10

Non-Perturbative Methods

We are now going to begin our study of higher-order contributions to physical processes, corresponding to Feynman diagrams involving one or more loops. It will be very useful in this work to have available a method of deriving results valid to all orders in perturbation theory (and in some cases beyond perturbation theory). In this chapter we will exploit the field equations and commutation relations of the interacting fields in the Heisenberg picture for this purpose. The essential bridge between the Heisenberg picture and the Feynman diagrams of perturbation theory is provided by the theorem proved in Section 6.4: the sum of all diagrams for a process $\alpha \to \beta$ with extra vertices inserted corresponding to operators $o_a(x)$, $o_b(y)$, etc. is given by the matrix element of the time-ordered product of the corresponding Heisenberg-picture operators

$$\left(\Psi_\beta{}^-, T\left\{-iO_a(x), -iO_b(y)\cdots\right\}\Psi_\alpha{}^+\right) .$$

As a special case, where the operators $O_a(x)$, $O_b(x)$, etc. are elementary particle fields, this matrix element equals the sum of all Feynman diagrams with incoming lines on the mass shell corresponding to the state α, outgoing lines on the mass shell corresponding to the state β, and lines off the mass shell (including propagators) corresponding to the operators $O_a(x)$, $O_b(x)$, etc. After exploring some of the non-perturbative results that can be obtained in this way we will be in a good position to take up the perturbative calculation of radiative corrections.

10.1 Symmetries

One obvious but important use of the theorem quoted above is to extend the application of symmetry principles from S-matrix elements, where all external lines have four-momenta on the mass shell, to parts of Feynman diagrams, with some or all external lines off the mass shell.

For instance, consider the symmetry of spacetime translational invariance. This symmetry has as a consequence the existence of a Hermitian

four-vector operator P^μ, with the property that, for any local function $O(x)$ of field operators and their canonical conjugates,

$$\left[P_\mu, O(x)\right] = i\frac{\partial}{\partial x^\mu}O(x)\,. \tag{10.1.1}$$

(See Eqs. (7.3.28) and (7.3.29).) Also, the states α and β are usually chosen to be eigenstates of the four-momentum:

$$P^\mu\Psi_\alpha{}^+ = p_\alpha^\mu\Psi_\alpha{}^+\,, \qquad P^\mu\Psi_\beta{}^- = p_\beta^\mu\Psi_\beta{}^-\,. \tag{10.1.2}$$

It follows that for any set of local functions $O_a(x)$, $O_b(x)$, etc. of fields and/or field derivatives

$$\begin{aligned}(p_{\beta\mu} - p_{\alpha\mu})&\left(\Psi_\beta{}^-, T\left\{O_a(x_1), O_b(x_2)\cdots\right\}\Psi_\alpha{}^+\right)\\ &= \left(\Psi_\beta{}^-, \left[P_\mu, T\left\{O_a(x_1), O_b(x_2)\cdots\right\}\right]\Psi_\alpha{}^+\right)\\ &= i\left(\frac{\partial}{\partial x_1^\mu} + \frac{\partial}{\partial x_2^\mu} + \cdots\right)\left(\Psi_\beta{}^- T\left\{O_a(x_1), O_b(x_2), \cdots\right\}\Psi_\alpha{}^+\right)\,.\end{aligned} \tag{10.1.3}$$

This has the solution

$$\begin{aligned}&\left(\Psi_\beta{}^-, T\left\{O_a(x_1), O_b(x_2), \cdots\right\}\Psi_\alpha{}^+\right)\\ &= \exp\left(i(p_\alpha - p_\beta)\cdot x\right) F_{ab\cdots}(x_1 - x_2, \cdots)\,,\end{aligned} \tag{10.1.4}$$

where x is any sort of average spacetime coordinate

$$x^\mu = c_1 x_1^\mu + c_2 x_2^\mu + \cdots\,, \qquad c_1 + c_2 + \cdots = 1 \tag{10.1.5}$$

and F depends only on differences among the xs. (In particular, a vacuum expectation value can depend only on the coordinate differences.) We can Fourier transform Eq. (10.1.4) by integrating separately over x^μ and the coordinate differences, with the result that

$$\begin{aligned}&\int d^4x_1\, d^4x_2 \cdots \left(\Psi_\beta{}^-, T\left\{O_a(x_1), O_b(x_2), \cdots\right\}\Psi_\alpha{}^+\right)\\ &\times \exp(-ik_1\cdot x_1 - ik_2\cdot x_2 - \cdots) \propto \delta^4(p_\alpha - p_\beta - k_1 - k_2 - \cdots)\,.\end{aligned} \tag{10.1.6}$$

We saw in Section 6.4 that the matrix element of the time-ordered product is given by applying the usual coordinate-space Feynman rules to the sum of all graphs with incoming particles corresponding to particles in α, outgoing particles in β, and external lines that simply terminate in vertices at $x_1, x_2, \cdots$. The Fourier transform (10.1.6) is correspondingly given by applying the momentum-space Feynman rules to the same sum of Feynman diagrams, with off-shell external lines carrying four-momenta $k_1, k_2, \cdots$ into the diagrams. Eq. (10.1.6) is then just the statement that this sum of Feynman graphs conserves four-momentum. The result is obvious in perturbation theory, because four-momentum is conserved at

every vertex, so it is not surprising to see the same result emerging without having to rely on perturbation theory.

With somewhat more effort, one can use the Lorentz transformation properties of the Heisenberg-picture fields and the 'in' and 'out' states to show that the sum of all graphs with a given set of on- and off-shell lines satisfies the same Lorentz transformation conditions as the lowest-order terms.

Similar arguments apply to the conservation of internal quantum numbers, like electric charge. As shown in Section 7.3, a field or other operator $O_a(x)$ that destroys a charge q_a (or creates a charge $-q_a$) will satisfy

$$[Q, O_a(x)] = -q_a O_a(x)$$

in the Heisenberg and interaction pictures alike. Also, if the free-particle states α and β have charges q_α and q_β, then so do the corresponding 'in' and 'out' states. We then have

$$\begin{aligned}
&(q_\beta - q_\alpha)\left(\Psi_\beta{}^-, T\left\{O_a(x), O_b(y), \cdots\right\}\Psi_\alpha{}^+\right) \\
&= \left(\Psi_\beta{}^-, \left[Q, T\left\{O_a(x), O_b(y), \cdots\right\}\right]\Psi_\alpha{}^+\right) \\
&= -(q_a + q_b + \cdots)\left(\Psi_\beta{}^-, T\left\{O_a(x), O_b(y), \cdots\right\}\Psi_\alpha{}^+\right) .
\end{aligned}$$

Thus the amplitude $\left(\Psi_\beta{}^-, T\left\{O_a(x), O_b(y), \cdots\right\}\Psi_\alpha{}^+\right)$ vanishes unless charge is conserved

$$q_\beta = q_\alpha - q_a - q_b - \cdots . \tag{10.1.7}$$

A somewhat less trivial example is provided by the symmetry of charge-conjugation invariance. As we saw in Chapter 5, there is an operator C that interchanges electron and positron operators

$$\begin{aligned}
\mathsf{C}\, a(\mathbf{p}, \sigma, e^-)\mathsf{C}^{-1} &= \xi^*\, a(\mathbf{p}, \sigma, e^+) , \\
\mathsf{C}\, a(\mathbf{p}, \sigma, e^+)\mathsf{C}^{-1} &= \xi\, a(\mathbf{p}, \sigma, e^-) ,
\end{aligned}$$

with ξ a phase factor. For the free electron field $\psi(x)$, this gives

$$\mathsf{C}\psi(x)\,\mathsf{C}^{-1} = -\xi^*\beta\mathscr{C}\psi(x)^* ,$$

where βC is a 4×4 matrix, which (for the Dirac matrix representation we have been using, with γ_5 diagonal) takes the form

$$\beta\, C = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} .$$

Applied to the free-particle electric current in spinor electrodynamics, this

gives

$$\mathsf{C}(\bar{\psi}\gamma^\mu\psi)\mathsf{C}^{-1} = -\bar{\psi}C\gamma^{\mu T}C\psi = -\bar{\psi}\gamma^\mu\psi\ .$$

If C is to be conserved in electrodynamics, it must then also be defined to anticommute with the free photon field

$$\mathsf{C}(a^\mu)\mathsf{C}^{-1} = -a^\mu\ .$$

In theories like electrodynamics for which C commutes with the interaction as well as H_0, it also commutes with the similarity transformation $\Omega(t)$ between the Heisenberg and interaction pictures, and so it anticommutes with the electric current of the interacting fields

$$\mathsf{C}(\bar{\Psi}\gamma^\mu\Psi)\mathsf{C}^{-1} = -\bar{\Psi}\gamma^\mu\Psi \tag{10.1.8}$$

and the electromagnetic field in the Heisenberg-picture

$$\mathsf{C}(A^\mu)\mathsf{C}^{-1} = -A^\mu\ . \tag{10.1.9}$$

It follows then that the vacuum expectation value of the time-ordered product of any odd number of electromagnetic currents and/or fields vanishes. Therefore the sum of all Feynman graphs with an odd number of external photon lines (off or on the photon mass shell) and no other external lines vanishes.

This result is known as *Furry's theorem.*[1] It can be proved perturbatively by noting that a graph consisting of electron loops ℓ, to each of which are attached n_ℓ photon lines, must have numbers I and E of internal and external photon lines related by an analog of Eq. (6.3.11):

$$2I + E = \sum_\ell n_\ell\ .$$

Hence if E is odd at least one of the loops must have attached an odd number of photon lines. For any such loop there is a cancellation between the two diagrams in which the electron arrows circulate around the loop in opposite directions. Hence Furry's theorem is a somewhat less trivial consequence of a symmetry principle than translation or Lorentz invariance; it is not true of individual diagrams, but rather of certain sums of diagrams. Figure 10.1 illustrates the application of Furry's theorem that was historically most important, its use to show that the scattering of a photon by an external electromagnetic field receives no contributions of first order (or any odd order) in the external field.

10.2 Polology

One of the most important uses of the non-perturbative methods described in this chapter is to clarify the pole structure of Feynman amplitudes as

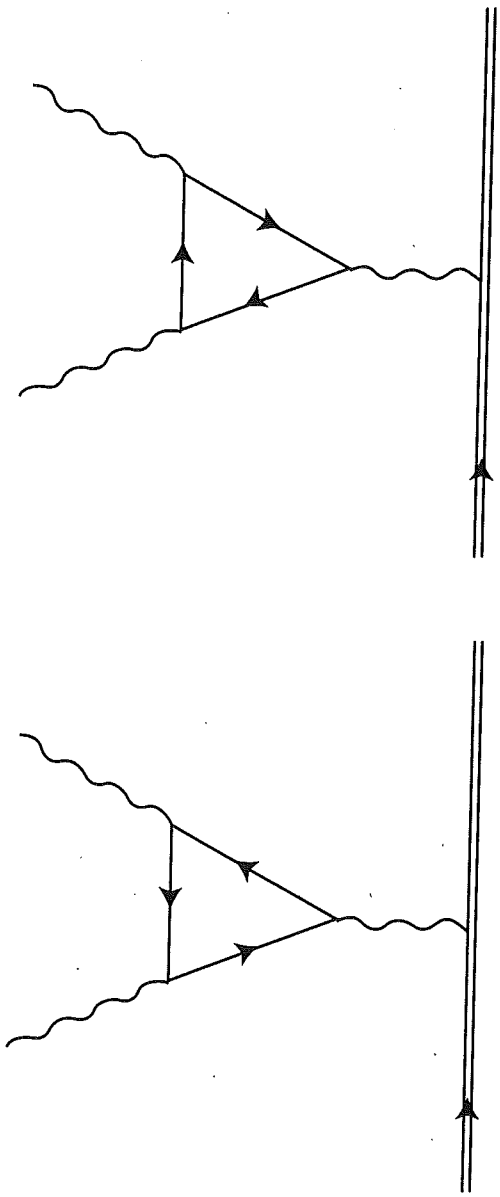

Figure 10.1. The lowest-order diagrams for the scattering of a photon by an electromagnetic field. Here straight lines represent virtual electrons; wavy lines represent real and virtual photons; and the double line represents a heavy particle like an atomic nucleus that serves as a source of an electromagnetic field. The contributions of these two diagrams cancel, as required by charge-conjugation invariance.

functions of the momenta carried by external lines. Often the S-matrix for a physical process can be well approximated by the contribution of a single pole. Also, an understanding of this pole structure will help us later in dealing with radiative corrections to particle propagators.

Consider the momentum-space amplitude

$$\int d^4x_1 \cdots d^4x_n \, e^{-iq_1\cdot x_1} \cdots e^{-iq_n\cdot x_n} \left\langle T\left\{A_1(x_1)\cdots A_n(x_n)\right\}\right\rangle_0 \equiv G(q_1 \cdots q_n) \,. \tag{10.2.1}$$

The As are Heisenberg-picture operators of arbitrary Lorentz type, and $\langle\cdots\rangle_0$ denotes the expectation value in the true vacuum $\Psi_0{}^+ = \Psi_0{}^- \equiv \Psi_0$. As discussed in Section 6.4, if $A_1, \cdots A_n$ are ordinary fields appearing in the Lagrangian, then (10.2.1) is a sum of the terms calculated using the ordinary Feynman rules, for all graphs with external lines corresponding to the fields $A_1, \cdots A_n$, carrying off-shell four-momenta $q_1 \cdots q_n$ into the graph. However, we will not be limited to this case; the A_i may be arbitrary local functions of fields and field derivatives.

We are interested in poles of G at certain values of the invariant squares of the total four-momenta carried by various subsets of the external lines. To be definite, let's consider G as a function of q^2, where

$$q \equiv q_1 + \cdots + q_r = -q_{r+1} - \cdots - q_n \tag{10.2.2}$$

with $1 \leq r \leq n-1$. We will show that G has a pole at $q^2 = -m^2$, where m is the mass of any one-particle state that has non-vanishing matrix elements with the states $A_1^\dagger \cdots A_r^\dagger \Psi_0$ and $A_{r+1} \cdots A_n \Psi_0$, and that the residue at this pole is given by

$$\begin{aligned} G \to \frac{-2i\sqrt{\mathbf{q}^2 + m^2}}{q^2 + m^2 - i\epsilon} (2\pi)^7 \delta^4(q_1 + \cdots + q_n) \\ \times \sum_\sigma M_{0|\mathbf{q},\sigma}(q_2 \cdots q_r) M_{\mathbf{q},\sigma|0}(q_{r+2} \cdots q_n) \end{aligned} \tag{10.2.3}$$

where the Ms are defined by*

$$\begin{aligned} &\int d^4x_1 \cdots d^4x_r \, e^{-iq_1\cdot x_1} \cdots e^{-iq_r\cdot x_r} \left(\Psi_0, T\left\{A_1(x_1) \cdots A_r(x_r)\right\}\Psi_{\mathbf{p},\sigma}\right) \\ &= (2\pi)^4 \, \delta^4(q_1 + \cdots + q_r - p) M_{0|\mathbf{p},\sigma}(q_2 \cdots q_r) \,, \end{aligned} \tag{10.2.4}$$

$$\begin{aligned} &\int d^4x_{r+1} \cdots d^4x_n \, e^{-iq_{r+1}\cdot x_{r+1}} \cdots e^{-iq_n\cdot x_n} \\ &\qquad \times \left(\Psi_{\mathbf{p},\sigma}, T\left\{A_{r+1}(x_{r+1}) \cdots A_n(x_n)\right\}\Psi_0\right) \\ &= (2\pi)^4 \, \delta^4(q_{r+1} + \cdots + q_n + p) M_{\mathbf{p},\sigma|0}(q_{r+2} \cdots q_n) \end{aligned} \tag{10.2.5}$$

(with $p^0 \equiv \sqrt{\mathbf{p}^2 + m^2}$), and the sum is over all spin (or other) states of the particle of mass m.

Before proceeding to the proof, it will help to clarify the significance of

* Recall that in the absence of time-varying external fields, there is no distinction between 'in' and 'out' one-particle states, so that $\Psi_{\mathbf{p},\sigma}{}^+ = \Psi_{\mathbf{p},\sigma}{}^- = \Psi_{\mathbf{p},\sigma}$.

(10.2.3) if we write it in the somewhat long-winded form

$$
\begin{aligned}
G(q_1 \cdots q_n) \to \sum_\sigma \int d^4k \\
\times \left[(2\pi)^4\, \delta^4(q_1 + \cdots + q_r - k)\,(2\pi)^{3/2} \left(2\sqrt{\mathbf{k}^2 + m^2}\right)^{1/2} M_{0|\mathbf{k},\sigma}(q_2 \cdots q_r)\right] \\
\times \left[\frac{-i}{(2\pi)^4}\, \frac{1}{k^2 + m^2 - i\epsilon}\right] \\
\times \left[(2\pi)^4\, \delta^4(k + q_{r+1} + \cdots + q_n)\,(2\pi)^{3/2}\right. \\
\left. \times \left(2\sqrt{\mathbf{k}^2 + m^2}\right)^{1/2} M_{\mathbf{k},\sigma|0}(q_{r+2} \cdots q_n)\right] .
\end{aligned}
\tag{10.2.6}
$$

This is just what we should expect from a Feynman diagram with a single internal line for a particle of mass m connecting the first r and the last $n-r$ external lines.** However, it is *not* necessary that the particle of mass m correspond to a field that appears in the Lagrangian of the theory. Eqs. (10.2.3) and (10.2.6) apply even if this particle is a bound state of the so-called elementary particles whose fields do appear in the Lagrangian. In this case, the pole arises not from single Feynman diagrams, like Figure 10.2, but rather from infinite sums of diagrams, such as the one shown in Figure 10.3. This is the first place where the methods of this chapter take us beyond results that could be derived as properties of each order of perturbation theory.

Now to the proof. Among the $n!$ possible orderings of the times $x_1^0 \cdots x_n^0$ in Eq. (10.2.1), there are $n!/r!(n-r)!$ for which the first r of the x_i^0 are *all* larger than the last $n-r$. Isolating the contribution of this part of the volume of integration in Eq. (10.2.1), we have

$$
\begin{aligned}
G(q_1 \cdots q_n) = \int d^4x_1 \cdots d^4x_n\, e^{-iq_1\cdot x_1} \cdots e^{-iq_n\cdot x_n} \\
\times \theta\left(\min\,[x_1^0 \cdots x_r^0] - \max\,[x_{r+1}^0 \cdots x_n^0]\right) \\
\times \left(\Psi_0,\ T\left\{A_1(x_1) \cdots A_r(x_r)\right\} T\left\{A_{r+1}(x_{r+1}) \cdots A_n(x_n)\right\}\Psi_0\right) \\
+ \text{OT},
\end{aligned}
\tag{10.2.7}
$$

where 'OT' denotes the other terms arising from different time-orderings. We can evaluate the matrix element here by inserting a complete set of

** See Figure 10.2. The factors $(2\pi)^{3/2}\left[2\sqrt{\mathbf{k}^2 + m^2}\right]^{1/2}$ just serve to remove kinematic factors associated with the mass m external line in $M_{0|\mathbf{k},\sigma}$ and $M_{\mathbf{k},\sigma|0}$. Also, the sum over σ of the product of coefficient-function factors from these two matrix elements yields the numerator of the propagator associated with the internal line in Figure 10.2.

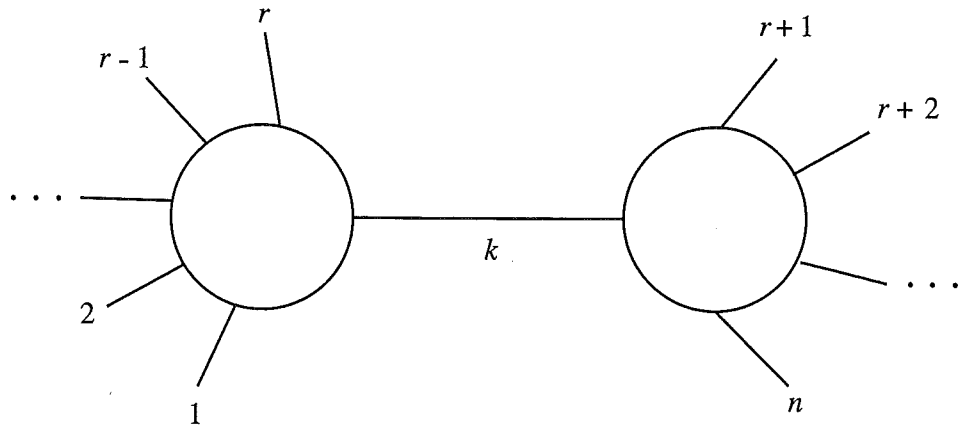

Figure 10.2. A Feynman diagram with the pole structure (10.2.6). Here the line carrying a momentum k represents an elementary particle, one whose field appears in the Lagrangian.

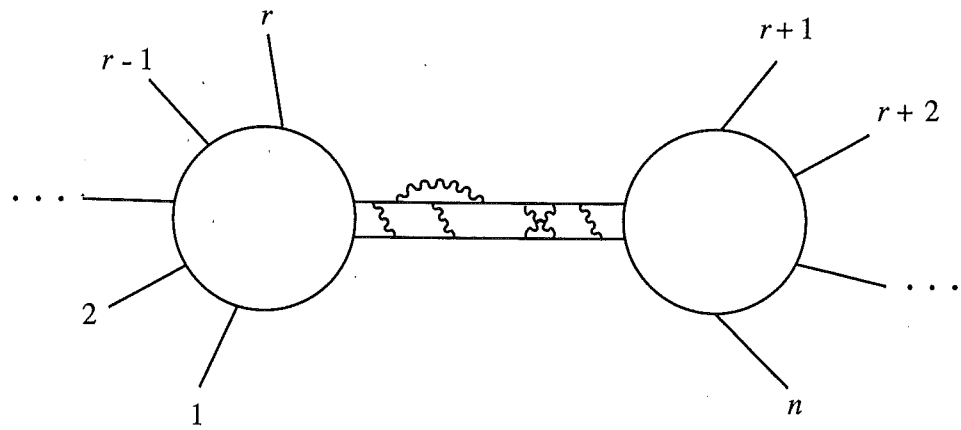

Figure 10.3. Figure 10.3. A Feynman diagram of the class whose sum has the pole structure (10.2.6). Here the pole is due to a composite particle, a bound state of two elementary particles. The elementary particles are represented by straight lines, and interact by the exchange of particles represented by wavy lines.

intermediate states between time-ordered products. Among these may be the single-particle state $\Psi_{\mathbf{p},\sigma}$ of a definite species of mass m. Further isolating the contribution of these one-particle intermediate states, we have

$$G(q_1 \cdots q_n) = \int d^4x_1 \cdots d^4x_n \, e^{-iq_1\cdot x_1} \cdots e^{-iq_n\cdot x_n}$$
$$\theta\left(\min\left[x_1^0 \cdots x_r^0\right] - \max\left[x_{r+1}^0 \cdots x_n^0\right]\right) \sum_\sigma \int d^3p$$
$$\left(\Psi_0,\, T\left\{A_1(x_1)\cdots A_r(x_r)\right\}\Psi_{\mathbf{p},\sigma}\right)\left(\Psi_{\mathbf{p},\sigma},\, T\left\{A_{r+1}(x_{r+1})\cdots A_n(x_n)\right\}\Psi_0\right)$$
$$+ \text{OT}, \tag{10.2.8}$$

where 'OT' now denotes other terms, here arising not only from other time-orderings, but also from other intermediate states. It will be convenient

to shift variables of integration, so that

$$x_i = x_1 + y_i\,, \qquad i = 2, 3, \cdots r\,,$$
$$x_i = x_{r+1} + y_i\,, \qquad i = r+2, \cdots n\,,$$

and use the results of the previous section to write

$$\begin{aligned}&\left(\Psi_0,\, T\left\{A_1(x_1)\cdots A_r(x_r)\right\}\Psi_{\mathbf{p},\sigma}\right)\\ &\quad = e^{ip\cdot x_1}\left(\Psi_0,\, T\left\{A_1(0)A_2(y_2)\cdots A_r(y_r)\right\}\Psi_{\mathbf{p},\sigma}\right)\,,\end{aligned} \tag{10.2.9}$$

$$\begin{aligned}&\left(\Psi_{\mathbf{p},\sigma},\, T\left\{A_{r+1}(x_{r+1})\cdots A_n(x_n)\right\}\Psi_0\right)\\ &\quad = e^{-ip\cdot x_{r+1}}\left(\Psi_{\mathbf{p},\sigma},\, T\left\{A_{r+1}(0)\cdots A_n(y_n)\right\}\Psi_0\right)\,.\end{aligned} \tag{10.2.10}$$

Also, the argument of the theta function becomes

$$\begin{aligned}&\min\,[x_1^0\cdots x_r^0] - \max\,[x_{r+1}^0\cdots x_n^0]\\ &\quad = x_1^0 - x_{r+1}^0 + \min\,[0\, y_2^0\cdots y_r^0] - \max\,[0\, y_{r+1}^0\cdots y_n^0]\,.\end{aligned}$$

We also insert the Fourier representation (6.2.15) of the step function

$$\theta(\tau) = -\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{d\omega\, e^{-i\omega\tau}}{\omega + i\epsilon}\,.$$

The integrals over x_1 and x_{r+1} now just yield delta functions:

$$\begin{aligned}G(q_1\cdots q_n) &= \int d^4y_2\cdots d^4y_r\, d^4y_{r+2}\cdots d^4y_n\\ &\times e^{-iq_2\cdot y_2}\cdots e^{-iq_r\cdot y_r}e^{-iq_{r+2}\cdot y_{r+2}}\cdots e^{-iq_n\cdot y_n}\\ &\times -\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{d\omega}{\omega+i\epsilon}\exp\left(-i\omega\left[\min[0\, y_2^0\cdots y_r^0] - \max[0\, y_{r+1}^0\cdots y_n^0]\right]\right)\\ &\times \sum_\sigma\int d^3p\left(\Psi_0,\, T\left\{A_1(0)\cdots A_r(y_r)\right\}\Psi_{\mathbf{p},\sigma}\right)\\ &\qquad\qquad\times\left(\Psi_{\mathbf{p},\sigma},\, T\left\{A_{r+1}(0)\cdots A_n(y_n)\right\}\Psi_0\right)\\ &\times (2\pi)^4\delta^3(\mathbf{p}-\mathbf{q}_1-\cdots-\mathbf{q}_r)\,\delta(\sqrt{\mathbf{p}^2+m^2}+\omega-q_1^0-\cdots-q_r^0)\\ &\times (2\pi)^4\,\delta^3(\mathbf{q}_{r+1}+\cdots+\mathbf{q}_n+\mathbf{p})\,\delta\left(q_{r+1}^0+\cdots+q_n^0+\sqrt{\mathbf{p}^2+m^2}+\omega\right)\\ &+\text{OT}\,.\end{aligned} \tag{10.2.11}$$

We are interested here only in the pole that arises from the vanishing of the denominator $\omega + i\epsilon$, so for our present purposes we can set the factor $\exp(-i\omega[\min - \max])$ equal to unity. The integrals over both $\mathbf{p}$ and ω are

now trivial, and yield the pole

$$G(q_1 \cdots q_n) \to i(2\pi)^7\, \delta^4(q_1 + \cdots + q_n) \left[q^0 - \sqrt{\mathbf{q}^2 + m^2} + i\epsilon\right]^{-1}$$
$$\times \sum_\sigma M_{0|\mathbf{q},\sigma}(q_2 \cdots q_n)\, M_{\mathbf{q},\sigma|0}(q_{r+2} \cdots q_n) + \cdots \tag{10.2.12}$$

where now

$$q \equiv q_1 + \cdots + q_r = -q_{r+1} - \cdots - q_n\,,$$

$$M_{0|\mathbf{q},\sigma}(q_2 \cdots q_r) \equiv \int d^4y_2 \cdots d^4y_r\, e^{-iq_2\cdot y_2} \cdots e^{-iq_r\cdot y_r}$$
$$\times \left(\Psi_0,\ T\left\{A_1(0)A_2(y_2)\cdots A_r(y_r)\right\}\Psi_{\mathbf{q},\sigma}\right)\,, \tag{10.2.13}$$

$$M_{\mathbf{q},\sigma|0}(q_{r+2} \cdots q_n) \equiv \int d^4y_{r+2} \cdots d^4y_n\, e^{-iq_{r+2}\cdot y_{r+2}} \cdots e^{-iq_n\cdot y_n}$$
$$\times \left(\Psi_{\mathbf{q},\sigma},\ T\left\{A_{r+1}(0)A_{r+2}(y_{r+2})\cdots A_n(y_n)\right\}\Psi_0\right)\,, \tag{10.2.14}$$

and the final '$\cdots$' in Eq. (10.2.12) denotes terms that do not exhibit this particular pole. (The 'other terms' arising from other single-particle states produce poles in q at different positions, while those arising from multi-particle states produce branch points in q, and those arising from other time-orderings produce poles and branch cuts in other variables.) Using Eqs. (10.2.9) and (10.2.10), it is easy to see that these Ms are the same as defined by Eqs. (10.2.4) and (10.2.5). Also, near the pole we can write

$$\frac{1}{q^0 - \sqrt{\mathbf{q}^2 + m^2} + i\epsilon} = \frac{-q^0 - \sqrt{\mathbf{q}^2 + m^2} + i\epsilon}{-(q^0)^2 + (\sqrt{\mathbf{q}^2 + m^2} - i\epsilon)^2} \to \frac{-2\sqrt{\mathbf{q}^2 + m^2}}{q^2 + m^2 - i\epsilon}\,.$$

(We again redefine ϵ by a positive factor $2\sqrt{\mathbf{q}^2 + m^2}$, which is permissible since ϵ stands for any positive infinitesimal.) Eq. (10.2.12) is thus the same as the desired result (10.2.3).

This result has a classic application to the theory of nuclear forces. Let $\Phi_a(x)$ be any real field or combination of fields (for instance, proportional to a quark–antiquark bilinear $\bar{q}\gamma_5\tau_a q$) that has a non-vanishing matrix element between a one-pion state of isospin a and the vacuum, normalized so that

$$\langle \mathrm{VAC}|\, \Phi_a(0)\, |\pi_b, \mathbf{p}\rangle = (2\pi)^{-3/2}(2p^0)^{-1/2}\delta_{ab}\,. \tag{10.2.15}$$

The matrix element of Φ_a between one-nucleon states with four-momenta p,p' then has a pole at $(p - p')^2 \to -m_\pi^2$ which isospin and Lorentz invariance (including space inversion invariance) dictate must take the

form**

$$\langle N', \sigma', \mathbf{p}' | \Phi_a(0) | N, \sigma, \mathbf{p} \rangle \to i(2\pi)^{-3} G_\pi \times \frac{\left(\bar{u}' \gamma_5 \tau_a u\right)}{(p-p')^2 + m_\pi^2}, \qquad (10.2.16)$$

where u and u' are the initial and final nucleon spinor coefficient functions, including the nucleon wave functions in isospin space, and τ_a with $a = 1, 2, 3$ are the 2×2 Pauli isospin matrices. The constant G_π is known as the *pion–nucleon coupling constant*. This pole is not actually in the physical region for the matrix element (10.2.16), for which $(p-p')^2 \geq 0$, but it can be reached by analytic extension of this matrix element, for instance by considering the off-shell matrix element

$$\int d^4x\, d^4x'\, e^{-ip\cdot x} e^{ip'\cdot x'} \langle T\{\Phi_a(0)\, \bar{N}(x)\, N'(x')\}\rangle_{\text{VAC}},$$

where N and N' are appropriate components of a field operator or product of field operators with non-vanishing matrix elements between one-nucleon states and the vacuum. The theorem proved above in this section shows then that exchange of a pion in the scattering of two nucleons with initial four-momenta p_1, p_2, and final four-momenta p_1', p_2' yields a pole at $(p_1 - p_1')^2 = (p_2 - p_2')^2 \to -m_\pi^2$:

$$S_{N_1' N_2', N_1 N_2} \to -i(2\pi)^4 \delta^4(p_1' + p_2' - p_1 - p_2) \frac{G_\pi^2}{(p_1 - p_1')^2 + m_\pi^2}$$
$$\times (2\pi)^{-3} \left(\bar{u}_{1'} \gamma_5 \tau_a u_1\right) \times (2\pi)^{-3} \left(\bar{u}_2' \gamma_5 \tau_a u_2\right). \qquad (10.2.17)$$

(The easiest way to get the phases and numerical factors right in such formulas is to use Feynman diagrams; our theorem just says that the pole structure is the same as would be found in a field theory in which the Lagrangian involved an elementary pion field.) Again, this pion pole is not actually in the physical region for scattering of nucleons on the mass shell, for which $(p_1 - p_1')^2 \geq 0$, but it can be reached by analytic extension of the S-matrix element, for instance by considering the off-shell matrix

** Lorentz and isospin invariance requires this matrix to take the form $(\bar{u}' \Gamma \tau_a u)$, where Γ is a 4×4 matrix for which the bilinear $(\bar{\psi}' \Gamma \psi)$ transforms as a pseudoscalar. Like any 4×4 matrix, Γ can be expanded as a sum of terms proportional to the Dirac matrices 1, γ_μ, $[\gamma_\mu, \gamma_\nu]$, $\gamma_5 \gamma_\mu$, and γ_5. The coefficients must be respectively pseudoscalar, pseudovector, pseudotensor, vector, and scalar. Out of the two momenta p and p' it is possible to construct no pseudoscalars or pseudovectors; just one pseudotensor, proportional to $\epsilon^{\mu\nu\rho\sigma} p_\rho p_\sigma'$; two independent vectors, proportional to p_μ or p_μ'; and a scalar proportional to unity, in each case with a proportionality factor depending on the only independent scalar variable, $(p-p')^2$. By using the momentum-space Dirac equations for u and u', it is easy to see that the tensor and pseudovector matrices in Γ give contributions proportional to γ_5.

element

$$\int d^4x_1\, d^4x_2\, d^4x_1'\, d^4x_2'\; e^{-ip_1\cdot x_1} e^{-ip_2\cdot x_2} e^{ip_1'\cdot x_1'} e^{ip_2'\cdot x_2'}$$
$$\times \langle\, T\{\bar{N}_1(x_1), \bar{N}_2(x_2), N_1'(x_1'), N_2'(x_2')\}\rangle_{\text{VAC}}\,.$$

Although this pole is not in the physical region for nucleon–nucleon scattering, the pion mass is small enough so that the pole is quite near the physical region, and under some circumstances may dominate the scattering amplitude, as for instance for large ℓ in the partial wave expansion.

Interpreted in coordinate space, a pole like this at $(p_1 - p_1')^2 = (p_2 - p_2')^2 \to -m_\pi^2$ implies a force of range $1/m_\pi$. For instance, in Yukawa's original theory[2] of nuclear force the exchange of mesons (then assumed scalar rather than pseudoscalar) produced a local potential of the form $\exp(-m_\pi r)/4\pi r$, which in the first Born approximation yields an S-matrix for non-relativistic nucleon scattering proportional to the Fourier transform:

$$\int d^3x_1\, d^3x_2\, d^3x_1'\, d^3x_2'\; e^{-i\mathbf{x}_1\cdot\mathbf{p}_1} e^{-i\mathbf{x}_2\cdot\mathbf{p}_2} e^{i\mathbf{x}_1'\cdot\mathbf{p}_1'} e^{i\mathbf{x}_2'\cdot\mathbf{p}_2'}$$
$$\times \frac{\exp\left(-m_\pi|\mathbf{x}_1 - \mathbf{x}_2|\right)}{4\pi|\mathbf{x}_1 - \mathbf{x}_2|}\,\delta^3(\mathbf{x}_1 - \mathbf{x}_1')\delta^3(\mathbf{x}_2 - \mathbf{x}_2')$$
$$= -(2\pi)^3\delta^3(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{p}_1' - \mathbf{p}_2')\frac{1}{(\mathbf{p}_1 - \mathbf{p}_1')^2 + m_\pi^2}\,.$$

The factor $1/[(\mathbf{p}_1 - \mathbf{p}_1')^2 + m_\pi^2]$ is just the non-relativistic limit of the propagator $1/[(p_1 - p_1')^2 + m_\pi^2]$ in (10.2.17). (In (10.2.17) the energy transfer $p_1^0 - p_1'^0$ for $|\mathbf{p}_1| \ll m_N$ and $|\mathbf{p}_1'| \ll m_N$ equals $[\mathbf{p}_1^2 - \mathbf{p}_1'^2]/2m_N$, which is negligible compared with the magnitude $|\mathbf{p}_1 - \mathbf{p}_1'|$ of the momentum transfer.) When Yukawa's theory was first proposed, it was generally supposed that this sort of momentum-dependence arises from the appearance of a meson field in the theory. It was not until the 1950s that it became generally understood that the existence of a pole at $(p_1 - p_1')^2 \to -m_\pi^2$ follows from the existence of a pion *particle* and has nothing to do with whether this is an elementary particle with its own field in the Lagrangian.

10.3 Field and Mass Renormalization

We will now use a special case of the result of the previous section to clarify the treatment of radiative corrections in the internal and external line of general processes.

The special case that concerns us here is the one in which the four-

momentum of a single external line approaches the mass shell. (In the notation of the previous section, this corresponds to taking $r = 1$.) We will consider a function

$$G_\ell(q_1 q_2 \cdots) = \int d^4x_1\, d^4x_2 \cdots\; e^{-iq_1\cdot x_1}\, e^{-iq_2\cdot x_2} \ldots \\ \times \left(\Psi_0,\, T\left\{\mathcal{O}_\ell(x_1),\, A_2(x_2), \cdots\right\}\Psi_0\right) , \qquad (10.3.1)$$

where $\mathcal{O}_\ell(x)$ is a Heisenberg-picture operator, with the Lorentz transformation properties of some sort of free field ψ_ℓ belonging to an irreducible representation of the homogeneous Lorentz group (or the Lorentz group including space inversion for theories that conserve parity), as labelled by the subscript ℓ, and A_2, A_3, etc. are arbitrary Heisenberg-picture operators. Suppose there is a one-particle state $\Psi_{\mathbf{q}_1,\sigma}$ that has non-vanishing matrix elements with the states $\mathcal{O}_\ell^\dagger \Psi_0$ and with $A_2 A_3 \cdots \Psi_0$. Then according to the theorem proved in the previous section, G_ℓ has a pole at $q_1^2 = -m^2$, with

$$G_\ell(q_1 q_2 \cdots) \to \frac{-2i\sqrt{\mathbf{q}_1{}^2 + m^2}}{q_1^2 + m^2 - i\epsilon}\, (2\pi)^3 \sum_\sigma \left(\Psi_0, \mathcal{O}_\ell(0)\Psi_{\mathbf{q}_1,\sigma}\right) \\ \times \int d^4x_2 \cdots e^{-iq_2\cdot x_2} \cdots \left(\Psi_{\mathbf{q}_1,\sigma}\, T\left\{A_2(x_2)\cdots\right\}\Psi_0\right) , \qquad (10.3.2)$$

We use Lorentz invariance to write

$$\left(\Psi_0, \mathcal{O}_\ell(0)\Psi_{\mathbf{q}_1,\sigma}\right) = (2\pi)^{-3/2}\, N\, u_\ell(\mathbf{q}_1, \sigma) , \qquad (10.3.3)$$

where $u_\ell(\mathbf{q}, \sigma)$ is (aside from the factor $(2\pi)^{-3/2}$) the coefficient function* appearing in the free field ψ_ℓ with the same Lorentz transformation properties as $\mathcal{O}_\ell$, and N is a constant. (It was in order to obtain Eq. (10.3.3) with a single free constant N that we had to assume that $\mathcal{O}_\ell$ transforms irreducibly.) We also define a 'truncated' matrix element M_ℓ by

$$\int d^4x_2 \cdots e^{-iq_2\cdot x_2} \cdots \left(\Psi_{\mathbf{q}_1,\sigma}\, T\left\{A_2(x_2)\cdots\right\}\Psi_0\right) \\ \equiv N^{-1}(2\pi)^{-3/2} \sum_\ell u_\ell^*(\mathbf{q}_1, \sigma)\, M_\ell(q_2 \cdots) . \qquad (10.3.4)$$

Eq. (10.3.2) then reads, for $q^2 \to -m_1^2$

$$G_\ell \to \frac{-2i\sqrt{\mathbf{q}_1{}^2 + m^2}}{q_1^2 + m^2 - i\epsilon} \sum_{\sigma,\ell'} u_\ell(\mathbf{q}_1, \sigma) u_{\ell'}^*(\mathbf{q}_1, \sigma) M_{\ell'} . \qquad (10.3.5)$$

According to Eqs. (6.2.2) and (6.2.18), the quantity multiplying $M_{\ell'}$ in (10.3.5) is the momentum space matrix propagator $-i\Delta_{\ell\ell'}(q_1)$ for the free

* For instance, for a conventionally normalized free scalar field, $u_\ell(\mathbf{q}_1, \sigma) = [2\sqrt{\mathbf{q}_1^2 + m^2}]^{-1/2}$.

field with the Lorentz transformation properties of $\mathcal{O}_\ell$ (or at least its limiting behavior for $q_1^2 \to -m^2$), so (10.3.5) allows us to identify M_ℓ as the sum of all graphs with external lines carrying momenta $q_1, q_2 \cdots$ corresponding to the operators $\mathcal{O}_\ell, A_2, \cdots$, but with the final propagator for the $\mathcal{O}_\ell$ line stripped away. Eq. (10.3.4) is then just the usual prescription for how to calculate the matrix element for emission of a particle from the sum of Feynman diagrams: strip away the particle propagator, and contract with the usual external line factor $(2\pi)^{-3/2} u_\ell^*$. The only discrepancy with the usual Feynman rules is the factor N.

The above theorem is a famous result due to Lehmann, Symanzik, and Zimmerman,[3] known as the *reduction formula*, which we have proved here by a somewhat different method that has allowed us easily to generalize this result to the case of arbitrary spin. One important aspect of this result is that it applies to any sort of operator; $\mathcal{O}_\ell$ need not be some field that actually appears in the Lagrangian, and the particle it creates may be a bound state composed of those particles whose fields do occur in the Lagrangian. It provides an important lesson even where $\mathcal{O}_\ell$ is some field Ψ_ℓ in the Lagrangian: if we are to use the usual Feynman rules to calculate S-matrix elements, then we should first redefine the normalization of the fields by a factor $1/N$, so that (with apologies for the multiple use of the symbol Ψ):

$$\left(\Psi_0, \Psi_\ell(0)\Psi_{\mathbf{q},\sigma}\right) = (2\pi)^{-3/2}\, u_\ell(\mathbf{q},\sigma)\,. \tag{10.3.6}$$

A field normalized as in Eq. (10.3.6) is called a *renormalized field.*

The field renormalization constant N shows up in another place. Suppose that there is just one of the operators $A_2, A_3, \cdots$ in Eq. (10.3.1), and take it to be the adjoint of a member of the same field multiplet as $\mathcal{O}_\ell$. Then Eq. (10.3.2) reads

$$\begin{aligned}
&\int d^4x_1 \int d^4x_2\, e^{-iq_1\cdot x_1}\, e^{-iq_2\cdot x_2} \left(\Psi_0,\, T\left\{\mathcal{O}_\ell(x_1)\mathcal{O}^\dagger_{\ell'}(x_2)\right\}\Psi_0\right) \\
&\overset{q_1^2\to -m^2}{\longrightarrow} \frac{-2i\sqrt{\mathbf{q}_1{}^2+m^2}\,(2\pi)^3}{q_1^2+m^2-i\epsilon} \sum_\sigma \left(\Psi_0, \mathcal{O}_\ell(0)\Psi_{\mathbf{q}_1,\sigma}\right) \\
&\qquad \times \int d^4x_2\, e^{-iq_2\cdot x_2} e^{-iq_1\cdot x_2} \left(\Psi_{\mathbf{q}_1,\sigma}, \mathcal{O}^\dagger_{\ell'}(0)\Psi_0\right) \\
&= \frac{-2i|N|^2\sqrt{\mathbf{q}_1{}^2+m^2}}{q_1^2+m^2-i\epsilon} \sum_\sigma u_\ell(\mathbf{q}_1,\sigma)\, u^*_{\ell'}(\mathbf{q}_1,\sigma)\,(2\pi)^4\,\delta^4(q_1+q_2)\,.
\end{aligned}$$

This is just the usual behavior of a propagator (the sum of all graphs with two external lines) near its pole, except for the factor $|N|^2$. According to Eq. (10.3.6), this factor is absent in the propagator of the renormalized field Ψ_ℓ. Thus *a renormalized field is one whose propagator has the same*

behavior near its pole as for a free field, and the renormalized mass is defined by the position of the pole.

To see how this works in practice, consider the theory of a real self-interacting scalar field Φ_B, the subscript B being added here to remind us that so far this is a 'bare' (i.e., unrenormalized) field. The Lagrangian density is taken as usual as

$$\mathscr{L} = -\tfrac{1}{2}\partial_\mu\Phi_B\partial^\mu\Phi_B - \tfrac{1}{2}m_B^2\Phi_B^2 - V_B(\Phi_B)\,. \tag{10.3.7}$$

In general there would be no reason to expect that the field Φ_B would satisfy condition (10.3.6), nor that the pole in q^2 would be at $-m_B^2$, so let us introduce a renormalized field and mass

$$\Phi \equiv Z^{-1/2}\Phi_B\,, \tag{10.3.8}$$

$$m^2 \equiv m_B^2 + \delta m^2\,, \tag{10.3.9}$$

with Z to be chosen so that Φ does satisfy Eq. (10.3.6), and δm^2 chosen so that the pole of the propagator is at $q^2 = -m^2$. (The use of the symbol Z in this context has become conventional; there is a different Z for each field in the Lagrangian.) The Lagrangian density (10.3.7) may then be rewritten

$$\mathscr{L} = \mathscr{L}_0 + \mathscr{L}_1, \tag{10.3.10}$$

$$\mathscr{L}_0 = -\tfrac{1}{2}\partial_\mu\Phi\partial^\mu\Phi - \tfrac{1}{2}m^2\Phi^2, \tag{10.3.11}$$

$$\mathscr{L}_1 = -\tfrac{1}{2}(Z-1)[\partial_\mu\Phi\partial^\mu\Phi + m^2\Phi^2] + \tfrac{1}{2}Z\delta m^2\Phi^2 - V(\Phi)\,, \tag{10.3.12}$$

where

$$V(\Phi) \equiv V_B(\sqrt{Z}\Phi)\,.$$

In calculating the corrections to the complete momentum space propagator of the renormalized scalar field, conventionally called $\Delta'(q)$, it is convenient to consider separately the *one-particle-irreducible* graphs: those connected graphs (excluding a graph consisting of a single scalar line) that cannot be disconnected by cutting through any one internal scalar line. An example is shown in Figure 10.4. It is conventional to write the sum of all such graphs, with the two external line propagator factors $-i(2\pi)^{-4}(q^2 + m^2 - i\epsilon)^{-1}$ omitted, as $i(2\pi)^4\Pi^*(q^2)$, with the asterisk to remind us that these are one-particle-irreducible graphs. Then the corrections to the complete propagator are given by a sum of chains of one, two, or more of these one-particle-irreducible subgraphs connected with the usual uncorrected

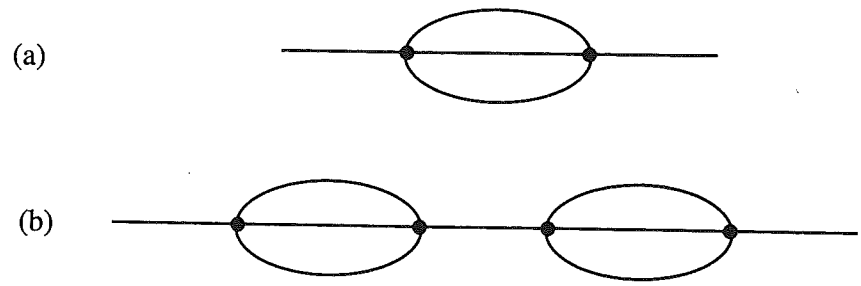

Figure 10.4. Diagrams that (a) are, or (b) are not, one-particle irreducible. These diagrams are drawn for a theory with some sort of quadrilinear interaction, like the theory of a scalar field ϕ with interaction proportional to ϕ^4.

propagator factors:

$$\begin{aligned}\frac{-i}{(2\pi)^4}\Delta'(q) &= \frac{-i}{(2\pi)^4}\frac{1}{q^2+m^2-i\epsilon} \\ &+\left[\frac{-i}{(2\pi)^4}\frac{1}{q^2+m^2-i\epsilon}\right]\left[i(2\pi)^4\Pi^*(q^2)\right]\left[\frac{-i}{(2\pi)^4}\frac{1}{q^2+m^2-i\epsilon}\right] \\ &+\left[\frac{-i}{(2\pi)^4}\frac{1}{q^2+m^2-i\epsilon}\right]\left[i(2\pi)^4\Pi^*(q^2)\right]\left[\frac{-i}{(2\pi)^4}\frac{1}{q^2+m^2-i\epsilon}\right] \\ &\times\left[i(2\pi)^4\Pi^*(q^2)\right]\left[\frac{-i}{(2\pi)^4}\frac{1}{q^2+m^2-i\epsilon}\right]+\cdots \qquad (10.3.13)\end{aligned}$$

or more simply

$$\begin{aligned}\Delta'(q) &= [q^2+m^2-i\epsilon]^{-1}+[q^2+m^2-i\epsilon]^{-1}\Pi^*(q^2)[q^2+m^2-i\epsilon]^{-1} \\ &+[q^2+m^2-i\epsilon]^{-1}\Pi^*(q^2)[q^2+m^2-i\epsilon]^{-1}\Pi^*(q^2)[q^2+m^2-i\epsilon]^{-1}+\cdots\,. \qquad (10.3.14)\end{aligned}$$

Summing the geometric series, this gives

$$\Delta'(q) = \left[q^2+m^2-\Pi^*(q^2)-i\epsilon\right]^{-1}\,. \qquad (10.3.15)$$

In calculating Π^*, we encounter a tree graph arising from a single insertion of vertices corresponding to the terms in Eq. (10.3.12) proportional to $\partial_\mu\Phi\partial^\mu\Phi$ and Φ^2, plus a term $\Pi^*_{\rm LOOP}$ arising from loop graphs like that in Figure 10.4(a):

$$\Pi^*(q^2) = -(Z-1)[q^2+m^2]+Z\,\delta m^2+\Pi^*_{\rm LOOP}(q^2)\,. \qquad (10.3.16)$$

The condition that m^2 is the true mass of the particle is that the pole of the propagator should be at $q^2=-m^2$, so that

$$\Pi^*(-m^2)=0\,. \qquad (10.3.17)$$

Also, the condition that the pole of the propagator at $q^2=-m^2$ should

have a unit residue (like the uncorrected propagator) is that

$$\left[\frac{d}{dq^2}\Pi^*(q^2)\right]_{q^2=-m^2} = 0\,. \tag{10.3.18}$$

These conditions allow us to evaluate Z and δm^2:

$$Z\,\delta m^2 = -\Pi^*_{\text{LOOP}}(-m^2)\,, \tag{10.3.19}$$

$$Z = 1 + \left[\frac{d}{dq^2}\Pi^*_{\text{LOOP}}(q^2)\right]_{q^2=-m^2}\,. \tag{10.3.20}$$

This incidentally shows that $Z\,\delta m^2$ and $Z-1$ are given by a series of terms containing one or more coupling constant factors, justifying the treatment of the first two terms in Eq. (10.3.12) as part of the interaction $\mathscr{L}_1$.

In actual calculations it is simplest just to say that from the loop terms $\Pi^*_{\text{LOOP}}(q^2)$ we must subtract a first-order polynomial in q^2 with coefficients chosen so that the difference satisfies Eqs. (10.3.17) and (10.3.18). As we shall see, this subtraction procedure incidentally cancels the infinities that arise from the momentum space integrals in Π^*_{LOOP}. However, as this discussion should make clear, *the renormalization of masses and fields has nothing directly to do with the presence of infinities, and would be necessary even in a theory in which all momentum space integrals were convergent.*

An important consequence of the conditions (10.3.17) and (10.3.18) is that it is not necessary to include radiative corrections in external lines on the mass shell. That is,

$$\Big[\Pi^*(q^2)[q^2+m^2-i\epsilon]^{-1} + \Pi^*(q^2)[q^2+m^2-i\epsilon]^{-1}\Pi^*(q^2)[q^2+m^2-i\epsilon]^{-1} + \cdots\Big]_{q^2\to -m^2} = 0\,. \tag{10.3.21}$$

Similar remarks apply to particles of arbitrary spin. For instance, for the 'bare' Dirac field the Lagrangian is

$$\mathscr{L} = -\bar{\Psi}_{\text{B}}[\not\partial + m_{\text{B}}]\Psi_{\text{B}} - V_{\text{B}}(\bar{\Psi}_{\text{B}}\Psi_{\text{B}})\,. \tag{10.3.22}$$

We introduce renormalized fields and masses

$$\Psi \equiv Z_2^{-1/2}\Psi_{\text{B}}\,, \tag{10.3.23}$$

$$m = m_{\text{B}} + \delta m\,. \tag{10.3.24}$$

(The subscript 2 on Z_2 is conventionally used to distinguish the renormalization constant of a fermion field.) The Lagrangian density is then rewritten

$$\mathscr{L} = \mathscr{L}_0 + \mathscr{L}_1\,, \tag{10.3.25}$$

$$\mathscr{L}_0 = -\bar{\Psi}[\not\partial + m]\Psi\,, \tag{10.3.26}$$

$$\mathscr{L}_1 = -(Z_2 - 1)[\bar{\Psi}[\not{\partial} + m]\Psi] + Z_2\delta m\bar{\Psi}\Psi - V_{\mathrm{B}}(Z_2\bar{\Psi}\Psi)\,. \tag{10.3.27}$$

Let $i(2\pi)^4\Sigma^*(\not{k})$ be the sum of all connected graphs, with one fermion line coming in with four-momentum k and one going out with the same four-momentum, that cannot be disconnected by cutting through any single internal fermion line, and with external line propagator factors $-i(2\pi)^{-4}$ and $[i\not{k} + m - i\epsilon]^{-1}$ omitted. (Lorentz invariance is being used to justify writing Σ^* as an ordinary function of the Lorentz scalar matrix $\not{k} \equiv k_\mu\gamma^\mu$.) Then the complete fermion propagator is

$$\begin{aligned} S'(k) &= [i\not{k} + m - i\epsilon]^{-1} + [i\not{k} + m - i\epsilon]^{-1}\Sigma^*(\not{k})[i\not{k} + m - i\epsilon]^{-1} \\ &\quad + [i\not{k} + m - i\epsilon]^{-1}\Sigma^*(\not{k})[i\not{k} + m - i\epsilon]^{-1}\Sigma^*(\not{k})[i\not{k} + m - i\epsilon]^{-1} + \cdots \\ &= [i\not{k} + m - \Sigma^*(\not{k}) - i\epsilon]^{-1}\,. \end{aligned} \tag{10.3.28}$$

In calculating $\Sigma^*(\not{k})$ we take into account the tree graphs from the terms in Eq. (10.3.27) proportional to $\bar{\Psi}\not{\partial}\Psi$ and $\bar{\Psi}\Psi$ as well as loop contributions:

$$\Sigma^*(\not{k}) = -(Z_2 - 1)[i\not{k} + m] + Z_2\delta m + \Sigma^*_{\mathrm{LOOP}}(\not{k})\,. \tag{10.3.29}$$

The condition that the complete propagator has a pole at $k^2 = -m^2$ with the same residue as the uncorrected propagator is then that

$$\Sigma^*(im) = 0\,, \tag{10.3.30}$$

$$\left.\frac{\partial\Sigma^*(\not{k})}{\partial\not{k}}\right|_{\not{k}=im} = 0\,, \tag{10.3.31}$$

and hence

$$Z_2\delta m = -\Sigma^*_{\mathrm{LOOP}}(im)\,, \tag{10.3.32}$$

$$Z_2 = 1 - i\left.\frac{\partial\Sigma^*_{\mathrm{LOOP}}(\not{k})}{\partial\not{k}}\right|_{\not{k}=im}\,. \tag{10.3.33}$$

Just as for scalars, the vanishing of $[i\not{k} + m]^{-1}\Sigma^*(\not{k})$ in the limit $\not{k} \to im$ tells us that radiative corrections may be ignored in external fermion lines. Corresponding results for the photon propagator will be derived in Section 10.5.

10.4 Renormalized Charge and Ward Identities

The use of the commutation and conservation relations of Heisenberg-picture operators allows us to make a connection between the charges (or other similar quantities) in the Lagrangian density and the properties of physical states. Recall that the invariance of the Lagrangian density

with respect to global gauge transformations $\Psi_\ell \to \exp(iq_\ell\alpha)\Psi_\ell$ (with α an arbitrary constant phase) implies the existence of a current

$$J^\mu = -i \sum_\ell \frac{\partial \mathscr{L}}{\partial(\partial_\mu \Psi_\ell)} q_\ell \Psi_\ell , \tag{10.4.1}$$

satisfying the conservation condition

$$\partial_\mu J^\mu = 0 . \tag{10.4.2}$$

This implies that the space-integral of the time component of J^μ is time-independent:

$$i \frac{d}{dt} Q = [Q, H] = 0 , \tag{10.4.3}$$

where

$$Q \equiv \int d^3x \, J^0 . \tag{10.4.4}$$

(There is a very important possible exception here, that the integral (10.4.4) may not exist if there are long-range forces due to massless scalars in the system. We will return to this point when we consider broken symmetries in Volume II.) Also, since it is a space-integral, Q is manifestly translation-invariant

$$[\mathbf{P}, Q] = 0 \tag{10.4.5}$$

and since J^μ is a four-vector, Q is invariant with respect to homogeneous Lorentz transformations

$$[J^{\mu\nu}, Q] = 0 . \tag{10.4.6}$$

It follows that Q acting on the true vacuum Ψ_0 must be another Lorentz-invariant state of zero energy and momentum, and hence (assuming no vacuum degeneracy) must be proportional to Ψ_0 itself. But the proportionality constant must vanish, because Lorentz invariance requires that $(\Psi_0, J_\mu \Psi_0)$ vanish. Hence

$$Q \Psi_0 = 0 . \tag{10.4.7}$$

Also, Q acting on any one-particle state $\Psi_{\mathbf{p},\sigma,n}$ must be another state with the same energy, momentum, and Lorentz transformation properties, and thus (assuming no degeneracy of one-particle states) must be proportional to the same one-particle state

$$Q \Psi_{\mathbf{p},\sigma,n} = q_{(n)} \Psi_{\mathbf{p},\sigma,n} . \tag{10.4.8}$$

The Lorentz invariance of Q ensures that the eigenvalue $q_{(n)}$ is independent of $\mathbf{p}$ and σ, depending only on the species of the particle. This eigenvalue is what is known as the electric charge (or whatever other quantum number

of which J^μ may be the current) of the one-particle state. To relate this to the q_ℓ parameters in the Lagrangian, we note that the canonical commutation relations give

$$\left[J^0(\mathbf{x},t),\ \Psi_\ell(\mathbf{y},t)\right] = -q_\ell \Psi_\ell(\mathbf{y},t)\delta^3(\mathbf{x}-\mathbf{y})\,, \tag{10.4.9}$$

or integrating over $\mathbf{x}$:

$$\left[Q, \Psi_\ell(y)\right] = -q_\ell \Psi_\ell(y)\,. \tag{10.4.10}$$

The same is true of any local function $F(y)$ of the fields and field derivatives and their adjoints, containing definite numbers of each:

$$\left[Q, F(y)\right] = -q_F F(y)\,, \tag{10.4.11}$$

where q_F is the sum of the q_ℓ for all fields and field derivatives in $F(y)$, minus the sum of the q_ℓ for all field adjoints and their derivatives. Taking the matrix element of this equation between a one-particle state and the vacuum, and using Eqs. (10.4.7) and (10.4.8), we have

$$\left(\Psi_0,\ F(y)\Psi_{\mathbf{p},\sigma,n}\right)(q_F - q_{(n)}) = 0\,. \tag{10.4.12}$$

Hence we must have

$$q_{(n)} = q_F \tag{10.4.13}$$

as long as

$$\left(\Psi_0,\ F(y)\,\Psi_{\mathbf{p},\sigma,n}\right) \neq 0\,. \tag{10.4.14}$$

As we saw in the previous section, Eq. (10.4.14) is the condition that assures that momentum space Green's functions involving F have poles corresponding to the one-particle state $\Psi_{\mathbf{p},\sigma,n}$. For a one-particle state corresponding to one of the fields in the Lagrangian we could take $F = \Psi_\ell$, in which case $q_F = q_\ell$, but our results here apply to general one-particle states, whether or not their fields appear in the Lagrangian.

This almost, but not quite, tells us that despite all the possible high-order graphs that affect the emission and absorption of photons by charged particles, the physical electric charge is just equal to a parameter q_ℓ appearing in the Lagrangian (or to a sum of such parameters, like q_F .) The qualification that has to be added here is that the requirement, that the Lagrangian be invariant under the transformations $\Psi_\ell \to \exp(iq_\ell\alpha)\Psi_\ell$, does nothing to fix the over-all scale of the quantities q_ℓ. The physical electric charges are those that determine the response of matter fields to a given *renormalized* electromagnetic field A^μ. That is, the scale of the q_ℓ is fixed by requiring that the renormalized electromagnetic field appears in the matter Lagrangian $\mathscr{L}_M$ in the linear combinations $[\partial_\mu - iq_\ell A_\mu]\Psi_\ell$,

so that the current J^μ is

$$J^\mu = \frac{\delta \mathscr{L}_M}{\delta A_\mu} . \tag{10.4.15}$$

But A^μ and q_ℓ are not the same as the 'bare electromagnetic field' $A^\mu_{\rm B}$ and 'bare charges' $q_{{\rm B}\ell}$ that appear in the Lagrangian when we write it in its simplest form

$$\mathscr{L} = -\tfrac{1}{4}(\partial_\mu A_{B\nu} - \partial_\nu A_{B\mu})(\partial^\mu A^\nu_{\rm B} - \partial^\nu A^\mu_{\rm B}) + \mathscr{L}_M\left(\Psi_\ell, [\partial_\mu - iq_{B\ell}A_{B\mu}]\Psi_\ell\right) . \tag{10.4.16}$$

The renormalized electromagnetic field (defined to have a complete propagator whose pole at $p^2 = 0$ has unit residue) is conventionally written in terms of $A^\mu_{\rm B}$ as

$$A^\mu = Z_3^{-1/2} A^\mu_{\rm B} , \tag{10.4.17}$$

so in order for the charge q_ℓ to characterize the response of the charged particles to a given renormalized electromagnetic field, we should define the renormalized charges by

$$q_\ell = \sqrt{Z_3}\, q_{{\rm B}\ell} . \tag{10.4.18}$$

We see that the physical electric charge q of any particle is just proportional to a parameter $q_{\rm B}$ related to those appearing in the Lagrangian, with a proportionality constant $\sqrt{Z_3}$ that is the same for all particles. This helps us to understand how a particle like the proton, that is surrounded by a cloud of virtual mesons and other strongly interacting particles, can have the same charge as the positron, whose interactions are all much weaker. It is only necessary to assume that for some reason the charges $q_{{\rm B}\ell}$ in the Lagrangian are equal and opposite for the electron and for those particles (two u quarks and one d quark) that make up the proton; the effect of higher-order corrections then appears solely in the *common* factor $\sqrt{Z_3}$.

In order for charge renormalization to arise only from radiative corrections to the photon propagator, there must be cancellations among the great variety of other radiative corrections to the propagators and electromagnetic vertices of the charged particles. We can see a little more deeply into the nature of these cancellations by making use of the celebrated relations between these charged particle propagators and vertices known as the *Ward identities.*

For instance, consider the Green's function for an electric current $J^\mu(x)$ together with a Heisenberg-picture Dirac field $\Psi_n(y)$ of charge q and its covariant adjoint $\bar{\Psi}_m(z)$. We define the electromagnetic vertex function Γ^μ

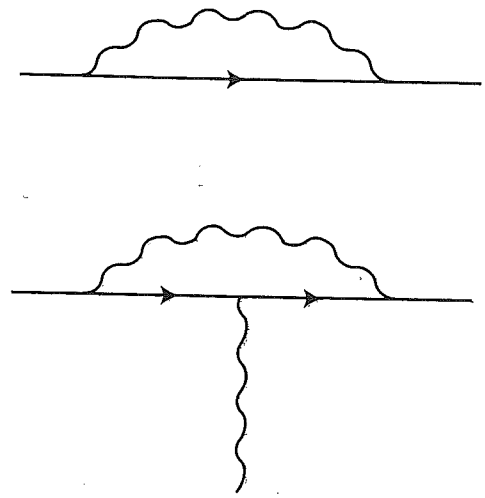

Figure 10.5. Diagrams for the first corrections to the electron propagator and vertex function in quantum electrodynamics. Here straight lines are electrons; wavy lines are photons.

of the charged particle by

$$\int d^4x\, d^4y\, d^4z\, e^{-ip\cdot x} e^{-ik\cdot y} e^{+i\ell\cdot z} \left(\Psi_0,\, T\left\{J^\mu(x)\,\Psi_n(y)\,\bar{\Psi}_m(z)\right\}\Psi_0\right)$$
$$\equiv -\, i(2\pi)^4 q S'_{nn'}(k)\Gamma^\mu_{n'm'}(k,\ell) S'_{m'm}(\ell)\,\delta^4(p+k-\ell)\,, \qquad (10.4.19)$$

where

$$-\, i(2\pi)^4 S'_{nm}(k)\,\delta^4(k-\ell) \equiv \int d^4y\, d^4z\, \left(\Psi_0,\, T\left\{\Psi_n(y)\bar{\Psi}_m(z)\right\}\Psi_0\right) e^{-ik\cdot y} e^{+i\ell\cdot z}\,. \qquad (10.4.20)$$

According to the theorem of Section 6.4, Eq. (10.4.20) gives the sum of all Feynman graphs with one incoming and one outgoing fermion line, i.e., the complete Dirac propagator. Also, Eq. (10.4.19) gives the sum of all such graphs with an extra photon line attached, so Γ^μ is the sum of "vertex" graphs with one incoming Dirac line, one outgoing Dirac line, and one photon line, but with the complete Dirac external line propagators and the bare photon external line propagator stripped away. To make the normalization of S' and Γ^μ perfectly clear, we mention that in the limit of no interactions, these functions take the values

$$S'(k) \to [i\gamma_\lambda k^\lambda + m - i\epsilon]^{-1}\,, \quad \Gamma^\mu(k,\ell) \to \gamma^\mu\,.$$

The one-loop diagrams that provide corrections to these limiting values are shown in Figure 10.5.

We can derive a relation between Γ^μ and S' by use of the identity

$$\begin{aligned}\frac{\partial}{\partial x^\mu} T\left\{J^\mu(x)\Psi_n(y)\bar{\Psi}_m(z)\right\} &= T\left\{\partial_\mu J^\mu(x)\Psi_n(y)\bar{\Psi}_m(z)\right\} \\ &+ \delta(x^0-y^0)\, T\left\{\left[J^0(x),\Psi_n(y)\right]\bar{\Psi}_m(z)\right\} \\ &+ \delta(x^0-z^0)\, T\left\{\Psi_n(y)\left[J^0(x),\bar{\Psi}_m(z)\right]\right\}, \end{aligned} \tag{10.4.21}$$

where the delta functions arise from time-derivatives of step functions. The conservation condition (10.4.2) tells us that the first term vanishes, while the second and third terms can be calculated using the commutation relations (10.4.9), which here give

$$\left[J^0(\mathbf{x},t),\ \Psi_n(\mathbf{y},t)\right] = -q\Psi_n(\mathbf{y},t)\delta^3(\mathbf{x}-\mathbf{y}) \tag{10.4.22}$$

and its adjoint

$$\left[J^0(\mathbf{x},t),\ \bar{\Psi}_n(\mathbf{y},t)\right] = q\bar{\Psi}_n(\mathbf{y},t)\delta^3(\mathbf{x}-\mathbf{y})\,. \tag{10.4.23}$$

Eq. (10.4.21) then reads

$$\begin{aligned}\frac{\partial}{\partial x^\mu}\, T\left\{J^\mu(x)\Psi_n(y)\bar{\Psi}_m(z)\right\} &= -q\,\delta^4(x-y)\, T\left\{\Psi_n(y)\bar{\Psi}_m(z)\right\} \\ &+ q\,\delta^4(x-z)\, T\left\{\Psi_n(y)\bar{\Psi}_m(z)\right\}. \end{aligned} \tag{10.4.24}$$

Inserting this in the Fourier transform (10.4.19) gives

$$(\ell-k)_\mu\, S'(k)\,\Gamma^\mu(k,\ell)\, S'(\ell) = i\, S'(\ell) \;-\; i S'(k)$$

or in other words

$$(\ell-k)_\mu\Gamma^\mu(k,\ell) = i\, S'^{-1}(k) - iS'^{-1}(\ell)\,. \tag{10.4.25}$$

This is known as the *generalized Ward identity*, first derived (by these methods) by Takahashi.[4] The original Ward identity, derived earlier by Ward[5] from a study of perturbation theory, can be obtained from Eq. (10.4.25) by letting ℓ approach k. In this limit, Eq. (10.4.25) gives

$$\Gamma^\mu(k,k) = -i\,\frac{\partial}{\partial k_\mu}\, S'^{-1}(k)\,. \tag{10.4.26}$$

The fermion propagator is related to the self-energy insertion $\Sigma^*(\not{k})$ by Eq. (10.3.28)

$$S'^{-1}(k) = i\,\not{k} + m - \Sigma^*(\not{k})\,,$$

so Eq. (10.4.26) may be written

$$\Gamma^\mu(k,k) = \gamma^\mu + i\,\frac{\partial}{\partial k_\mu}\,\Sigma^*(\not{k})\,. \tag{10.4.27}$$

For a *renormalized* Dirac field, Eqs. (10.3.31) and (10.4.27) tell us that on the mass shell

$$\bar{u}'_k \, \Gamma^\mu(k,k) \, u_k = \bar{u}'_k \, \gamma^\mu \, u_k \, , \tag{10.4.28}$$

where $[i\gamma_\mu k^\mu + m]u_k = [i\gamma_\mu k^\mu + m]u'_k = 0$. Thus the renormalization of the fermion field ensures that the radiative corrections to the vertex function Γ_μ cancel when a fermion on the mass shell interacts with an electromagnetic field with zero momentum transfer, as is the case when we set out to measure the fermion's electric charge. If we had not used a renormalized fermion field then the corrections to the vertex function would have just cancelled the corrections due to radiative corrections to the external fermion lines, leaving the electric charge again unchanged.

10.5 Gauge Invariance

The conservation of electric charge may be used to prove a useful result for the quantities

$$M^{\mu\mu'\cdots}_{\beta\alpha}(q,q',\cdots) \equiv \int d^4x \int d^4x' \cdots e^{-iq\cdot x} e^{-iq'\cdot x'} \cdots \\ \times \left(\Psi^-_\beta, \, T\left\{J^\mu(x), J^{\mu'}(x') \cdots\right\}\Psi^+_\alpha\right) . \tag{10.5.1}$$

In theories like spinor electrodynamics in which the electromagnetic interaction is linear in the field A^μ, this is the matrix element for emission (and/or absorption) of on- or off-shell photons having four-momenta q, q', etc. (and/or $-q$, $-q'$, etc.), with external line photon coefficient functions or propagators omitted, in an arbitrary transition $\alpha \to \beta$. Our result is that Eq. (10.5.1) vanishes when contracted with any one of the photon four-momenta:

$$q_\mu M^{\mu\mu'\cdots}_{\beta\alpha}(q,q',\cdots) = q'_{\mu'} M^{\mu\mu'\cdots}_{\beta\alpha}(q,q',\cdots) \\ = \cdots = 0 \, . \tag{10.5.2}$$

Since M is defined symmetrically with respect to the photon lines, it will be sufficient to show the vanishing of the first of these quantities.

For this purpose, note that by an integration by parts

$$q_\mu \, M^{\mu\mu'\cdots}_{\beta\alpha}(q,q',\cdots) = -i \int d^4x \int d^4x' \cdots \\ \times e^{-iq\cdot x} e^{-iq'\cdot x'} \cdots \left(\Psi^-_\beta, \frac{\partial}{\partial x^\mu} \, T\left\{J^\mu(x), J^{\mu'}(x') \cdots\right\}\Psi^+_\alpha\right) . \tag{10.5.3}$$

The electric current $J^\mu(x)$ is conserved, but this does not immediately imply that Eq. (10.5.3) vanishes, because we still have to take account

of the x^0-dependence contained in the theta functions that appear in the definition of the time-ordered product. For instance, for just two currents

$$T\left\{J^\mu(x)J^\nu(y)\right\} = \theta(x^0 - y^0)J^\mu(x)J^\nu(y) + \theta(y^0 - x^0)J^\nu(y)J^\mu(x)$$

so, taking account of the conservation of $J^\mu(x)$:

$$\begin{aligned}\frac{\partial}{\partial x^\mu}\, T\left\{J^\mu(x)\, J^\nu(y)\right\} &= \delta(x^0 - y^0)\, J^0(x)\, J^\nu(y) - \delta(y^0 - x^0)\, J^\nu(y)\, J^0(x) \\ &= \delta(x^0 - y^0)\left[J^0(x),\, J^\nu(y)\right]. \end{aligned} \tag{10.5.4}$$

With more than two currents, we get an equal-time commutator like this (inside the time-ordered product) for each current aside from $J^\mu(x)$ itself. To evaluate this commutator, we recall that (as shown in the previous section) for any product F of field operators and their adjoints and/or derivatives

$$\left[J^0(\vec{x}, t),\, F(\vec{y}, t)\right] = -q_F\, F(\vec{x}, t)\, \delta^3(\vec{x} - \vec{y})\,,$$

where q_F is the sum of the q_ℓs for the fields and field derivatives in F, minus the sum of the q_ℓs for the field adjoints and their derivatives. For the electric current, q_J is zero; $J^\nu(y)$ is itself an electrically neutral operator. It follows that

$$\left[J^0(\vec{x}, t),\, J^\nu(\vec{y}, t)\right] = 0 \tag{10.5.5}$$

and therefore Eq. (10.5.4) vanishes, so that Eq. (10.5.3) gives

$$q_\mu\, M^{\mu\mu'\cdots}_{\beta\alpha}(q, q', \cdots) = 0 \tag{10.5.6}$$

as was to be proved.

There is an important qualification here. In deriving Eq. (10.5.5) we should take into account the fact that a product of fields at the same spacetime point y like the current operator $J^\nu(y)$ can only be properly defined through some regularization procedure that deals with the infinities in such products. In many cases it turns out that there are non-vanishing contributions to the commutator of $J^0(\vec{x}, t)$ with the regulated current $J^i(\vec{y}, t)$, known as *Schwinger terms*.[6] Where the current includes terms arising from a charged scalar field Φ, there are additional regulator-independent Schwinger terms involving $\Phi^\dagger\Phi$. However, all these Schwinger terms are cancelled in multi-photon amplitudes by the contribution of additional interactions that are quadratic in the electromagnetic field, either arising from the regulator procedure (if gauge-invariant) or, as for charged scalars, directly from terms in the Lagrangian. We will be dealing mostly with charged spinor fields, and will use a regularization procedure (dimensional regularization) that does not lead to Schwinger terms, so

in what follows we will ignore this issue and continue to use the naive commutation relation (10.5.5).

The same argument yields a result like Eq. (10.5.2) even if other particles besides photons are off the mass shell, provided that all *charged* particles are taken on the mass shell, i.e., kept in the states Ψ_β^- and Ψ_α^+. Otherwise, the left-hand side of Eq. (10.5.2) receives contributions from non-vanishing equal-time commutators, such as those we encountered in the derivation of the Ward identity in the previous section.

One consequence of Eq. (10.5.2) is that S-matrix elements are unaffected if we change any photon propagator $\Delta_{\mu\nu}(q)$ by

$$\Delta_{\mu\nu}(q) \to \Delta_{\mu\nu}(q) + \alpha_\mu q_\nu + q_\mu \beta_\nu \tag{10.5.7}$$

or if we change any photon polarization vector by

$$e_\rho(\mathbf{k}, \lambda) \to e_\rho(\mathbf{k}, \lambda) + c k_\rho \,, \tag{10.5.8}$$

where $k^0 \equiv |\mathbf{k}|$, and α_μ, β_ν, and c are entirely arbitrary (not necessarily constants, and not necessarily the same for all propagators or polarization vectors.) This is (somewhat loosely) called the gauge invariance of the S-matrix.

To prove this result it is only necessary to display the explicit dependence of the S-matrix on photon polarization vectors and propagators

$$\begin{aligned} S_{\beta\alpha} \propto \int d^4q_1\, d^4q_2 \cdots \Delta_{\mu_1\nu_1}(q_1)\Delta_{\mu_2\nu_2}(q_2)\cdots \\ \times e^*_{\rho_1}(\mathbf{k}'_1\lambda'_1)e^*_{\rho_2}(\mathbf{k}'_2\lambda'_2)\cdots e_{\sigma_1}(\mathbf{k}_1\lambda_1)e_{\sigma_2}(\mathbf{k}_2\lambda_2)\cdots \\ \times M_{ba}^{\mu_1\mu_2\cdots\nu_1\nu_2\cdots\rho_1\rho_2\cdots\sigma_1\sigma_2\cdots}(-q_1, -q_2, \cdots, q_1, q_2, \cdots, -k'_1, -k'_2\cdots, k_1, k_2\cdots) \end{aligned} \tag{10.5.9}$$

where $M^{\rho\sigma\cdots}$ is the matrix element (10.5.1) calculated in the absence of electromagnetic interactions.* The invariance of Eq. (10.5.9) under the 'gauge transformations' (10.5.7) and (10.5.8) follows immediately from the conservation conditions (10.5.2). (In Section 9.6 we used the path-integral formalism to prove a special case of this theorem, that vacuum expectation values of time-ordered products of gauge-invariant operators are independent of the constant α in the propagator (9.6.21).) This result is not as elementary as it looks, as it applies not to individual diagrams, but only to sums of diagrams in which the current vertices are inserted in all possible places in the diagrams.

There is a particularly important application of Eq. (10.5.2) to the calculation of the photon propagator. The complete photon propagator,

* The states a and b are the same as α and β, but with photons deleted. Note that the arguments of M are all taken to be *incoming* four-momenta, which is why we have to insert various signs for some of the arguments of M in Eq. (10.5.9).

conventionally called $\Delta'_{\mu\nu}(q)$, takes the form

$$\Delta'_{\mu\nu}(q) = \Delta_{\mu\nu}(q) + \Delta_{\mu\rho}(q)M^{\rho\sigma}(q)\Delta_{\sigma\nu}(q)\,, \tag{10.5.10}$$

where $M^{\rho\sigma}$ is proportional to the matrix element (10.5.1) with two currents and α and β both the vacuum state, and $\Delta_{\mu\nu}$ is the bare photon propagator, written here in a general Lorentz-invariant gauge as

$$\Delta_{\mu\nu}(q) \equiv \frac{\eta_{\mu\nu} - \xi(q^2)q_\mu q_\nu/q^2}{q^2 - i\epsilon}\,. \tag{10.5.11}$$

From Eq. (10.5.2) we have here $q^\mu M_{\mu\nu}(q) = 0$, so that

$$q^\mu \Delta'_{\mu\nu}(q) = q^\mu \Delta_{\mu\nu}(q) = \frac{q_\nu(1 - \xi(q^2))}{q^2 - i\epsilon}\,. \tag{10.5.12}$$

On the other hand, just as we did for scalar and spinor fields in Section 10.3, we may express the complete photon propagator in terms of a sum $\Pi^*(q)$ of graphs with two external photon lines that (unlike M) are one-photon-irreducible:

$$\begin{aligned}\Delta'(q) &= \Delta(q) + \Delta(q)\Pi^*(q)\Delta(q) + \Delta(q)\Pi^*(q)\Delta(q)\Pi^*(q)\Delta(q) + \cdots \\ &= [\Delta(q)^{-1} - \Pi^*(q)]^{-1}\end{aligned} \tag{10.5.13}$$

or in other words

$$\Delta'_{\mu\nu}(q) = \Delta_{\mu\nu}(q) + \Delta_{\mu\rho}(q)\Pi^{*\rho\sigma}(q)\Delta'_{\sigma\nu}(q)\,. \tag{10.5.14}$$

Then in order to satisfy Eq. (10.5.12), we must have

$$q_\rho \Pi^{*\rho\sigma}(q) = 0\,. \tag{10.5.15}$$

This together with Lorentz invariance tells us that $\Pi^*(q)$ must take the form

$$\Pi^{*\rho\sigma}(q) = (q^2\eta^{\rho\sigma} - q^\rho q^\sigma)\pi(q^2)\,. \tag{10.5.16}$$

Then Eq. (10.5.13) yields a complete propagator of the form

$$\Delta'_{\mu\nu}(q) = \frac{\eta_{\mu\nu} - \tilde{\xi}(q^2)q_\mu q_\nu/q^2}{[q^2 - i\epsilon][1 - \pi(q^2)]}\,, \tag{10.5.17}$$

where

$$\tilde{\xi}(q^2) = \xi(q^2)[1 - \pi(q^2)] + \pi(q^2)\,. \tag{10.5.18}$$

Now, because $\Pi^*_{\mu\nu}(q)$ receives contributions only from one-photon-irreducible graphs, it is expected not to have any pole at $q^2 = 0$. (There is an important exception in the case of broken gauge symmetry, to be discussed in Volume II.) In particular, the absence of poles at $q^2 = 0$ in the $q_\mu q_\nu$ term in $\Pi^*_{\mu\nu}(q)$ tells us that the function $\pi(q^2)$ in Eq. (10.5.16) also has no such pole, and so the pole in the complete propagator (10.5.17) is

still at $q^2 = 0$, indicating that *radiative corrections do not give the photon a mass.*

For a renormalized electromagnetic field, radiative corrections should also not alter the gauge-invariant part of the residue of the photon pole in Eq. (10.5.17), so

$$\pi(0) = 0 . \tag{10.5.19}$$

This condition leads to a determination of the electromagnetic field renormalization constant Z_3. Recall that when expressed in terms of the renormalized field (10.4.17), the electrodynamic Lagrangian takes the form

$$\mathscr{L} = -\tfrac{1}{4}Z_3(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu) + \mathscr{L}_M\left(\Psi_\ell, [\partial_\mu - iq_\ell A_\mu]\Psi_\ell\right) .$$

The function $\pi(q^2)$ in the one-photon-irreducible amplitude is then

$$\pi(q^2) = 1 - Z_3 + \pi_{\text{LOOP}}(q^2) , \tag{10.5.20}$$

where π_{LOOP} is the contribution of loop diagrams. It follows that

$$Z_3 = 1 + \pi_{\text{LOOP}}(0) . \tag{10.5.21}$$

In practice, we just calculate the loop contributions and subtract a constant in order to make $\pi(0)$ vanish.

Incidentally, Eq. (10.5.18) shows that for $q^2 \neq 0$ the gauge term in the photon propagator *is* altered by radiative corrections. The one exception is the case of Landau gauge, for which $\tilde{\xi} = \xi = 1$ for all q^2.

10.6 Electromagnetic Form Factors and Magnetic Moment

Suppose that we want to calculate the scattering of a particle by an external electromagnetic field (or by the electromagnetic field of another particle), to first order in this electromagnetic field, but to all orders in all other interactions (including electromagnetic) of our particle. For this purpose, we need to know the sum of the contributions of all Feynman diagrams with one incoming and one outgoing particle line, both on the mass shell, plus a photon line, which may be on or off the mass shell. According to the theorem of Section 6.4, this sum is given by the one-particle matrix element of the electromagnetic current $J^\mu(x)$. Let us see what governs the general form of this matrix element.

According to spacetime translation invariance, the one-particle matrix element of the electromagnetic current takes the form

$$\left(\Psi_{\mathbf{p}',\sigma'},\ J^\mu(x)\Psi_{\mathbf{p},\sigma}\right) = \exp(i(p-p')\cdot x)\left(\Psi_{\mathbf{p}',\sigma'}, J^\mu(0)\Psi_{\mathbf{p},\sigma}\right) . \tag{10.6.1}$$

The current conservation condition $\partial_\mu J^\mu = 0$ then requires

$$(p'-p)_\mu\left(\Psi_{\mathbf{p}',\sigma'},\ J^\mu(0)\Psi_{\mathbf{p},\sigma}\right) = 0 . \tag{10.6.2}$$

Also, setting $\mu = 0$ and integrating over all $\mathbf{x}$ gives

$$\left(\Psi_{\mathbf{p}',\sigma'},\ Q\Psi_{\mathbf{p},\sigma}\right) = (2\pi)^3\delta^3(\mathbf{p}-\mathbf{p}')\left(\Psi_{\mathbf{p}',\sigma'},\ J^0(0)\Psi_{\mathbf{p},\sigma}\right).$$

Using Eq. (10.4.8), this gives

$$\left(\Psi_{\mathbf{p},\sigma'},\ J^0(0)\Psi_{\mathbf{p},\sigma}\right) = (2\pi)^{-3} q\, \delta_{\sigma'\sigma}\,, \tag{10.6.3}$$

where q is the particle charge.

We also have at our disposal the constraints on the current matrix elements imposed by Lorentz invariance. To explore these, we will limit ourselves to the simplest cases: spin zero and spin $\frac{1}{2}$. The analysis presented here provides an example of techniques that are useful for other currents, such as those of the semi-leptonic weak interactions.

Spin Zero

For spin zero, Lorentz invariance requires the one-particle matrix element of the current to take the general form

$$\left(\Psi_{\mathbf{p}'},\ J^\mu(0)\Psi_{\mathbf{p}}\right) = q(2\pi)^{-3}(2p'^0)^{-1/2}(2p^0)^{-1/2}\mathscr{J}^\mu(p',p), \tag{10.6.4}$$

where p^0 and p'^0 are the mass-shell energies ($p^0 = \sqrt{\mathbf{p}^2+m^2}$), and $\mathscr{J}^\mu(p',p)$ is a four-vector function of the two four-vectors p'^μ and p^μ. (We have extracted a factor of the charge q of the particle from $\mathscr{J}$ for future convenience.) Obviously, the most general such four-vector function takes the form of a linear combination of p'^μ and p^μ, or equivalently of $p'^\mu + p^\mu$ and $p'^\mu - p^\mu$, with scalar coefficients. But the scalars p^2 and p'^2 are fixed at the values $p^2 = p'^2 = -m^2$, so the scalar variables that can be formed from p^μ and p'^μ can be taken as functions only of $p\cdot p'$, or equivalently of

$$k^2 \equiv (p-p')^2 = -2m^2 - 2p\cdot p'\,. \tag{10.6.5}$$

Thus the function $\mathscr{J}^\mu(p',p)$ must take the form

$$\mathscr{J}^\mu(p',p) = (p'+p)^\mu F(k^2) + i(p'-p)^\mu H(k^2)\,. \tag{10.6.6}$$

The fact that J^μ is Hermitian implies that $\mathscr{J}^\mu(p',p)^* = \mathscr{J}^\mu(p,p')$, so that both $F(k^2)$ and $H(k^2)$ are real.

Now $(p'-p)\cdot(p'+p)$ vanishes, while $(p'-p)^2 = k^2$ is not generally zero, so the condition of current conservation is simply

$$H(k^2) = 0\,. \tag{10.6.7}$$

Also, setting $\mathbf{p}' = \mathbf{p}$ and $\mu = 0$ in Eq. (10.6.4), and comparing with Eq. (10.6.3), we find that

$$F(0) = 1\,. \tag{10.6.8}$$

The function $F(k^2)$ is called the *electromagnetic form factor* of the particle.

Spin $\frac{1}{2}$

For spin $\frac{1}{2}$, Lorentz invariance requires the one-particle matrix element of the current to take the general form

$$\left(\Psi_{\mathbf{p}',\sigma'},\, J^\mu(0)\Psi_{\mathbf{p},\sigma}\right) = iq\,(2\pi)^{-3}\,\bar{u}(\mathbf{p}',\sigma')\Gamma^\mu(p',p)u(\mathbf{p},\sigma) \tag{10.6.9}$$

where Γ^μ is a four-vector 4×4 matrix function of p^ν, p'^ν, and γ^ν, and u is the usual Dirac coefficient function. We have extracted a factor iq to make the normalization of Γ^μ the same as in the previous section.

Just as for any 4×4 matrix, we may expand Γ^μ in the 16 covariant matrices 1, γ_ρ, $[\gamma_\rho,\gamma_\sigma]$, $\gamma_5\gamma_\rho$, and γ_5. The most general four-vector Γ^μ can therefore be written as a linear combination of

$$\begin{array}{rl}
1: & p^\mu\,,\ p'^\mu \\
\gamma_\rho: & \gamma^\mu\,,\ p^\mu\not{p}\,,\ p'^\mu\not{p}\,,\ p^\mu\not{p}'\,,\ p'^\mu\not{p}' \\
[\gamma_\rho,\gamma_\sigma]: & [\gamma^\mu,\ \not{p}],\ [\gamma^\mu,\ \not{p}'],\ [\not{p},\ \not{p}']p^\mu,\ [\not{p},\ \not{p}']p'^\mu \\
\gamma_5\gamma_\rho: & \gamma_5\gamma_\rho\,\epsilon^{\rho\mu\nu\sigma}p_\nu p'_\sigma \\
\gamma_5: & \text{NONE}
\end{array}$$

with the coefficient of each term a function of the only scalar variable in the problem, the quantity (10.6.5). This can be greatly simplified by using the Dirac equations satisfied by u and $\bar{u}$:

$$\bar{u}(\mathbf{p}',\sigma')\,(i\not{p}'+m)=0\,,\qquad (i\not{p}+m)\,u(\mathbf{p},\sigma)=0\,.$$

In consequence, we can drop* all but the first three entries: p^μ, p'^μ, and γ^μ. We conclude that, on the fermion mass shell, Γ^μ may be expressed as a linear combination of γ^μ, p^μ, and p'^μ, which we choose to write as

$$\begin{aligned}\bar{u}(\mathbf{p}',\sigma')\Gamma^\mu(p',p)u(\mathbf{p},\sigma) &= \bar{u}(\mathbf{p}',\sigma')\Big[\gamma^\mu F(k^2) \\ &\quad -\frac{i}{2m}\,(p+p')^\mu G(k^2)+\frac{(p-p')^\mu}{2m}H(k^2)\Big]u(\mathbf{p},\sigma).\end{aligned} \tag{10.6.10}$$

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* This is obvious for the terms $p^\mu\not{p}$, $p'^\mu\not{p}$, $p^\mu\not{p}'$, and $p'^\mu\not{p}'$, which may be replaced respectively with imp^μ, imp'^μ, imp^μ, and imp'^μ, which are the same as terms already on our list. Also, we can write

$$[\gamma^\mu,\ \not{p}] = 2\gamma^\mu\not{p}-\{\gamma^\mu,\ \not{p}\} = 2\gamma^\mu\not{p}-2p^\mu\,,$$

which may be replaced with $2im\gamma^\mu-2p^\mu$, a linear combination of terms already on our list. The same applies to $[\gamma^\mu,\ \not{p}']$. Also,

$$[\not{p},\ \not{p}'] = -2\not{p}'\not{p}+\{\not{p},\ \not{p}'\} = -2\not{p}'\not{p}+2p\cdot p'\,,$$

which may be replaced with $2m^2+2p\cdot p' = -k^2$. Hence the terms $[\not{p},\ \not{p}']p^\mu$ and $[\not{p},\ \not{p}']p'^\mu$ give nothing new. Finally, to deal with the last term we may use the relation

$$\gamma_5\gamma_\rho\,\epsilon^{\rho\mu\nu\sigma} = \tfrac{1}{6}i\Big(\gamma^\mu\gamma^\nu\gamma^\sigma+\gamma^\sigma\gamma^\mu\gamma^\nu+\gamma^\nu\gamma^\sigma\gamma^\mu-\gamma^\nu\gamma^\mu\gamma^\sigma-\gamma^\mu\gamma^\sigma\gamma^\nu-\gamma^\sigma\gamma^\nu\gamma^\mu\Big)\,.$$

Contracting this with p_ν and p'_σ and then moving all $\not{p}$ factors to the right and $\not{p}'$ factors to the left again gives a linear combination of p^μ, p'^μ, and γ^μ.

The hermiticity of $J^\mu(0)$ implies that

$$\beta\,\Gamma^{\mu\dagger}(p',p)\beta = -\Gamma^\mu(p,p')\,, \tag{10.6.11}$$

so that $F(k^2)$, $G(k^2)$, and $H(k^2)$ must all be *real* functions of k^2.

The conservation condition (10.6.2) is automatically satisfied by the first two terms in Eq. (10.6.10), because

$$(p'-p)_\mu\gamma^\mu = -i\left[(i\not{p}'+m)-(i\not{p}+m)\right]$$

and

$$(p'-p)\cdot(p'+p) = p'^2 - p^2\,.$$

On the other hand, $(p'-p)^2$ does not in general vanish, so current conservation requires the third term to vanish

$$H(k^2) = 0\,. \tag{10.6.12}$$

Also, letting $\mathbf{p}' \to \mathbf{p}$ in Eqs. (10.6.9) and (10.6.10), we find

$$\left(\Psi_{\mathbf{p},\sigma'}J^\mu(0)\Psi_{\mathbf{p},\sigma}\right) = i\,q(2\pi)^{-3}\bar{u}(\mathbf{p},\sigma')\left[\gamma^\mu F(0) - \frac{i}{m}\,p^\mu G(0)\right]u(\mathbf{p},\sigma)\,.$$

Using the identity $\{\gamma^\mu, i\not{p}+m\} = 2m\gamma^\mu + 2ip^\mu$, we also have

$$\bar{u}(\mathbf{p},\sigma')\gamma^\mu u(\mathbf{p},\sigma) = -\frac{ip^\mu}{m}\,\bar{u}(\mathbf{p},\sigma')u(\mathbf{p},\sigma)\,.$$

Recall also that

$$\bar{u}(\mathbf{p},\sigma')u(\mathbf{p},\sigma) = \delta_{\sigma'\sigma}m/p^0$$

and therefore

$$\left(\Psi_{\mathbf{p},\sigma'},\, J^\mu(0)\Psi_{\mathbf{p},\sigma}\right) = q(2\pi)^{-3}(p^\mu/p^0)\delta_{\sigma'\sigma}\left[F(0)+G(0)\right]\,. \tag{10.6.13}$$

Comparing this with Eq. (10.6.3) yields the normalization condition

$$F(0)+G(0) = 1\,. \tag{10.6.14}$$

It may be useful to note that the electromagnetic vertex matrix Γ^μ is commonly written in terms of two other matrices, as

$$\begin{aligned}\bar{u}(\mathbf{p}',\sigma')\Gamma^\mu(p',p)u(\mathbf{p},\sigma) = \bar{u}(\mathbf{p}',\sigma')\,\Big[&\gamma^\mu F_1(k^2)\\ &+\tfrac{1}{2}i\,[\gamma^\mu,\gamma^\nu]\,(p'-p)_\nu F_2(k^2)\Big]\,u(\mathbf{p},\sigma)\,.\end{aligned} \tag{10.6.15}$$

As already mentioned, we may rewrite the matrix appearing in the second term in terms of those used in defining $F(k^2)$ and $G(k^2)$:

$$\begin{aligned}&\bar{u}(\mathbf{p}',\sigma')\;\tfrac{1}{2}i\,[\gamma^\mu,\gamma^\nu]\,(p'-p)_\nu\,u(\mathbf{p},\sigma)\\ &\quad= \bar{u}(\mathbf{p}',\sigma')\left[-i\not{p}'\gamma^\mu + \tfrac{1}{2}i\{\gamma^\mu,\not{p}'\} - i\gamma^\mu\not{p} + \tfrac{1}{2}i\{\gamma^\mu,\not{p}\}\right]u(\mathbf{p},\sigma)\\ &\quad= \bar{u}(\mathbf{p}',\sigma')\left[i(p'^\mu+p^\mu)+2m\gamma^\mu\right]u(\mathbf{p},\sigma)\,.\end{aligned} \tag{10.6.16}$$

Comparing Eq. (10.6.15) with Eq. (10.6.10), we find

$$F(k^2) = F_1(k^2) + 2m\,F_2(k^2) \tag{10.6.17}$$

$$G(k^2) = -2m\,F_2(k^2)\,. \tag{10.6.18}$$

The normalization condition (10.6.14) now reads

$$F_1(0) = 1\,.$$

In order to evaluate the magnetic moment of our particle in terms of its form factors, let us consider the spatial part of the vertex function in the case of small momenta, $|\mathbf{p}|, |\mathbf{p}'| \ll m$. For this purpose, it is useful to use Eq. (10.6.16) to rewrite Eq. (10.6.10) (with $H = 0$) in a third form:

$$\bar{u}(\mathbf{p}',\sigma')\Gamma^\mu(p',p)u(\mathbf{p},\sigma) = \frac{-i}{2m}\bar{u}(\mathbf{p}',\sigma')\left[(p+p')^\mu\{F(k^2)+G(k^2)\}\right.$$
$$\left. - \frac{1}{2}[\gamma^\mu,\gamma^\nu](p'-p)_\nu F(k^2)\right]u(\mathbf{p},\sigma)\,. \tag{10.6.19}$$

For zero momenta the matrix elements of the commutators of Dirac matrices are given by (5.4.19) and (5.4.20) as

$$\bar{u}(0,\sigma')[\gamma^i,\gamma^j]u(0,\sigma) = 4i\epsilon_{ijk}\left(J_k^{(\frac{1}{2})}\right)_{\sigma',\sigma}\,, \qquad \bar{u}(0,\sigma')[\gamma^i,\gamma^0]u(0,\sigma) = 0\,,$$

where $\mathbf{J}^{(\frac{1}{2})} = \frac{1}{2}\boldsymbol{\sigma}$ is the angular momentum matrix for spin $\frac{1}{2}$. Hence to first order in the small momenta,

$$\bar{u}(\mathbf{p}',\sigma')\boldsymbol{\Gamma}(p',p)u(\mathbf{p},\sigma) \to \frac{-i}{2m}(\mathbf{p}+\mathbf{p}')\delta_{\sigma',\sigma} + \frac{1}{m}[(\mathbf{p}-\mathbf{p}')\times\mathbf{J}^{(\frac{1}{2})}]_{\sigma'\sigma}\,F(0)\,. \tag{10.6.20}$$

In a very weak time-independent external vector potential $A(\mathbf{x})$ the matrix element of the interaction Hamiltonian $H' = -\int d^3x\,\mathbf{J}(\mathbf{x})\cdot A(\mathbf{x})$ between one-particle states of small momentum is therefore

$$(\Psi_{\mathbf{p}',\sigma'}, H'\Psi_{\mathbf{p},\sigma}) = \frac{-iqF(0)}{m(2\pi)^3}\int d^3x\, e^{i(\mathbf{p}-\mathbf{p}')\cdot\mathbf{x}}A(\mathbf{x})\cdot[(\mathbf{p}-\mathbf{p}')\times\mathbf{J}^{(\frac{1}{2})}]_{\sigma'\sigma}$$
$$= -\frac{qF(0)}{m(2\pi)^3}\int d^3x\, e^{i(\mathbf{p}-\mathbf{p}')\cdot\mathbf{x}}(\mathbf{J}^{(\frac{1}{2})})_{\sigma'\sigma}\cdot\mathbf{B}(\mathbf{x})\,, \tag{10.6.21}$$

where $\mathbf{B} = \nabla\times A$ is the magnetic field. Hence in the limit of a slowly varying weak magnetic field, the matrix element of the interaction Hamiltonian is

$$(\Psi_{\mathbf{p}',\sigma'}, H'\Psi_{\mathbf{p},\sigma}) = -\frac{qF(0)}{m}(\mathbf{J}^{(\frac{1}{2})})_{\sigma'\sigma}\cdot\mathbf{B}\,\delta^3(\mathbf{p}-\mathbf{p}')\,. \tag{10.6.22}$$

The magnetic moment μ for an arbitrary particle of general spin j is defined by the statement that the matrix element of the interaction of the particle with a weak static slowly varying magnetic field is

$$(\Psi_{\mathbf{p}',\sigma'}, H'\Psi_{\mathbf{p},\sigma}) = -\frac{\mu}{j}(\mathbf{J}^{(j)})_{\sigma'\sigma}\cdot\mathbf{B}\,\delta^3(\mathbf{p}-\mathbf{p}')\,. \tag{10.6.23}$$

Hence Eq. (10.6.22) gives the magnetic moment of a particle of charge q, mass m, and spin $\frac{1}{2}$ as:

$$\mu = \frac{qF(0)}{2m} . \tag{10.6.24}$$

This contains as a special case the celebrated Dirac result[7] $\mu = q/2m$ for a spin $\frac{1}{2}$ particle without radiative corrections.

We mention without proof that the form factors $F(k^2)$ and $G(k^2)$ of the proton may be measured for $k^2 > 0$ by comparison of experimental data for electron–proton scattering with the Rosenbluth formula[8] for the laboratory frame differential cross-section:

$$\frac{d\sigma}{d\Omega} = \frac{e^4}{4(4\pi)^2 E_0^2} \frac{\cos^2(\theta/2)}{\sin^4(\theta/2)} \left[1 + \frac{2E_0}{m} \sin^2(\theta/2)\right]^{-1}$$
$$\times \left\{ \left(F(k^2) + G(k^2)\right)^2 + \frac{k^2}{4m^2} \left(2F^2(k^2)\tan^2(\theta/2) + G^2(k^2)\right) \right\} ,$$

where E_0 is the energy of the incident electron (taken here with $E_0 \gg m_e$); θ is the scattering angle; and

$$k^2 = \frac{4E_0^2 \sin^2(\theta/2)}{1 + (2E_0/m)\sin^2(\theta/2)} .$$

10.7 The Källen–Lehmann Representation*

We saw in Section 10.2 that the presence of one-particle intermediate states leads to poles in Fourier transforms of matrix elements of time-ordered products, like (10.2.1). Multi-particle intermediate states lead to more complicated singularities, which are difficult to describe in general. But in the special case of a vacuum expectation value involving just two operators, we have a convenient representation that explicitly displays the analytic structure of the Fourier transform. In particular, this representation may be used for propagators, where the two operators are the fields of elementary particles. When combined with the positivity requirements of quantum mechanics, this representation yields interesting bounds on the asymptotic behavior of propagators and the magnitude of renormalization constants.

Consider a complex scalar Heisenberg-picture operator $\Phi(x)$, which may or may not be an elementary particle field. The vacuum expectation

* This section lies somewhat out of the book's main line of development, and may be omitted in a first reading.

value of a product $\Phi(x)\Phi^\dagger(y)$ may be expressed as

$$\langle\Phi(x)\Phi^\dagger(y)\rangle_0 = \sum_n \langle 0|\Phi(x)|n\rangle\ \langle n|\Phi^\dagger(y)|0\rangle\ , \tag{10.7.1}$$

where the sum runs over any complete set of states. (Here the sum over n includes integrals over continuous labels as well as sums over discrete labels.) Choosing these states as eigenstates of the momentum four-vector P^μ, translational invariance tells us that

$$\begin{aligned}\langle 0|\Phi(x)|n\rangle &= \exp(ip_n\cdot x)\langle 0|\Phi(0)|n\rangle\ ,\\ \langle n|\Phi^\dagger(y)|0\rangle &= \exp(-ip_n\cdot y)\langle n|\Phi^\dagger(0)|0\rangle\end{aligned} \tag{10.7.2}$$

and so

$$\langle\Phi(x)\Phi^\dagger(y)\rangle_0 = \sum_n \exp(ip_n\cdot(x-y))\,|\langle 0|\Phi(0)|n\rangle|^2\ . \tag{10.7.3}$$

It is convenient to rewrite this in terms of a *spectral function*. Note that the sum $\sum_n \delta^4(p-p_n)\,|\langle 0|\Phi(0)|n\rangle|^2$ is a scalar function of the four-vector p^μ, and therefore may depend only on p^2 and (for $p^2 \le 0$) on the step function $\theta(p^0)$. In fact, the intermediate states in Eq. (10.7.3) all have $p^2 \le 0$ and $p^0 > 0$, so this sum takes the form

$$\sum_n \delta^4(p-p_n)\,|\langle 0|\Phi(0)|n\rangle|^2 = (2\pi)^{-3}\,\theta(p^0)\,\rho(-p^2) \tag{10.7.4}$$

with $\rho(-p^2) = 0$ for $p^2 > 0$. (The factor $(2\pi)^{-3}$ is extracted from ρ for future convenience.) The spectral function $\rho(-p^2)$ is clearly real and positive. With this definition, we can rewrite Eq. (10.7.3) as

$$\begin{aligned}\langle\Phi(x)\Phi^\dagger(y)\rangle_0 &= (2\pi)^{-3}\int d^4p\ \exp[ip\cdot(x-y)]\ \theta(p^0)\,\rho(-p^2)\\ &= (2\pi)^{-3}\int d^4p\int_0^\infty d\mu^2\ \exp[ip\cdot(x-y)]\,\theta(p^0)\\ &\qquad\times\rho(\mu^2)\,\delta(p^2+\mu^2)\ .\end{aligned} \tag{10.7.5}$$

Interchanging the order of integration over p^μ and μ^2, this may be expressed as

$$\langle\Phi(x)\Phi^\dagger(y)\rangle_0 = \int_0^\infty d\mu^2\ \rho(\mu^2)\,\Delta_+(x-y;\mu^2)\ , \tag{10.7.6}$$

where Δ_+ is the familiar function

$$\Delta_+(x-y;\mu^2) \equiv (2\pi)^{-3}\int d^4p\ \exp[ip\cdot(x-y)]\,\theta(p^0)\,\delta(p^2+\mu^2)\ . \tag{10.7.7}$$

In just the same way, we can show that

$$\langle\Phi^\dagger(y)\Phi(x)\rangle_0 = \int_0^\infty d\mu^2\ \bar\rho(\mu^2)\,\Delta_+(y-x;\mu^2) \tag{10.7.8}$$

with a second spectral function $\bar{\rho}(\mu^2)$ defined by

$$\sum_n \delta^4(p - p_n)\,|\langle n|\Phi(0)|0\rangle|^2 = (2\pi)^{-3}\,\theta(p^0)\,\bar{\rho}(-p^2)\,. \tag{10.7.9}$$

We now make use of the causality requirement, that the commutator $[\Phi(x), \Phi^\dagger(y)]$ must vanish for space-like separations $x - y$. The vacuum expectation value of the commutator is

$$\langle[\Phi(x), \Phi^\dagger(y)]\rangle_0 = \int_0^\infty d\mu^2 \left(\rho(\mu^2)\,\Delta_+(x-y;\mu^2) - \bar{\rho}(\mu^2)\,\Delta_+(y-x;\mu^2)\right)\,. \tag{10.7.10}$$

As noted in Section 5.2, for $x - y$ space-like the function $\Delta_+(x - y)$ does not vanish, but it does become *even*. In order for (10.7.10) to vanish for arbitrary space-like separations, it is thus necessary that

$$\rho(\mu^2) = \bar{\rho}(\mu^2)\,. \tag{10.7.11}$$

This is a special case of the CPT theorem, proved here without the use of perturbation theory; for whatever states with $p^2 = -\mu^2$ have the quantum numbers of the operator Φ, there must be corresponding states with $p^2 = -\mu^2$ that have the quantum numbers of the operator $\Phi^\dagger$.

Using Eq. (10.7.11), the vacuum expectation of the time-ordered product is

$$\langle T\left\{\Phi(x)\Phi^\dagger(y)\right\}\rangle_0 = -i\int_0^\infty d\mu^2\,\rho(\mu^2)\,\Delta_F(x-y;\mu^2)\,, \tag{10.7.12}$$

where $\Delta_F(x - y;\mu^2)$ is the Feynman propagator for a spinless particle of mass μ:

$$-i\Delta_F(x-y;\mu^2) \equiv \theta(x^0-y^0)\Delta_+(x-y;\mu^2) - \theta(y^0-x^0)\Delta_+(y-x;\mu^2)\,. \tag{10.7.13}$$

Borrowing the notation introduced in Section 10.3 for complete propagators, we introduce the momentum space function

$$-i\Delta'(p) \equiv \int d^4x\,\exp[-ip\cdot(x-y)]\,\langle T\left\{\Phi(x)\Phi^\dagger(y)\right\}\rangle_0\,. \tag{10.7.14}$$

Recall that

$$\int d^4x\,\exp[-ip\cdot(x-y)]\,\Delta_F(x-y;\mu^2) = \frac{1}{p^2+\mu^2-i\epsilon}\,. \tag{10.7.15}$$

This yields our spectral representation:[9]

$$\Delta'(p) = \int_0^\infty \rho(\mu^2)\,\frac{d\mu^2}{p^2+\mu^2-i\epsilon}\,. \tag{10.7.16}$$

One immediate consequence of this result and the positivity of $\rho(\mu^2)$ is

that $\Delta'(p)$ cannot vanish for $|p^2| \to \infty$ faster** than the bare propagator $1/(p^2 + m^2 - i\epsilon)$. From time to time the suggestion is made to include higher derivative terms in the unperturbed Lagrangian, which would make the propagator vanish faster than $1/p^2$ for $|p^2| \to \infty$, but the spectral representation shows that this would necessarily entail a departure from the positivity postulates of quantum mechanics.

We can use the spectral representation together with equal-time commutation relations to derive an interesting sum rule for the spectral function. If $\Phi(x)$ is a conventionally normalized (not renormalized) canonical field operator, then

$$\left[\frac{\partial\Phi(\mathbf{x},t)}{\partial t}, \Phi^\dagger(\mathbf{y},t)\right] = -i\delta^3(\mathbf{x}-\mathbf{y}) . \tag{10.7.17}$$

We note that

$$\frac{\partial}{\partial x^0}\Delta_+(x-y)\bigg|_{x^0=y^0} = -i\delta^3(\mathbf{x}-\mathbf{y})$$

so the spectral representation (10.7.10) and the commutation relations (10.7.17) together tell us that

$$\int_0^\infty \rho(\mu^2)\, d\mu^2 = 1 . \tag{10.7.18}$$

This implies that for $|p^2| \to \infty$, the momentum space propagator (10.7.16) of the unrenormalized fields has the free-field asymptotic behavior

$$\Delta'(p) \to \frac{1}{p^2} .$$

This result is only meaningful within a suitable scheme for regulating ultraviolet divergences; in perturbation theory the unrenormalized fields have infinite matrix elements, and their propagator is ill-defined.

Now consider the possibility that there is a one-particle state $|\mathbf{k}\rangle$ of mass m with a non-vanishing matrix element with the state $\langle 0|\Phi(0)$. Lorentz

** In fact, it is not even certain that $\Delta'(p)$ vanishes for $|p^2| \to \infty$ at all, even though this would seem to follow from the spectral representation. The problem arises from the interchange of the integrals over p^μ and μ^2. What is certain is that $\Delta'(p)$ is an analytic function of $-p^2$ with a discontinuity across the positive real axis $-p^2 = \mu^2$ given by $\pi\rho(\mu^2)$, as can be shown by the methods of the next section. From this, it follows that $\Delta'(p)$ is given by a dispersion relation with spectral function $\rho(\mu^2)$ and possible subtractions:

$$\Delta'(p) = P(p^2) + (-p^2 + \mu_0^2)^n \int_0^\infty \frac{\rho(\mu^2)}{(\mu^2+\mu_0^2)^n} \frac{d\mu^2}{p^2+\mu^2-i\epsilon} ,$$

where n is a positive integer, μ_0^2 is an arbitrary positive constant, and $P(p^2)$ is a μ_0^2-dependent polynomial in p^2 of order $n-1$ that is absent for $n = 0$.

invariance requires this matrix element to take the form

$$\langle 0|\Phi(0)|\mathbf{k}\rangle = (2\pi)^{-3/2}\left(2\sqrt{\mathbf{k}^2+m^2}\right)^{-1/2} N\,, \tag{10.7.19}$$

where N is a constant. According to the general results of Section 10.3, the propagator $\Delta'(p)$ of the unrenormalized fields should have a pole at $p^2 \to -m^2$ with residue $Z \equiv |N|^2 > 0$. That is,

$$\rho(\mu^2) = Z\,\delta(\mu^2 - m^2) + \sigma(\mu^2)\,, \tag{10.7.20}$$

where $\sigma(\mu^2) \geq 0$ is the contribution of multi-particle states. Together with Eq. (10.7.18), this has the consequence that

$$1 = Z + \int_0^\infty \sigma(\mu^2)\,d\mu^2 \tag{10.7.21}$$

and so

$$Z \leq 1 \tag{10.7.22}$$

with the equality reached only for a free particle, for which $\langle 0|\Phi(x)$ has no matrix elements with multi-particle states.

Because Z is positive, Eq. (10.7.21) can also be regarded as providing an upper bound on the coupling of the field Φ to multi-particle states:

$$\int_0^\infty \sigma(\mu^2)\,d\mu^2 \leq 1 \tag{10.7.23}$$

with the equality reached for $Z = 0$. The limit $Z = 0$ has an interesting interpretation as a condition for a particle to be composite rather than elementary.[10] In this context, a 'composite' particle may be understood to be one whose field does not appear in the Lagrangian. Consider such a particle, say a neutral particle of spin zero, and suppose that its quantum numbers allow it to be destroyed by an operator $F(\Psi)$ constructed out of other fields. We can freely introduce a field Φ for this particle by adding a term to the Lagrangian density of the form† $\Delta\mathscr{L} = (\Phi - F(\Psi))^2$, because the path integral over Φ can be done by setting it equal to the stationary point $\Phi = F(\Psi)$, at which $\Delta\mathscr{L} = 0$. But suppose instead we write $\Delta\mathscr{L} = \Delta\mathscr{L}_0 + \Delta\mathscr{L}_1$, where $\Delta\mathscr{L}_0 \equiv -\frac{1}{2}\partial_\mu\Phi\partial^\mu\Phi - \frac{1}{2}m^2\Phi^2$ is the usual free-field Lagrangian, and treat $\Delta\mathscr{L}_1 \equiv \Delta\mathscr{L} - \Delta\mathscr{L}_0$ as an interaction. A term $\frac{1}{2}\partial_\mu\Phi\partial^\mu\Phi$ in the interaction is nothing new. We encountered such a term in Eq. (10.3.12), multiplied by a factor $(1-Z)$; the only new thing is that here $Z = 0$. Instead of adjusting Z to satisfy the field renormalization condition $\Pi^{*\prime}(0) = 0$, here we must regard this as a condition on the

† This is known in condensed matter physics as a 'Hubbard–Stratonovich transformation'.[11] It will be used to introduce fields for pairs of electrons in our discussion of superconductivity in Volume II.

coupling constants of the composite particle. Unfortunately, it has not been possible to implement this procedure in quantum field theories, because as we have seen $Z = 0$ means that the particle couples as strongly as possible to its constituents, and this rules out the use of perturbation theory. The condition $Z = 0$ does prove useful in non-relativistic quantum mechanics; for instance, it fixes the coupling of the deuteron to the neutron and proton.[12]

Although the spectral representation has been derived here only for a spinless field, it is easy to generalize these results to other fields. Indeed, in the next chapter we shall show that to order e^2, the Z-factor for the electromagnetic field (conventionally called Z_3) is given by

$$Z_3 = 1 - \frac{e^2}{12\pi^2} \ln\left(\frac{\Lambda^2}{m_e^2}\right)$$

(where $\Lambda \gg m_e$ is an ultraviolet cutoff), in agreement with the bound (10.7.22).

10.8 Dispersion Relations*

The failure of early attempts to apply perturbative quantum field theory to the strong and weak nuclear forces had led theorists by the late 1950s to attempt the use of the analyticity and unitarity of scattering amplitudes as a way of deriving general non-perturbative results that would not depend on any particular field theory. This started with a revival of interest in dispersion relations. In its original form,[13] a dispersion relation was a formula giving the real part of the index of refraction in terms of an integral over its imaginary part. It was derived from an analyticity property of the index of refraction as a function of frequency, which followed from the condition that electromagnetic signals in a medium cannot travel faster than light in a vacuum. By expressing the index of refraction in terms of the forward photon scattering amplitude, the dispersion relation could be rewritten as a formula for the real part of the forward scattering amplitude as an integral of its imaginary part, and hence via unitarity in terms of the total cross-section. One of the exciting things about this relation was that it provided an alternative to conventional perturbation theory; given the scattering amplitude to order e^2, one could calculate the cross-section and the imaginary part of the scattering amplitude to order e^4, and then use the dispersion relation to

* This section lies somewhat out of the book's main line of development, and may be omitted in a first reading.

calculate the real part of the forward scattering amplitude to this order, without having ever to calculate a loop graph.

The modern approach to dispersion relations began in 1954 with the work of Gell-Mann, Goldberger, and Thirring.[14] Instead of considering the propagation of light in a medium, they derived the analyticity of the scattering amplitude directly from the condition of microscopic causality, which states that commutators of field operators vanish when the points at which the operators are evaluated are separated by a space-like interval. This approach allowed Goldberger[15] soon thereafter to derive a very useful dispersion relation for the forward pion–nucleon scattering amplitude.

To see how to use the principle of microscopic causality, consider the forward scattering in the laboratory frame of a massless boson of any spin on an arbitrary target α of mass $m_\alpha > 0$ and $\mathbf{p}_\alpha = 0$. (This has important applications to the scattering not only of photons but also pions in the limit $m_\pi = 0$, to be discussed in Volume II.) By a repeated use of Eq. (10.3.4) or the Lehmann–Symanzik–Zimmerman theorem,[3] the S-matrix element here is

$$S = \frac{1}{(2\pi)^3\sqrt{4\omega\omega'}|N|^2} \lim_{k^2\to 0} \lim_{k'^2\to 0} \times \int d^4x \int d^4y\, e^{-ik'\cdot y} e^{ik\cdot x}(i\Box_y)(i\Box_x)\, \langle\alpha|T\{A^\dagger(y), A(x)\}|\alpha\rangle\,. \quad (10.8.1)$$

Here k and k' are the initial and final boson four-momenta, with $\omega = k^0$, $\omega' = k'^0$; $A(x)$ is any Heisenberg-picture operator with a non-vanishing matrix element $\langle \mathrm{VAC}|A(x)|k\rangle = (2\pi)^{-3/2}(2\omega)^{-1/2}Ne^{ik\cdot x}$ between the one-boson state $|k\rangle$ and the vacuum; and N is the constant in this matrix element. In photon scattering $A(x)$ would be one of the transverse components of the electromagnetic field, while for massless pion scattering it would be a pseudoscalar function of hadron fields. The differential operators $-i\Box_x$ and $-i\Box_y$ are inserted to supply factors of ik'^2 and ik^2 that are needed to cancel the external line boson propagators. Letting these operators act on $A^\dagger(y)$ and $A(x)$, we have

$$S = \frac{-1}{(2\pi)^3\sqrt{4\omega\omega'}|N|^2} \lim_{k^2\to 0} \lim_{k'^2\to 0} \times \int d^4x \int d^4y\, e^{-ik'\cdot y} e^{ik\cdot x}\langle\alpha|T\{J^\dagger(y), J(x)\}|\alpha\rangle + \mathrm{ETC}\,, \quad (10.8.2)$$

where $J(x) \equiv \Box_x A(x)$, and 'ETC' denotes the Fourier transform of equal time commutator terms arising from the derivative acting on the step functions in the time-ordered product. The commutators of operators like $A(x)$ and $A^\dagger(y)$ (or their derivatives) vanish for $x^0 = y^0$ unless $\mathbf{x} = \mathbf{y}$, so the 'ETC' term is the Fourier transform of a differential operator acting on $\delta^4(x-y)$, and is hence a polynomial function of the boson

four-momenta. We are concerned here with the analytic properties of the S-matrix element, so the details of this polynomial will be irrelevant.

Using translation invariance, Eq. (10.8.2) gives the S-matrix element as $S = -2\pi i\delta^4(k' - k)M(\omega)$, where

$$M(\omega) = \frac{-i}{2\omega|N|^2}\, F(\omega)\,, \tag{10.8.3}$$

$$F(\omega) \equiv \int d^4x\, e^{i\omega\,\ell\cdot x}\langle\alpha|T\{J^\dagger(0)\,,\, J(x)\}|\alpha\rangle \;+\; \text{ETC}\,, \tag{10.8.4}$$

it now being understood that $k^\mu = \omega\ell^\mu$, where ℓ is a fixed four-vector with $\ell^\mu\ell_\mu = 0$ and $\ell^0 = 1$.

The time-ordered product can be rewritten in terms of commutators in two different ways:

$$\begin{aligned} T\{J^\dagger(0), J(x)\} &= \theta(-x^0)[J^\dagger(0), J(x)] + J(x)J^\dagger(0) \\ &= -\theta(x^0)[J^\dagger(0), J(x)] + J^\dagger(0)J(x)\,. \end{aligned} \tag{10.8.5}$$

Correspondingly, we can write

$$F(\omega) = F_A(\omega) + F_+(\omega) = F_R(\omega) + F_-(\omega)\,, \tag{10.8.6}$$

where

$$F_A(\omega) \equiv \int d^4x\, \theta(-x^0)\, \langle\alpha|[J^\dagger(0)\,,\, J(x)]|\alpha\rangle\, e^{i\omega\,\ell\cdot x} \;+\; \text{ETC}\,, \tag{10.8.7}$$

$$F_R(\omega) \equiv -\int d^4x\, \theta(x^0)\, \langle\alpha|[J^\dagger(0)\,,\, J(x)]|\alpha\rangle\, e^{i\omega\,\ell\cdot x} \;+\; \text{ETC}\,, \tag{10.8.8}$$

$$F_+(\omega) \equiv \int d^4x\, \langle\alpha|J(x)\, J^\dagger(0)|\alpha\rangle\, e^{i\omega\,\ell\cdot x}\,, \tag{10.8.9}$$

$$F_-(\omega) \equiv \int d^4x\, \langle\alpha|J^\dagger(0)\, J(x)|\alpha\rangle\, e^{i\omega\,\ell\cdot x}\,. \tag{10.8.10}$$

Microscopic causality tells us that the integrands in (10.8.7) and (10.8.8) vanish unless x^μ is within the light cone, and the step functions then require that x^μ is in the backward light cone in (10.8.7), so that $x\cdot\ell > 0$, and in the forward light cone in Eq. (10.8.8), so that $x\cdot\ell < 0$. We conclude that $F_A(\omega)$ is analytic for $\operatorname{Im}\omega > 0$ and $F_R(\omega)$ is analytic for $\operatorname{Im}\omega < 0$, because in both cases the factor $e^{i\omega\,\ell\cdot x}$ provides a cutoff for the integral over x^μ. (Recall that the 'ETC' term is a polynomial, and hence analytic at all finite points.) We may then define a function

$$\mathscr{F}(\omega) \equiv \begin{cases} F_A(\omega) & \operatorname{Im}\omega > 0 \\ F_R(\omega) & \operatorname{Im}\omega < 0 \end{cases} \tag{10.8.11}$$

which is analytic in the whole complex ω plane, except for a cut on the real axis.

We can now derive the dispersion relation. According to Eq. (10.8.6), the discontinuity of $\mathscr{F}(\omega)$ across the cut at any real E is

$$\mathscr{F}(E+i\epsilon)-\mathscr{F}(E-i\epsilon)=F_A(E)-F_R(E)=F_-(E)-F_+(E)\,. \tag{10.8.12}$$

If $\mathscr{F}(\omega)/\omega^n$ vanishes as $|\omega|\to\infty$ in the upper or lower half-plane, then by dividing by any polynomial $P(\omega)$ of order n we obtain a function that vanishes for $|\omega|\to\infty$ and is analytic except for the cut on the real axis and poles at the zeroes ω_ν of $P(\omega)$. (Where $\mathscr{F}(\omega)$ itself vanishes as $|\omega|\to\infty$, we can take $P(\omega)=1$.) According to the method of residues, we then have

$$\frac{\mathscr{F}(\omega)}{P(\omega)}+\sum_\nu\frac{\mathscr{F}(\omega_\nu)}{(\omega_\nu-\omega)P'(\omega_\nu)}=\frac{1}{2\pi i}\oint_C\frac{\mathscr{F}(z)\,dz}{(z-\omega)\,P(z)}\,, \tag{10.8.13}$$

where ω is any point off the real axis, and C is a contour consisting of two segments: one running just above the real axis from $-\infty+i\epsilon$ to $+\infty+i\epsilon$ and then around a large semi-circle back to $-\infty+i\epsilon$, and the other just below the real axis from $+\infty-i\epsilon$ to $-\infty-i\epsilon$ and then around a large semi-circle back to $+\infty-i\epsilon$. Because the function $\mathscr{F}(z)/P(z)$ vanishes for $|z|\to\infty$, we can neglect the contribution from the large semi-circles. Using Eq. (10.8.12), Eq. (10.8.13) becomes

$$\mathscr{F}(\omega)=Q(\omega)+\frac{P(\omega)}{2\pi i}\int_{-\infty}^{+\infty}\frac{F_-(E)-F_+(E)}{(E-\omega)\,P(E)}\,dE\,, \tag{10.8.14}$$

where $Q(\omega)$ is the $(n-1)$th-order polynomial

$$Q(\omega)\equiv-P(\omega)\sum_\nu\frac{\mathscr{F}(\omega_\nu)}{(\omega_\nu-\omega)\,P'(\omega_\nu)}\,.$$

A dispersion relation of this form, with $P(\omega)$ and $Q(\omega)$ of order n and $n-1$ respectively, is said to have n *subtractions*. If we can take $P=1$ then $Q=0$, and the dispersion relation is said to be unsubtracted.

If we now let ω approach the real axis from above, Eq. (10.8.14) gives

$$F_A(\omega)=Q(\omega)+\frac{P(\omega)}{2\pi i}\int_{-\infty}^{+\infty}\frac{F_-(E)-F_+(E)}{(E-\omega-i\epsilon)\,P(E)}\,dE\,. \tag{10.8.15}$$

Recalling Eqs. (10.8.6) and (3.1.25), this is

$$F(\omega)=Q(\omega)+\frac{1}{2}F_-(\omega)+\frac{1}{2}F_+(\omega)+\frac{P(\omega)}{2\pi i}\int_{-\infty}^{+\infty}\frac{F_-(E)-F_+(E)}{(E-\omega)\,P(E)}\,dE \tag{10.8.16}$$

with $1/(E-\omega)$ now interpreted as the principal value function $\mathscr{P}/(E-\omega)$.

This result is useful because the functions $F_\pm(E)$ may be expressed in terms of measurable cross-sections. Summing over a complete set of multi-particle intermediate states β in Eqs. (10.8.9) and (10.8.10) (including integrations over the momenta of the particles in β) and using translation invariance again, we have

$$F_+(E) = (2\pi)^4 \sum_\beta \left|\langle\beta|J(0)^\dagger|\alpha\rangle\right|^2 \delta^4(-p_\alpha + E\ell + p_\beta)\,, \tag{10.8.17}$$

$$F_-(E) = (2\pi)^4 \sum_\beta |\langle\beta|J(0)|\alpha\rangle|^2 \delta^4(p_\alpha + E\ell - p_\beta)\,. \tag{10.8.18}$$

But the matrix elements for the absorption of the massless scalar boson B in $B + \alpha \to \beta$ or its antiparticle B^c in $B^c + \alpha \to \beta$ are

$$-2i\pi M_{B^c+\alpha\to\beta} = \frac{(2\pi)^4}{(2\pi)^{3/2}\sqrt{2E_{B^c}}N}\langle\beta|J^\dagger(0)|\alpha\rangle\,, \tag{10.8.19}$$

$$-2i\pi M_{B+\alpha\to\beta} = \frac{(2\pi)^4}{(2\pi)^{3/2}\sqrt{2E_B}N}\langle\beta|J(0)|\alpha\rangle\,. \tag{10.8.20}$$

Comparing with Eq. (3.4.15), we see that $F_\pm(E)$ may be expressed in terms of total cross-sections** at energies $\mp E$:

$$F_+(E) = \theta(-E)\frac{2|E||N|^2}{(2\pi)^3}\sigma_{\alpha+B^c}(|E|)\,, \tag{10.8.21}$$

$$F_-(E) = \theta(E)\frac{2E|N|^2}{(2\pi)^3}\sigma_{\alpha+B}(E)\,. \tag{10.8.22}$$

The scattering amplitude (10.8.3) is now, for real $\omega > 0$,

$$M(\omega) = \frac{-iQ(\omega)}{2\omega|N|^2} - \frac{i}{2(2\pi)^3}\sigma_{\alpha+B}(\omega) - \frac{P(\omega)}{\omega(2\pi)^4}\int_0^\infty \left[\frac{\sigma_{\alpha+B}(E)}{(E-\omega)P(E)} + \frac{\sigma_{\alpha+B^c}(E)}{(E+\omega)P(-E)}\right] E\,dE\,. \tag{10.8.23}$$

It is more usual to express this dispersion relation in terms of the amplitude $f(\omega)$ for forward scattering in the laboratory frame, defined so that the laboratory frame differential cross-section in the forward direction is $|f(\omega)|^2$. This amplitude is given in terms of $M(\omega)$ by

** In some cases where selection rules allow the transition $\alpha \to \alpha + B$ and $\alpha \to \alpha + B^c$, the functions $F_\pm(E)$ also contain terms proportional to $\delta(E)$ arising from the contribution of the one-particle state α in the sum over intermediate states β. This does not occur for transversely polarized photons, or for pseudoscalar pions in the limit $m_\pi \to 0$.

$f(\omega) = -4\pi^2\omega\, M(\omega) = 2\pi^2 i F(\omega)/|N|^2$, so Eq. (10.8.23) now reads

$$f(\omega) = R(\omega) + \frac{i\omega}{4\pi}\sigma_{\alpha+B}(\omega) + \frac{P(\omega)}{4\pi^2}\int_0^\infty \left[\frac{\sigma_{\alpha+B}(E)}{(E-\omega)P(E)} + \frac{\sigma_{\alpha+B^c}(E)}{(E+\omega)P(-E)}\right] E\, dE\,,$$

where $R(\omega) \equiv 2i\pi^2\, Q(\omega)/|N|^2$. The optical theorem (3.6.4) tells us that the second term on the right-hand side equals $i\mathrm{Im}\, f(\omega)$, so this can just as well be written in the more conventional form

$$\mathrm{Re}\, f(\omega) = R(\omega) + \frac{P(\omega)}{4\pi^2}\int_0^\infty \left[\frac{\sigma_{\alpha+B}(E)}{(E-\omega)P(E)} + \frac{\sigma_{\alpha+B^c}(E)}{(E+\omega)P(-E)}\right] E\, dE\,, \quad (10.8.24)$$

In particular, we see that $R(\omega)$ is real if we choose $P(\omega)$ real.

The forward scattering amplitude also satisfies an important symmetry condition. By changing the integration variable in Eqs. (10.8.7) and (10.8.8) from x to $-x$ and then using the translation-invariance property

$$\langle\alpha|[J^\dagger(0), J(-x)]|\alpha\rangle = \langle\alpha|[J^\dagger(x), J(0)]|\alpha\rangle$$

we see that for $\mathrm{Im}\,\omega \le 0$, $F_A(-\omega)$ is the same as $F_R(\omega)$, except for an interchange of J with $J^\dagger$. That is,

$$F_A(-\omega) = F_R^c(\omega) \quad \text{for} \quad \mathrm{Im}\,\omega \le 0\,,$$

where a superscript c indicates that the amplitude refers to the scattering of the antiparticle B^c on α. (We leave it to the reader to show that this relation is not upset by the equal-time commutator terms in Eqs. (10.8.7) and (10.8.8).) In the same way, we find

$$F_R(-\omega) = F_A^c(\omega) \quad \text{for} \quad \mathrm{Im}\,\omega \ge 0\,,$$

and for real ω

$$F_\pm(-\omega) = F_\mp(\omega)\,.$$

Using these relations in (10.8.6), and recalling that $f(\omega)$ is proportional to $F(\omega)$, we find the *crossing symmetry* relation, that for real ω

$$f(-\omega) = f^c(\omega). \quad (10.8.25)$$

We are free to take $P(\omega)$ as any polynomial of sufficiently high order, but $R(\omega)$ then depends not only on $P(\omega)$ but also on the values of $\mathscr{F}(\omega)$ at the zeroes of $P(\omega)$. For $P(\omega)$ real and of nth order, the only free parameters in Eq. (10.8.16) are the n real coefficients in the real $(n-1)$th order polynomial $R(\omega)$. Hence Eq. (10.8.16) contains just n unknown real independent constants, the coefficients in the polynomial $R(\omega)$ for a given $P(\omega)$. We therefore wish to take the order n of the otherwise arbitrary polyomial $P(\omega)$ to be as small as possible.

We might try taking $P(\omega) = 1$, but this doesn't work. The analysis of Section 3.7 suggests that the forward scattering amplitude should grow like ω or perhaps as fast as $\omega \ln^2 \omega$. In this case for $f(\omega)/P(\omega)$ to vanish as $\omega \to \infty$, it is sufficient to take $P(\omega)$ as a second order polynomial, so that $R(\omega)$ is linear in ω. Choosing $P(E) = E^2$ for convenience, Eq. (10.8.24) then becomes

$$\begin{aligned} \operatorname{Re} f(\omega) &= a + b\omega \\ &\quad + \frac{\omega^2}{4\pi^2} \int_0^\infty \left[\frac{\sigma_{\alpha+B}(E)}{(E-\omega)} + \frac{\sigma_{\alpha+B^c}(E)}{(E+\omega)} \right] \frac{dE}{E} , \end{aligned} \tag{10.8.26}$$

with a and b unknown real constants. The crossing symmetry condition (10.8.25) tells us that the corresponding constants in the dispersion relation for the antiparticle scattering amplitude $f^c(\omega)$ are

$$a^c = a , \qquad b^c = -b . \tag{10.8.27}$$

If we assume for instance that the cross-sections $\sigma_{\alpha+B}(E)$ and $\sigma_{\alpha+B^c}(E)$ behave for $E \to \infty$ as different constants times $(\ln E)^r$, then (10.8.26) would give

$$\operatorname{Re} f(\omega) \sim [\sigma_{\alpha+B}(\omega) - \sigma_{\alpha+B^c}(\omega)]\omega \ln \omega \sim \omega(\ln \omega)^{r+1} \tag{10.8.28}$$

so the real part of the scattering amplitude would grow faster than the imaginary part by a factor $\ln \omega$. This is implausible; we saw in Section 3.7 that the real part of the forward scattering amplitude is expected to become much smaller than the imaginary part for $\omega \to \infty$, as confirmed by experiment. We conclude that if $\sigma_{\alpha+B}(E)$ and $\sigma_{\alpha+B^c}(E)$ do behave for $E \to \infty$ as constants times $(\ln E)^r$ then the constants must be the same. Because we are concerned here with the high-energy limit, this result does not depend on the assumption that B is a massless boson, so in the same sense, *the ratio of the cross-sections of any particle and its antiparticle on a fixed target should approach unity at high energy.* This result is a somewhat generalized version of what is known as Pomeranchuk's theorem.[16] (Pomeranchuk considered only the case $r = 0$, while Section 3.7 and the observed behavior of cross-sections both suggest that $r = 2$ is more likely.)

Although Pomeranchuk took his estimates of the asymptotic behavior of scattering amplitudes from arguments like those of Section 3.7, today high energy behavior is usually inferred from Regge pole theory.[17] It would take us too far from our subject to go into details about this; suffice it to say that for hadronic processes the asymptotic behavior of $f(\omega)$ as ω goes to infinity is a sum over terms proportional to $\omega^{\alpha_n(0)}$, where $\alpha_n(t)$ are a set of 'Regge trajectories', each representing the exchange of an infinite family of different one-hadron states in the collision process. The leading trajectory (actually, a complex of many trajectories) in hadron–

hadron scattering is the 'Pomeron,' for which $\alpha(0)$ is close to unity. It is this trajectory that gives cross-sections that are approximately constant for $E \to \infty$. According to Pomeranchuk's theorem, the Pomeron couples equally to any hadron and its antiparticle. We can estimate $\alpha_n(0)$ for the lower Regge trajectories from the spectrum of hadronic states. A necessary though not sufficient condition[18] for a mesonic resonance of spin j to occur at a mass m is that m^2 equals the value of t where one of trajectories $\alpha_n(t)$ equals j. Apart from the Pomeron, the leading trajectory in pion–nucleon scattering is that on which we find the $j = 1$ ρ meson at $m = 770$ MeV, the $j = 3$ g meson at $m = 1690$ MeV, and a $j = 5$ meson at $m = 2350$ MeV. Extrapolating these values of $\alpha(t)$ down to $t = 0$, we can estimate that this trajectory has $\alpha(0) \approx 0.5$. This trajectory couples with opposite sign to π^+ and π^-, so for pion–nucleon scattering we expect $f(\omega) - f^c(\omega)$ to behave roughly like $\sqrt{\omega}$.

For photon scattering there is no distinction between B and B^c, so here Eq. (10.8.27) gives $b = 0$, and Eq. (10.8.26) reads

$$f(\omega) = a + \frac{\omega^2}{2\pi^2} \int_0^\infty \frac{\sigma(E)}{E^2 - \omega^2} dE \,. \tag{10.8.29}$$

This is essentially the original Kramers–Kronig[13] relation. As we shall see in Section 13.5, for a target of charge e and mass m the constant a has the known value $\mathrm{Re}\, f(0) = -e^2/m$.

Problems

1. Consider a neutral vector field $v_\mu(x)$. What conditions must be imposed on the sum $\Pi^*_{\mu\nu}(k)$ of one-particle-irreducible graphs with two external vector field lines in order that the field should be properly renormalized and describe a particle of renormalized mass m? How do we split the free-field and interacting terms in the Lagrangian to achieve this?

2. Derive the generalized Ward identity that governs the electromagnetic vertex function of a charged scalar field.

3. What is the most general form of the matrix element $\langle \mathbf{p}_2\sigma_2 | J^\mu(x) | \mathbf{p}_1\sigma_1 \rangle$ of the electromagnetic current $J^\mu(x)$ between two spin $\frac{1}{2}$ one-particle states of *different* masses m_1 and m_2 and equal parity? What if the parities were opposite? (Assume parity conservation throughout.)

4. Derive the spectral (Källen–Lehmann) representation for the vacuum expectation value $\langle T\{J^\mu(x)\ J^\nu(y)^\dagger\}\rangle_0$, where $J^\mu(x)$ is a complex conserved current.

5. Derive the spectral (Källen–Lehmann) representation for the vacuum expectation value $\langle T\{\psi_n(x)\,\bar{\psi}_m(y)\}\rangle_0$, where $\psi(x)$ is a Dirac field.

6. Without using any assumptions about the asymptotic behavior of the scattering amplitude or cross-sections, show that it is impossible for forward photon scattering amplitudes to satisfy unsubtracted dispersion relations.

7. Derive the spectral (Källen–Lehmann) representation for a complex scalar field by using the methods of dispersion theory.

8. Use dispersion theory and the results of Section 8.7 to calculate the amplitude for forward photon–electron scattering in the electron rest frame to order e^4.

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