(0, s') J_{s's}^{i*} = - J_{e'e}^i v_e(0, s).$$

Let's assume that our particles and our fields ψ^+ , ψ^- , and $\psi = \psi^+ + \psi^-$ transform under irreducible representations of the rotation group. We now apply Schur's lemma:

- (1) If $D_1(g)A = A D_2(g)$ for all $g \in G$, and if D_1 & D_2 are inequivalent irreducible representations of G , then $A = 0$.
- (2) If for a finite-dimensional, irreducible representation $D(g)$ of a group G , we have $D(g)A = A D(g)$ for all $g \in G$, then $A = c I$.

We apply Schur's lemma to (12 & 13). First (12) tells us that if $D^{(i)}$ and D are irred. reps. of $SO(3)$, then they had better be equivalent, that is

$$D^{(i)}(R) = S D(R) S^{-1}$$

where S is a non-singular $(2j+1) \times (2j+1)$ square matrix. So we might as well take

$$D(R) = D^{(i)}(R).$$

Now part (2) of Schur's lemma applied to

$$u(0) D^{(j)}(R) = D^{(j)}(R) u(0)$$

tells us that $u(0) = c \mathbf{I}$. In the simplest non-trivial case, this means that for $D^{(1/2, 0)}$ and $D^{(0, 1/2)}$, $u(0)$ is proportional to the 2×2 unit matrix. So for them

$$D^{(1/2, 0)} \quad u(0, \frac{1}{2}) = c_+ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad u(0, -\frac{1}{2}) = c_+ \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$D^{(0, 1/2)} \quad u(0, \frac{1}{2}) = c_- \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad u(0, -\frac{1}{2}) = c_- \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Equation (13) $v(0) D^{(j)}(R) = D(R)v(0)$ is unknown. But since $D(R) = D^{(j)}(R)$, we have

$$v(0) D^{(j)*}(R) = D^{(j)}(R) v(0).$$

So $D^{(j)*}$ and $D^{(j)}$ must be equivalent.

$$D^{(j)*}(R) = S D^{(j)}(R) S^{-1}.$$

So

$$v(0) S D^{(j)}(R) S^{-1} = D^{(j)}(R) v(0)$$

or

$$v(0) S D^{(j)}(R) = D^{(j)}(R) v(0) S.$$

Thus by (2) of Schur's lemma $v(0) S = c \mathbf{I}$ for some constant c .

In the simplest non-trivial cases

$$\begin{aligned} D^{(j)}(R) &= D^{(1/2)}(\vec{\theta}) = e^{-i\theta \cdot \vec{\sigma}/2} \\ D^{(1/2)*}(\theta) &= e^{i\theta \cdot \vec{\sigma}^*/2} = \sigma_2 e^{-i\theta \cdot \vec{\sigma}/2} \quad \sigma_2 = \sigma_2 D(\vec{\theta}) \sigma_2. \end{aligned}$$

So $v(0) \sigma_2 = c I$ in these cases,
that is for $D^{(1/2; 0)}$ and $D^{(0, 1/2)}$. So

$$v(0) = c \sigma_2 = c \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$

So for $D^{(1/2, v)}$ we have

$$v(0, \frac{1}{2}) = d_+ \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad v(0, -\frac{1}{2}) = -d_+ \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

and for $D^{(0, 1/2)}$

$$v(0, \frac{1}{2}) = d_- \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad v(0, -\frac{1}{2}) = -d_- \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

And at finite momentum, the spinors are

$$u(\vec{p}, s) = \sqrt{\frac{m}{p_0}} D(L(p)) u(0, s)$$

$$v(\vec{p}, s) = \sqrt{\frac{m}{p_0}} D(L(p)) v(0, s).$$

For a $D^{(1/2, 0)}$ field $\xi(x)$, we may choose $c+$ as we like. In fact, in P&S's notation, we'd include a factor of $(2\pi)^3$ and \sqrt{m} and $\sqrt{2}$, etc.. So the real question, for this $(1/2, 0)$ case, is the ratio of $d+ \& c+$.

This ratio is determined by the requirement that the field

$$\xi(x) = \xi^{(+)}(x) + \xi^{(-)}(x)$$

commute or anti-commute with itself and with $\xi^+(x)$ at space-like separations. As we will see, this means that $d+ = c+$. Similarly, for $\eta(x)$ of $(0, 1/2)$, we need $d- = -c-$.

If, further, we wish to combine ξ and η into a ψ that under parity goes as

$$P \psi^{(+)}(x) P^{-1} = \gamma^k \beta \psi^{(+)}(Px)$$

$$P \psi^{(-c)}(x) P^{-1} = -\gamma^c \beta \psi^{(-c)}(Px)$$

where $Px = P(x^0, \vec{x}) = (x^0, -\vec{x})$ and γ and γ^c are constants and $\beta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ (Here c means anti-particle and applies if the field are complex linear combinations of two "real" fields.), then

we get $c_+ = c_-$ and so in Dirac's notation

$$u(\vec{0}, \frac{1}{2}) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$u(\vec{0}, -\frac{1}{2}) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

$$v(\vec{0}, \frac{1}{2}) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}$$

$$v(\vec{0}, -\frac{1}{2}) = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

and

$$\psi_e(x) = \frac{1}{(2\pi)^{3/2}} \sum_s \int d^3 p [u_e(p, s) e^{-ipx} + v_e(p, s) e^{ipx}]$$

in the PS metric. Here c means anti-particle.

$$u(p, s) = \sqrt{\frac{m}{p^0}} D(L(p)) u(0, s)$$

$$v(p, s) = \sqrt{\frac{m}{p^0}} D(L(p)) v(0, s)$$

$$D(L(p)) = \frac{1}{\sqrt{2m(p^0 + m)}} \begin{pmatrix} p^0 + m - p \cdot \sigma & 0 \\ 0 & p^0 + m + p \cdot \sigma \end{pmatrix}$$

$$= \frac{m + \cancel{p} \cdot \gamma^0}{\sqrt{2m(p^0 + m)}}.$$

In PS's notation, this is

$$\Psi(x) = \frac{1}{(2\pi)^3} \sum_s \int \frac{d^3 p}{\sqrt{2p^0}} [a_s^s(p) a_p^s e^{-ipx} + v_s^s(p) b_p^s e^{ipx}],$$

The ω 's differ because PS have

$$a_p^s = a(p, s) (2\pi)^{3/2}.$$

So

$$a_s^s(p) = \sqrt{\frac{1}{(p^0+m)}} (m + \not{p} r^0) u(\vec{p}, s)$$

$$= \sqrt{2p^0} u(\vec{p}, s)$$

and

$$v_s^s(p) = \sqrt{2p^0} v(\vec{p}, s).$$

$$v_s^s(p) = \sqrt{\frac{1}{p^0+m}} (m + \not{p} r^0) v(0, s).$$

We want ξ to anti-commute with η self and with ξ^* and with ζ and ζ^* at space-like separations. We can transfer the constants to the fields

$$\xi^{(+)}(x) \rightarrow c + \xi^{(+)}(x) \quad \xi^{(-)} \rightarrow d + \xi^{(+)}(x)$$

$$\zeta^{(+)}(x) \rightarrow c - \zeta^{(+)}(x) \quad \zeta^{(-)} \rightarrow d - \zeta^{(+)}(x).$$

Then in W's notation, the 4-spinsors are as on p. 17d of these notes. That is, for $\xi^{(+)}$ and $\xi^{(-)}$

$$u(0, \frac{1}{2}) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad u(0, -\frac{1}{2}) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$v(0, \frac{1}{2}) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad v(0, -\frac{1}{2}) = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 0 \end{pmatrix}$$

and so for the $(1/2, 0)$ representation $\rightarrow (1/2, 0)$

$$u(p, s) = \frac{1}{\sqrt{2p^0(p^0+m)}} (p^0 + m - p \cdot \sigma) u(0, s)$$

i.e. the spin matrices are

$$u(p) = \frac{1}{2} \frac{1}{\sqrt{p^0(p^0+m)}} (p^0 + m - p \cdot \sigma) = (u(p, \frac{1}{2}), u(p, -\frac{1}{2}))$$

$$v(p) = \frac{1}{2} \frac{-i}{\sqrt{p^0(p^0+m)}} (p^0 + m - p \cdot \sigma) \sigma_2 = (v(p, \frac{1}{2}), v(p, -\frac{1}{2})).$$

We assume (PS notation)

$$\{\alpha_p^s, \alpha_{p'}^{s'}\} = (2\pi)^3 \delta_{ss'} \delta^{(3)}(\vec{p} - \vec{p}').$$

The PS spin matrices are

$$u_p = \sqrt{2p^0} u(p) = \frac{p^0 + m - p \cdot \sigma}{\sqrt{2(p^0 + m)}}$$

$$v_p = \sqrt{2p^0} v(p) = -i \frac{(p^0 + m - p \cdot \sigma)}{\sqrt{2(p^0 + m)}} \sigma_2.$$

$$\xi(x) = \sum_s \int \frac{d^3 p}{(2\pi)^3 \sqrt{2p^0}} [c_p^s u_p^s a_p e^{-ipx} + d_p^s v_p^s a_p^* e^{ipx}]$$

S_0

$$\begin{aligned} \{\xi_s(x), \xi_t(y)\} &= \int \frac{d^3 p d^3 q}{(2\pi)^6 \sqrt{2p^0 2q^0}} \left[\{u_{ps}, a_{ps}\}, v_{qt}^*, a_{qt}^* \right] e^{-ipx + iqy} \\ &\quad + \left[\{v_{ps}, a_{ps}^*\}, u_{qt}^*, a_{qt}^* \right] e^{ipx - iqy} \Big]_{ct+dt} \\ &= c \int \frac{d^3 p}{(2\pi)^3 2p^0} \left[u_{ps}, v_{pt}^* e^{-ip(x-y)} + v_{ps}, u_{pt}^* e^{ip(x-y)} \right]_{ct} \\ &= c_{ct} \int \frac{d^3 p}{(2\pi)^3 2p^0} \left(e^{-ip(x-y)} \left(U_p V_p^T + V_p U_p^T \right) e^{ip(x-y)} \right). \end{aligned}$$

Still for $\Omega^{(1/2,0)}$,

$$\begin{aligned}
 UV^T &= \frac{(p^0 + m - p \cdot \sigma)(-\sigma_2^\top)(p^0 + m - p \cdot \sigma^\top)}{2(p^0 + m)} \\
 &= i(p^0 + m - p \cdot \sigma)\sigma_2(p^0 + m - p \cdot \sigma^\top)/2(p^0 + m) \\
 &= \frac{i}{2(p^0 + m)} [(p^0 + m)^2 - \vec{p}^2]\sigma_2 \\
 &= \frac{i}{2(p^0 + m)} (p^{0^2} + m^2 + 2m p^0 - \vec{p}^2)\sigma_2 \\
 &= \frac{i}{2(p^0 + m)} (2m^2 + 2m p^0)\sigma_2 = im\sigma_2
 \end{aligned}$$

and so $VU^T = -im\sigma_2$. So

$$\{\xi_s(x), \xi_t(y)\} = c + dt - im\sigma_2 \int \frac{d^3 p}{(2\pi)^3 2p^0} (e^{-ip(x-y)} - e^{-ip(y-x)})$$

which vanishes when $(x-y)^2 < 0$ [cf. PS (2.53)].

The reason is that

$$\Delta + (x-y) \equiv \int \frac{d^3 p}{(2\pi)^3 2p^0} e^{-ip(x-y)}$$

is Lorentz invariant.

So go to a frame where $x^0 = y^0$. Then flip $\vec{p} \rightarrow -\vec{p}$.

$$\text{We get } \Delta_+(x-y) = \Delta_+(y-x)$$

If $x-y$ is spacelike. Thus for space-like $x-y$, the function

$$\Delta(x-y) = \Delta_+(x-y) - \Delta_+(y-x) \text{ vanishes.}$$

So $\xi_s(x)$ and $\xi_t(y)$ anti-commute at space-like separations no matter what $c+$ and $d+$ are. What about $\xi_s(x)$ and $\xi_t^+(y)$?

$$\{\xi_s(x), \xi_t^+(y)\} = \sum_{s't'} \int \frac{d^3 q d^3 p}{(2\pi)^6 \sqrt{2p^0 2q^0}} \left[$$

$$\times \left\{ C + U_{ps}, a_{ps}, e^{-ipx} + d + V_{ps}, a_{ps}^*, e^{ipx}, \right. \\ \left. c_{q+e}^*, a_{q+e}^+, a_{q+e}^+, e^{iqy} + d_{q+e}^* V_{q+e}^*, a_{q+e}^+, e^{-iqy} \right\} \right]$$

$$= \int \frac{d^3 p}{(2\pi)^3 2p^0} \left((C + U_p U_p^*) e^{-ip(x-y)} + (d + V_p V_p^*) e^{ip(x-y)} \right)$$

$$U_p U_p^* = \frac{-(p^0 + m - p \cdot \sigma)^2}{2(p^0 + m)} = \frac{p^0{}^2 + 2mp^0 + m^2 + p^2 - 2(p^0 + m)p \cdot \sigma}{2(p^0 + m)}$$

$$= p^0 - p \cdot \sigma$$

$$V_p V_p^* = \frac{-i(p^0 + m - p \cdot \sigma) \sigma_2 \cdot \sigma_2 (p^0 + m - p \cdot \sigma)}{2(p^0 + m)} = p^0 - p \cdot \sigma.$$

So

$$\{\xi_s(x), \xi_t^+(y)\} = \int \frac{d^3 p}{(2\pi)^2 p^0} \left(iC + i^2 (p^0 - p \cdot \sigma) e^{-ip(x-y)} \right. \\ \left. + iD + i^2 (p^0 - p \cdot \sigma) e^{ip(x-y)} \right).$$

This vanishes at space-like separations if and only if $|C| + |D| = |D| + |D|$. For then

$$\{\xi_s(x), \xi_t^+(y)\} = iC + i^2 \int \frac{d^3 p}{(2\pi)^3 2p^0} (p^0 - p \cdot \sigma) \left(e^{-ip(x-y)} + e^{ip(x-y)} \right)$$

$$= iC + i^2 \left(iI \underline{\partial}_{\vec{x}} + i\vec{\nabla} \cdot \vec{\sigma} \right) \Delta(x-y).$$

But $\Delta(x-y) \equiv 0$ for $(x-y)^2 < 0$.

The case $B^{(+, 1/2)}$ is similar!

We need $|C| - |D| = |D| - |D|$. The spinors for this case in PS notation are

$$u_p = \sqrt{2p^0} u(p) = \frac{p^0 + m + p \cdot \sigma}{\sqrt{2(p^0 + m)}}$$

$$v_p = \sqrt{2p^0} v(p) = \frac{-i(p^0 + m + p \cdot \sigma) \sigma_2}{\sqrt{2(p^0 + m)}}.$$

$$\gamma(x) = c \gamma^{(+)}(x) + d \gamma^{(-)}(x).$$

Equal-time anti-commutators: In general

$$\{\xi_s(x), \xi_{s'}(y)\} = c + d + i m \delta_{ss'} \Delta(x-y)$$

This anti-commutator is zero when $(x-y)^2 < 0$ and also at equal times $x^0 = y^0$. So

$$\{\xi_s(x^0, \vec{x}), \xi_{s'}(x^0, \vec{x}')\} = 0.$$

Note that if $m=0$, then the anti-commutator $\{\xi(x), \xi(y)\} = 0$ vanishes identically

even when $x-y$ is time-like. If $|d| > |c|$, then

$$\{\xi_s(x), \xi_{s'}^+(y)\} = |c|^2 (\delta_0 + \nabla \cdot \sigma)_{ss'} \Delta(x-y)$$

which vanishes when $x-y$ is space-like. In general this is

$$\{\xi_s(x), \xi_{s'}^+(y)\} = i |c|^2 (\delta_0 + \nabla \cdot \sigma)_{ss'} \int \frac{d^3 p}{(2\pi)^3 2p^0} (e^{-ip^0 x^0 + i\vec{p} \cdot (x-y) + ip^0 y^0} - e^{ip^0 x^0 + i\vec{p} \cdot (x-y) - ip^0 y^0})$$

where the derivatives are with respect to x (or y) $x-y$.

When $x^0 = y^0$, the time derivative gives

$$|c|^2 \int \frac{d^3 p}{(2\pi)^3 2p^0} p^0 (e^{i\vec{p} \cdot (x-y)} + e^{-i\vec{p} \cdot (x-y)}) = |c|^2 \delta^{(3)}(\vec{x-y}).$$

The space derivatives give at $x^0 = y^0$

$$-iC\gamma^1 \int \frac{d^3 p}{(2\pi)^3 2p^0} [p \cdot \sigma e^{i\vec{p} \cdot (\vec{x} - \vec{y})} + p \cdot \sigma e^{-i\vec{p} \cdot (\vec{x} - \vec{y})}]$$

$= 0$ if we note $\vec{p} \not\propto \vec{p}$ in the second term. So at equal times

$$\{\xi_s(x^0, \vec{x}), \xi_t^+(x^0, \vec{x}')\} = iC\gamma^1 \delta^{(3)}(\vec{x} - \vec{y}) \delta_{st}$$

So we choose $iC\gamma^1 = 1 = Id + I$. The phases don't seem to matter. Then

$$\{\xi_s(t, \vec{x}), \xi_s^+(t, \vec{y})\} = \delta_{ss'} \delta^{(3)}(\vec{x}' - \vec{y}).$$

For the $D^{(0,1/2)}$ representation, u and v are

$$u_p = \frac{p^0 + m + p \cdot \sigma}{\sqrt{2(p^0 + m)}}$$

$$v_p = \frac{-i(p^0 + m + p \cdot \sigma)\sigma_2}{\sqrt{2(p^0 + m)}}$$

But again

$$u_p v_p^\top = i m \sigma_2 \quad \text{and} \quad v_p u_p^\top = -i m \sigma_2$$

So

$$\{\xi_s(x), \xi_t(y)\} = (c\gamma^1) i m \sigma_2 \delta_{st} \Delta(x - y)$$

which vanishes for spacelike $x - y$. Also

$$u_p u_p^\top = p^0 + p \cdot \sigma \quad \text{and} \quad v_p v_p^\top = p^0 + p \cdot \sigma$$

so we set $Id - I = iC\gamma^1 = 1$ and get

$$\{\xi_s(x), \xi_t^+(y)\} = i |C| \gamma^1 (Q_0 - \nabla \cdot \sigma) \Delta(x - y) = i (I \partial_0 - \nabla \cdot \sigma) \Delta(x - y).$$

So at equal times,

$$\{ \gamma_s(t, x), \gamma_{s'}^+(t, x') \} = \delta_{ss'} \delta^{(3)}(\vec{x} - \vec{x}').$$

Just a reminder: here are our spin matrices

$$(1/2, 0)$$

$$u_{pss'} = c + \frac{(p^0 + m - \vec{p} \cdot \vec{\sigma})_{ss'}}{\sqrt{2(p^0 + m)}}$$

$$v_{pss'} = -i d + \frac{[(p^0 + m - \vec{p} \cdot \vec{\sigma}) \sigma_2]_{ss'}}{\sqrt{2(p^0 + m)}}$$

and for

$$(0, 1/2)$$

$$u_{pss'} = c - \frac{(p^0 + m + \vec{p} \cdot \vec{\sigma})_{ss'}}{\sqrt{2(p^0 + m)}}$$

$$v_{pss'} = -i d - \frac{[(p^0 + m + \vec{p} \cdot \vec{\sigma}) \sigma_2]_{ss'}}{\sqrt{2(p^0 + m)}}$$

$$\begin{aligned} \left\{ \xi_s^{(+)}(x), \eta_t^{(-)}(y) \right\} &= \left\{ c + \xi_s^{(+)}(x) + d_+ \xi_s^{(-)}(x), c - \eta_t^{(+)}(y) + d_- \eta_t^{(-)}(y) \right\} \\ &= c + d_- \left\{ \xi_s^{(+)}(x), \eta_t^{(-)}(y) \right\} + c - d_+ \left\{ \xi_s^{(-)}(x), \eta_t^{(+)}(y) \right\} \end{aligned}$$

Now

$$\begin{aligned} \left\{ \xi_s^{(+)}(x), \eta_t^{(-)}(y) \right\} &= \frac{\int d^3 p d^3 q}{(2\pi)^6 / 2 p^0 2 q^0} \left\{ u_{ps,}^{(1/2, 0)} a_{ps} e^{-ipx}, v_{qt,}^{(0, 1/2)} a_{qt} e^{iqy} \right\} \\ &= \int \frac{d^3 p}{(2\pi)^3 2 p^0} u_p^{1/2, 0} v_p^{0, 1/2} e^{-ip(x-y)} \\ &= \int \frac{d^3 p}{(2\pi)^3 2 p^0} \frac{(p^0 + m - p \cdot \sigma)(-i\sigma_2^\top)(p^0 + m + p \cdot \sigma^\top)}{2(p^0 + m)} e^{-ip(x-y)} \\ &= i \int \frac{d^3 p}{(2\pi)^3 2 p^0} \frac{(p^0 + m - p \cdot \sigma)^2 \sigma_2}{2(p^0 + m)} e^{-ip(x-y)} \\ &= i \int \frac{d^3 p}{(2\pi)^3 2 p^0} \frac{(2p_0^2 + 2p^0 m - 2(p^0 + m)p \cdot \sigma)\sigma_2}{2(p^0 + m)} e^{-ip(x-y)} \\ &= i \int \frac{d^3 p}{(2\pi)^3 2 p^0} (p^0 - p \cdot \sigma)\sigma_2 e^{-ip(x-y)} \\ \left\{ \xi_s^{(-)}(x), \eta_t^{(+)}(y) \right\} &= \int \frac{d^3 p d^3 q}{(2\pi)^6 / 2 p^0 2 q^0} \left\{ v_{ps,}^{(1/2, 0)} + u_{ps,}^{(0, 1/2)} a_{ps} e^{-ipx}, v_{qt,}^{(0, 1/2)} a_{qt} e^{iqy} \right\} \\ &= \int \frac{d^3 p}{(2\pi)^3 2 p^0} v_p^{1/2, 0} u_p^{0, 1/2} e^{ip(x-y)} \end{aligned}$$

$$\sqrt{\frac{1}{2} \sigma^0} \mathbf{U}_p^T \mathbf{U}_p^{0\frac{1}{2}} = -i \frac{(\sigma^0 + m - p \cdot \sigma) \sigma_2 (\sigma^0 + m + p \cdot \sigma)^T}{2(\sigma^0 + m)}$$

$$= -i \frac{(\sigma^0 + m - p \cdot \sigma)^2 \sigma_2}{2(\sigma^0 + m)} = -i (\sigma^0 - p \cdot \sigma) \sigma_2$$

So

$$\{ \xi_s(x), \zeta_t(y) \} = i \frac{\frac{d^3 p}{(2\pi)^3 2p^0} [(\sigma^0 - p \cdot \sigma) \sigma_2]_{st}}{(\sigma^0 + m)} (e^{c+d_- - c_- d_+} - e^{-c+d_- - c_- d_+})$$

So if $c_- d_+ = -c_+ d_-$, then

$$\{ \xi_s(x), \zeta_t(y) \} = i c_+ d_- \int \frac{d^3 p}{(2\pi)^3 2p^0} [(\sigma^0 - p \cdot \sigma) \sigma_2]_{st} (e^{i p(x-y)} + e^{-i p(x-y)})$$

$$= i c_+ d_- \left[(i \vec{p} \cdot \vec{\sigma}_0 + i \nabla \cdot \vec{\sigma}) \sigma_2 \right]_{st} \int \frac{d^3 p}{(2\pi)^3 2p^0} e^{i p(x-y)} - e^{-i p(x-y)}$$

$$= -c_+ d_- \left[(\vec{\sigma}_0 + \nabla \cdot \vec{\sigma}) \sigma_2 \right]_{st} \Delta(x-y)$$

which vanishes at space-like separations but is singular when $x = y$.

$$\left\{ \xi_s(x), \zeta_t^+(y) \right\} = \left\{ c + \xi_s^{(+)}(x) + d_+ \xi_s^{(-)}(x), c^* - \zeta_t^{(+)}(y) + d_- \zeta_t^{(-)}(y) \right\}$$

$$= c_+ c_-^* \left\{ \xi_s^{(+)}(x), \zeta_t^{(+)}(y) \right\} + d_+ d_-^* \left\{ \xi_s^{(-)}(x), \zeta_t^{(-)}(y) \right\}$$

$$\left\{ \xi_s^{(+)}(x), \zeta_t^{(+)}(y) \right\} = \frac{\int d^3 p d^3 q}{\sqrt{(2\pi)^6 2p^0 2q^0}} \left\{ u_{pss}^{1/2}, a_{psr} e^{-ipx}, u_{qtt}^{0/2}, a_{qtr}^* e^{iqy} \right\}$$

$$= \frac{\int d^3 p}{(2\pi)^3 2p^0} u_p^{1/2} u_p^{0/2} e^{-ip(x-y)}$$

$$u_p^{1/2} u_p^{0/2} = \frac{(p^0 + m - p \cdot \sigma)(p^0 + m + p \cdot \sigma)}{2(p^0 + m)}$$

$$= \frac{(p^0 + m)^2 - \vec{p}^2}{2(p^0 + m)} = \frac{p^0 - \vec{p}^2 + 2p^0 m + m^2}{2(p^0 + m)} = m$$

So

$$\left\{ \xi_s^{(+)}(x), \zeta_t^{(+)}(y) \right\} = m \int \frac{d^3 p}{(2\pi)^3 2p^0} e^{-ip(x-y)} \quad \text{similarly}$$

$$\left\{ \xi_s^{(-)}(x), \zeta_t^{(-)}(y) \right\} = \int \frac{d^3 p d^3 q}{(2\pi)^6 \sqrt{2p^0 2q^0}} \left\{ v_{pss}^{1/2}, a_{psr}^* e^{ipx}, v_{qtt}^{0/2}, a_{qtr}^* e^{-iqy} \right\}$$

$$= \int \frac{d^3 p}{(2\pi)^3 2p^0} u v_p^{1/2} v_p^{0/2} e^{ip(x-y)}$$

$$v_p^{1/2} v_p^{0/2} = -i \frac{(p^0 + m - p \cdot \sigma) \sigma_2 + \sigma_2 (p^0 + m + p \cdot \sigma)}{2(p^0 + m)}$$

$$= m \quad \text{So} \quad \left[c + c_-^* e^{-ip(x-y)} + d + d_-^* e^{ip(x-y)} \right]$$

$$\left\{ \xi_s(x), \zeta_t^+(y) \right\} = m \int \frac{d^3 p}{(2\pi)^3 2p^0} \left[c + c_-^* e^{-ip(x-y)} + d + d_-^* e^{ip(x-y)} \right]$$

$$= c_+ c_-^* m \Delta(x-y) \quad \text{if } d + d_-^* = -c_+ c_-^*$$

So our conditions are

$$|C+|=|d_+|=1$$

$$c_- d_+ = -c_+ d_-$$

$$|C-|=|d_-|=1$$

$$d_+ d_-^* = -c_+ c_-^*$$

$$\text{So } i\theta$$

$$i\phi$$

$$i\psi$$

$$i\omega$$

$$c_+ = e^{i\theta} \quad c_- = e^{i\phi}$$

$$d_+ = e^{i\psi} \quad d_- = e^{i\omega}$$

$$e^{i(\phi+\psi)} = -e^{i(\theta+\omega)}$$

$$\text{So } \phi + \psi = \theta + \omega \pm \pi$$

$$e^{i(4-\omega)} = -e^{i(\theta-\phi)}$$

$$\text{So } 4 - \omega = \theta - \phi \pm \pi$$

$$\phi + \psi = \theta + \omega \pm \pi$$

So set $\psi = \pi$, all other θ , or $\omega = \pi$ all other θ .

$$\text{So } c_+ = c_- = 1$$

$$d_+ = -d_- = 1$$

So

$$u(0, \frac{1}{2}) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$u(0, -\frac{1}{2}) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$$

$$v(0, \frac{1}{2}) = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}$$

$$v(0, -\frac{1}{2}) = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

The other three phases come from the overall phases for $\Xi(x)$ and $\bar{\Xi}(x)$ and the phase of α_s .

Now that we've fixed the details of ξ and $\bar{\xi}$, we can pick some low energy twist. We can put ξ and $\bar{\xi}$ into a 4-component spinor with the 4-spinors of p. 30.

$$\chi = \begin{pmatrix} \xi \\ \bar{\xi} \end{pmatrix} = \int \frac{d^3 p}{(2\pi)^3} \sqrt{2p_0} \left[u_{ps} a_{ps} e^{-ipr} + v_{ps} \bar{a}_{ps} c^{ipx} \right]$$

Note that these u and v are given by

$$u_{ps} = \frac{1}{\sqrt{2(p^0+m)}} \begin{pmatrix} p^0+m - \vec{p} \cdot \vec{\sigma} \\ p^0+m + \vec{p} \cdot \vec{\sigma} \end{pmatrix}$$

$$v_{ps} = \frac{-i}{\sqrt{2(p^0+m)}} \begin{pmatrix} (p^0+m-p \cdot \sigma) \sigma_2 \\ -(p^0+m+p \cdot \sigma) \sigma_2 \end{pmatrix} \text{ or by}$$

$$u_p = \frac{1}{\sqrt{2(p^0+m)}} \begin{pmatrix} p^0+m-p \cdot \sigma & 0 \\ 0 & p^0+m+p \cdot \sigma \end{pmatrix} \begin{pmatrix} I \\ I \end{pmatrix}$$

$$v_p = \frac{-i}{\sqrt{2(p^0+m)}} \begin{pmatrix} p^0+m-p \cdot \sigma & 0 \\ 0 & p^0+m+p \cdot \sigma \end{pmatrix} \begin{pmatrix} \sigma_2 \\ -\sigma_2 \end{pmatrix}$$

that is

$$u = \frac{1}{\sqrt{2(p^0+m)}} (m + p \cdot \gamma^0) \begin{pmatrix} I \\ I \end{pmatrix}$$

$$v = \frac{-i}{\sqrt{2(p^0+m)}} (m + p \cdot \gamma^0) \begin{pmatrix} \sigma_2 \\ -\sigma_2 \end{pmatrix}.$$

But $\begin{pmatrix} \xi \\ \bar{\xi} \end{pmatrix}$ is an eigenvector of γ^0
with eigenvalue +1

$$\gamma^0 \begin{pmatrix} \xi \\ \bar{\xi} \end{pmatrix} = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \begin{pmatrix} \xi \\ \bar{\xi} \end{pmatrix} = \begin{pmatrix} \xi \\ \bar{\xi} \end{pmatrix}$$

so $u = \frac{1}{\sqrt{2(p^0+m)}} (m+\not{p}) \begin{pmatrix} \xi \\ \bar{\xi} \end{pmatrix}$.

And $\begin{pmatrix} \sigma_2 \\ -\sigma_2 \end{pmatrix}$ is an e.v. of γ^0 with e.v -1:

$$\gamma^0 \begin{pmatrix} \sigma_2 \\ -\sigma_2 \end{pmatrix} = - \begin{pmatrix} \sigma_2 \\ -\sigma_2 \end{pmatrix} \quad \text{so}$$

$$v = \frac{1}{\sqrt{2(p^0+m)}} (m-\not{p}) \begin{pmatrix} \sigma_2 \\ -\sigma_2 \end{pmatrix}.$$

But this means that our Majorana
4-spinor χ is

$$\chi(x) = \begin{pmatrix} \xi(x) \\ \bar{\xi}(x) \end{pmatrix} = \int \frac{d^3 p}{(2\pi)^3 \sqrt{p^0(p^0+m)}} \left[(m+\not{p}) \begin{pmatrix} \xi \\ \bar{\xi} \end{pmatrix} a_p e^{-ipx} + (m-\not{p}) \begin{pmatrix} \sigma_2 \\ -\sigma_2 \end{pmatrix} a_p^+ e^{ipx} \right]$$

$$\text{But } \not{p} e^{-ipx} = i \not{e} \quad \text{and} \quad \not{p} e^{ipx} = i \not{e}.$$

So $\chi(x) = (m + i\gamma) \phi(x)$, that is,

$$\chi(x) = (m + i\gamma) \int \frac{d^3 p}{(2\pi)^3 2\sqrt{(p^0 + m)p^1}} \left[\begin{pmatrix} I \\ I \end{pmatrix} a_p e^{-ip^k} + \begin{pmatrix} 0_2 \\ -0_2 \end{pmatrix} e a_p^\dagger \right]$$

But this means that $\chi(x)$ satisfies the Dirac equation

$$(i\gamma - m)\chi(x) = (i\gamma - m)(i\gamma + m)\phi(x)$$

$$= -(\gamma^2 + m^2)\phi(x)$$

$$= -(\partial_a \partial_b \gamma^a \gamma^b + m^2)\phi(x) \quad \text{(since } \partial_a \partial_b = \partial_b \partial_a \text{)}$$

$$= -(\partial_a \partial_b \tfrac{1}{2}\{\gamma^a \gamma^b\} + m^2)\phi(x)$$

$$= -(\partial_a \partial_b \eta^{ab} + m^2)\phi(x)$$

$$= -(\partial_0^2 - \vec{\nabla}^2 + m^2)\phi(x)$$

$$= -(\square + m^2)\phi(x) = 0$$

since by construction $(\square + m^2)\phi(x) = 0$

because $(\square + m^2)e^{\frac{\pm ipx}{m}} = 0$.

The other piece of low hanging fruit is that with these phase conventions (pp. 31)

$$\xi(x) = \int \frac{d^3 p}{(2\pi)^3 2\sqrt{p^0(p^0+m)}} \left[(p^0 + m - p \cdot \sigma) a_p e^{-ipx} - i(p^0 + m - p \cdot \sigma) \sigma_2 a_p^+ e^{ipx} \right]$$

and

$$\eta(x) = \int \frac{d^3 p}{(2\pi)^3 2\sqrt{p^0(p^0+m)}} \left[(p^0 + m + p \cdot \sigma) a_p e^{-ipx} + i(p^0 + m + p \cdot \sigma) \sigma_2 a_p^+ e^{ipx} \right].$$

In fact, $\eta(x) = i\sigma_2 \xi^*(x)$ (ie $\eta_s(x) = i\sigma_{2s} \xi_s^*(x)$)

Fn

$$i\sigma_2 \xi^*(x) = \int \frac{d^3 p}{(2\pi)^3 2\sqrt{p^0(p^0+m)}} \left[i\sigma_2 (p^0 + m - p \cdot \sigma^*) a_p^* e^{ipx} + \sigma_2 (p^0 + m - p \cdot \sigma^*) \sigma_2 a_p e^{-ipx} \right]$$

$$= \int \frac{d^3 p}{(2\pi)^3 2\sqrt{p^0(p^0+m)}} \left[i(p^0 + m + p \cdot \sigma) \sigma_2 a_p^* e^{ipx} + (p^0 + m + p \cdot \sigma) a_p e^{-ipx} \right]$$

$$= \eta(x).$$

So the right-handed spinor $\eta = i\sigma_2 \xi^*$

And so $\xi = -i\sigma_2 \eta^*$.

Now suppose there are two kinds of particle with the same mass. Then we can make two 4-spinors χ_1 and χ_2 . Since they are made from different a 's and a^\dagger 's, they anti-commute with each other

$$\{\chi_1(x), \chi_2(y)\} = 0 \quad \{\chi_1(x), \chi_2^+(y)\} = 0.$$

So the Dirac spinor

$$\psi(x) = \frac{1}{\sqrt{2}} (\chi_1(x) + i \chi_2(x))$$

satisfies the equal-time a.c. relations

$$\{\psi_s(x), \psi_{s'}(y)\} = 0 \text{ at equal times and}$$

$$\{\psi_s(x), \psi_{s'}^+(y)\} = \delta_{ss'} \delta^3(\vec{x} - \vec{y}) \text{ at equal times.}$$

More generally

$$\{\chi_{ia}(x), \chi_{jb}(y)\} = i \delta_{ij} (m + i \gamma)_{ab} \Delta(x-y)$$

and

$$\{\bar{\chi}_{ia}(x), \bar{\chi}_{jb}(y)\} = \delta_{ij} (m + i \gamma)_{ab} \Delta(x-y),$$

where $\bar{\chi}_{jb} = \chi_{j_b}^\dagger \gamma^0 b^\dagger = \chi^\dagger \gamma^0$.

so the Dirac field is

$$\Psi(x) = \int \frac{d^3 p}{(2\pi)^3 \sqrt{2p^0}} [u_{ps} a_{ps} e^{-ipx} + v_{ps} b_{ps}^\dagger e^{ipx}]$$

$$\text{where } a_{ps} = \frac{1}{\sqrt{2}} (a_{1ps} + i a_{2ps})$$

$$\text{and } b_{ps}^\dagger = \frac{1}{\sqrt{2}} (a_{1ps}^\dagger + i a_{2ps}^\dagger).$$

In 2-component notation

$$\xi = \frac{1}{\sqrt{2}} (\xi_1 + i \xi_2)$$

$$\zeta = \frac{1}{\sqrt{2}} (\zeta_1 + i \zeta_2)$$

$$= \frac{1}{\sqrt{2}} (i \sigma_2 \xi_1^* + i \sigma_2 \xi_2^*)$$

$$= i \sigma_2 \xi^*$$

$$\text{and } \xi = -i \sigma_2 \zeta^*.$$

The spinors $u(p, 1/2)$ $u(p, -1/2)$ naturally form the 4×2 matrix

$$u(p) = \frac{1}{\sqrt{2(p^0+m)}} \begin{pmatrix} p^0+m-p \cdot \sigma \\ p^0+m+p \cdot \sigma \end{pmatrix}.$$

In terms of this 4×2 matrix,

$$u u^\dagger = \begin{pmatrix} p^0 + m - p \cdot \sigma \\ p^0 + m + p \cdot \sigma \end{pmatrix} \begin{pmatrix} p^0 + m - p \cdot \sigma & p^0 + m + p \cdot \sigma \end{pmatrix} / (2(p^0 + m))$$

$$= \begin{pmatrix} (p^0 + m - p \cdot \sigma)^2 & (p^0 + m)^2 - \vec{p}^2 \\ (p^0 + m)^2 - \vec{p}^2 & (p^0 + m + p \cdot \sigma)^2 \end{pmatrix} / 2(p^0 + m)$$

$$= \frac{1}{2(p^0 + m)} \begin{pmatrix} p^0 + m^2 + 2p^0 m + \vec{p}^2 - 2(p^0 + m)p \cdot \sigma & 2m^2 + 2mp^0 \\ 2m^2 + 2mp^0 & 2p^0 + 2mp^0 + 2(p^0 + m)p \cdot \sigma \end{pmatrix}$$

$$= \frac{2(m + p^0)}{2(m + p^0)} \begin{pmatrix} p^0 - p \cdot \sigma & m \\ m & p^0 + p \cdot \sigma \end{pmatrix}$$

So

$$u(p) u^\dagger(p) = \begin{pmatrix} p^0 - p \cdot \sigma & m \\ m & p^0 + p \cdot \sigma \end{pmatrix}.$$

This is called a spin sum.

$$\text{Similarly, } V(p) V^+(p) = \frac{1}{2(p^0 + m)}$$

$$X \left(\begin{array}{l} (p^0 + m - p \cdot \alpha) \sigma_2 \\ -(p^0 + m + p \cdot \alpha) \bar{\sigma}_2 \end{array} \right) \left(\sigma_2 (p^0 + m - p \cdot \bar{\alpha}) - \bar{\sigma}_2 (p^0 + m + p \cdot \alpha) \right)$$

$$= \frac{1}{2(p^0 + m)} \left(\begin{array}{l} ((p^0 + m - p \cdot \alpha)^2 - (p^0 + m)^2 + \vec{p}^2) \\ -(p^0 + m)^2 + \vec{p}^2 \end{array} \right) (p^0 + m + p \cdot \alpha)^2$$

$$= \frac{2(m + p^0)}{2(m + p^0)} \left(\begin{array}{cc} p^0 - p \cdot \bar{\alpha} & -m \\ -m & p^0 + p \cdot \alpha \end{array} \right)$$

So

$$V(p) V^+(p) = \left(\begin{array}{cc} p^0 - p \cdot \bar{\alpha} & -m \\ -m & p^0 + p \cdot \alpha \end{array} \right)$$

These often are written as

$$u \bar{u} = u u^+ \gamma^0 = \begin{pmatrix} p^0 - p \cdot \bar{\alpha} & m \\ m & p^0 + p \cdot \alpha \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= m + \not{p} \quad \text{and as}$$

$$v \bar{v} = v v^+ \gamma^0 = \not{p} - m.$$