Ward-Takahashi Identity

We start with W's definitions of S' and Γ in (10.4.20) and (10.4.19) and use (10.4.9) as well as (10.4.21). Then

$$G \equiv (-i)(ip_\mu)(2\pi)^4 q\, S'_{nn'}(k)\, \Gamma^\mu_{n'm'}(k,\ell)\, S'_{m'm}(\ell)\, \delta^4(p+k-\ell)$$

$$= \int d^4x\, d^4y\, d^4z \left(-\frac{\partial}{\partial x^\mu} e^{-ipx}\right) e^{-iky+i\ell z} \langle 0|T\{J^\mu(x)\psi_n(y)\bar{\psi}_m(z)\}|0\rangle$$

$$= \int d^4x\, d^4y\, d^4z\, e^{-ipx-iky+i\ell z} \frac{\partial}{\partial x^\mu} \langle 0|T\{J^\mu(x)\psi_n(y)\bar{\psi}_m(z)\}|0\rangle$$

$$= \int d^4x\, d^4y\, d^4z\, e^{-ipx-iky+i\ell z} \Big[\langle 0|\delta(x^0-y^0)\, T\{[J^0(x),\psi_n(y)]\bar{\psi}_m(z)\}|0\rangle + \langle 0|\delta(x^0-z^0)\, T\{\psi_n(y)[J^0(x),\bar{\psi}_m(z)]\}|0\rangle \Big]$$

Now (10.4.9) implies (10.4.22) & (10.4.23):

$$[J^0(\mathbf{x},t), \psi_n(\mathbf{y},t)] = -q\psi_n(\mathbf{y},t)\,\delta^3(\mathbf{x}-\mathbf{y})$$

$$[J^0(\mathbf{x},t), \bar{\psi}_n(\mathbf{y},t)] = q\bar{\psi}_n(\mathbf{y},t)\,\delta^3(\mathbf{x}-\mathbf{y})$$

So

$$G = \int d^4x\, d^4y\, d^4z\, e^{-ipx-iky+i\ell z} \langle 0|\left(-\delta^4(x-y)+\delta^4(x-z)\right) q\, T\{\psi_n(y)\bar{\psi}_m(z)\}|0\rangle$$

$$= -q\int d^4y\, d^4z \left(e^{-i(p+k)y+i\ell z} - e^{-iky+i(\ell-p)z} \right) \langle 0|T\{\psi_n(y)\bar{\psi}_m(z)\}|0\rangle$$

$$= -q(-i)(2\pi)^4 \left(S'_{nm}(p+k) - S'_{nm}(k) \right) \delta^4(p+k-\ell)$$

$$(\ell-k)_\mu (2\pi)^4 q \, S'_{nn'}(k)\, \Gamma^\mu_{n'm'}(k,\ell)\, S'_{m'm}(\ell)\, \delta^4(p+k-\ell)$$

$$= i q (2\pi)^4 \, S'_{nm}(\ell)\, \delta^4(p+k-\ell)$$

$$- i q (2\pi)^4 \, S'_{nm}(k)\, \delta^4(p+k-\ell)$$

So

$$(\ell-k)_\mu \, S'_{nn'}(k)\, \Gamma^\mu_{n'm'}(k,\ell)\, S'_{m'm}(\ell)$$

$$= i S'_{nm}(\ell) - i S'_{nm}(k)$$

or in matrix notation

$$(\ell-k)_\mu \, S'(k)\, \Gamma^\mu(k,\ell)\, S'(\ell) = i S'(\ell) - i S'(k)$$

or

$$(\ell-k)_\mu \, \Gamma^\mu(k,\ell) = i S'^{-1}(k) - i S'^{-1}(\ell)$$

which is a Ward-Takahashi identity.

Let $\ell = k+\epsilon$, then

$$\Gamma^\mu(k,k) = -i \frac{\partial}{\partial k_\mu} S'^{-1}(k),$$

which is Ward's form.

Now $\qquad S'^{-1}(k) = i\not{k} + m - \Sigma^*(\not{k})$

So

$$\frac{\partial S'^{-1}(k)}{\partial k_\mu} = i\gamma^\mu - \frac{\partial}{\partial k_\mu}\Sigma^*(\not{k})$$

so Ward's identity gives

$$\Gamma^\mu(k,k) = \gamma^\mu + i\frac{\partial}{\partial k_\mu}\Sigma^*(\not{k})$$

But if $\quad \not{k} = im$, then $\quad k^2 = -m^2$ and

$$\left.\frac{\partial\Sigma^*}{\partial\not{k}}\right|_{\not{k}=im} = 0$$

so for k on the electron's mass shell

$$\Gamma^\mu(k,k) = \gamma^\mu \qquad \text{for} \quad k^2 = -m^2.$$

That is, if $\quad (i\not{k}+m)u(k) = (i\not{k}+m)u'(k) = 0$
then

$$\bar{u}'(k)\,\Gamma^\mu(k,k)u(k) = \bar{u}'(k)\gamma^\mu u(k).$$

So radiative corrections cancel when photons of zero momentum scatter off a real electron.