$Hg = gH \ \forall g \in G$, and so the left cosets are the same sets as the right cosets. In this case, the coset space G/H is itself a group with multiplication defined by

$$\begin{aligned}(Hg_1)\,(Hg_2) &= \{h_i g_1 h_j g_2 | h_i, h_j \in H\} \\ &= \left\{h_i g_1 h_j g_1^{-1} g_1 g_2 | h_i, h_j \in H\right\} \\ &= \{h_i h_k g_1 g_2 | h_i, h_k \in H\} \\ &= \{h_\ell g_1 g_2 | h_\ell \in H\} = H g_1 g_2 \end{aligned} \tag{9.17}$$

which is the multiplication rule of the group G. This group G/H is called the **factor group of G by H**.

9.6 Morphisms

An **isomorphism** is a one-to-one map between groups that respects their multiplication laws. For example, the relation between two equivalent representations

$$D'(g) = S^{-1} D(g) S \tag{9.18}$$

is an isomorphism (problem 7). An **automorphism** is an isomorphism between a group and itself. The map $g_i \to g\, g_i\, g^{-1}$ is one to one because $g\, g_1\, g^{-1} = g\, g_2\, g^{-1}$ implies that $g\, g_1 = g\, g_2$, and so that $g_1 = g_2$. This map also preserves the law of multiplication since $g\, g_1\, g^{-1}\, g\, g_2\, g^{-1} = g\, g_1\, g_2\, g^{-1}$. So the map

$$G \to gGg^{-1} \tag{9.19}$$

is an automorphism. It is called an **inner automorphism** because g is an element of G. An automorphism not of this form (9.19) is called an **outer automorphism**.

9.7 Schur's Lemma

Part 1: If $D_1(g)A = AD_2(g)$ for all $g \in G$, and if D_1 & D_2 are inequivalent irreducible representations, then $A = 0$.

Proof: First suppose that A annihilates some vector $|x\rangle$, that is, $A|x\rangle = 0$. Let P be the projection operator P into the subspace that A annihilates, which is of at least one dimension. This subspace, incidentally, is called the **null space** $\mathcal{N}(A)$ or the **kernel** of the matrix A. The representation D_2 must leave this null space $\mathcal{N}(A)$ invariant since

$$AD_2(g)P = D_1(g)AP = 0. \tag{9.20}$$

If $\mathcal{N}(A)$ were a proper subspace, then the representation D_2 would be reducible, which is contrary to our assumption that D_1 and D_2 are irreducible. So the null space $\mathcal{N}(A)$ must be the whole space upon which A acts, that is, $A = 0$.

A similar argument shows that if $\langle y|A = 0$ for some bra $\langle y|$, then $A = 0$.

So either A is zero or it annihilates no ket and no bra. In the latter case, A must be square and invertible, which would imply that $D_2(g) = A^{-1}D_1(g)A$, that is, that D_1 and D_2 are equivalent representations, which is contrary to our assumption that they are inequivalent. The only way out is that A vanishes.

Part 2: If for a finite-dimensional, irreducible representation $D(g)$ of a group G, we have $D(g)A = AD(g)$ for all $g \in G$, then $A = cI$. That is, any matrix that commutes with every element of a finite-dimensional, irreducible representation must be a multiple of the identity matrix.

Proof: Every square matrix A has at least one eigenvector $|x\rangle$ and eigenvalue c so that $A|x\rangle = c|x\rangle$ because its characteristic equation $\det(A-cI) = 0$ always has at least one root by the fundamental theorem of algebra (5.89). So the null space $\mathcal{N}(A-cI)$ has dimension greater than zero. Now $D(g)A = AD(g)$ for all $g \in G$ implies that $D(g)(A-cI) = (A-cI)D(g)$ for all $g \in G$. Let P be the projection operator onto the null space $\mathcal{N}(A-cI)$. Then we have $(A-cI)D(g)P = D(g)(A-cI)P = 0$ for all $g \in G$ which implies that $D(g)P$ maps vectors into the null space $\mathcal{N}(A-cI)$. This null space is therefore invariant under $D(g)$, which means that D is reducible unless the null space $\mathcal{N}(A-cI)$ is the whole space. Since by assumption D is irreducible, it follows that $\mathcal{N}(A-cI)$ is the whole space, that is, that $A = 0$.

Example and Application: Suppose an arbitrary observable O is invariant under the action of the rotation group $SU(2)$ represented by unitary operators $U(g)$ for $g \in SU(2)$

$$U^\dagger(g)OU(g) = O \quad \text{or} \quad [O, U(g)] = 0. \tag{9.21}$$

These unitary rotation operators commute with the square $\boldsymbol{J}^2$ of the angular momentum $[\boldsymbol{J}^2, U] = 0$. Suppose that they also leave the hamiltonian H unchanged $[H, U] = 0$. Then as shown in Sec. 9.3, the state $U|E,j,m\rangle$ is a sum of states all with the same values of j and E. It follows that

$$\begin{aligned}\sum_{m'}\langle E,j,m|O|E',j',m'\rangle\langle E',j',m'|U(g)|E',j',m''\rangle = \\ \sum_{m'}\langle E,j,m|U(g)|E,j,m'\rangle\langle E,j,m'|O|E',j',m''\rangle\end{aligned} \tag{9.22}$$

or more simply in view of (9.11)

$$\sum_{m'} \langle E, j, m|O|E', j', m'\rangle D^{j'}(g)_{m'm''} = \sum_{m'} D^{(j)}(g)_{mm'} \langle E, j, m'|O|E', j', m''\rangle. \tag{9.23}$$

Now Part 1 of Schur's lemma tells us that the matrix $\langle E, j, m|O|E', j', m'\rangle$ must vanish unless the representations are equivalent, which is to say unless $j = j'$. So we have

$$\sum_{m'} \langle E, j, m|O|E', j, m'\rangle D^{j}(g)_{m'm''} = \sum_{m'} D^{(j)}(g)_{mm'} \langle E, j, m'|O|E', j, m''\rangle. \tag{9.24}$$

Now Part 2 of Schur's lemma tells us that the matrix $\langle E, j, m|O|E', j, m'\rangle$ must be a multiple of the identity. Thus the symmetry of $\mathcal{O}$ under rotations simplifies the matrix element to

$$\langle E, j, m|O|E', j', m'\rangle = \delta_{jj'}\delta_{mm'} O_j(E, E'). \tag{9.25}$$

This result is a special case of the **Wigner-Eckart theorem** (Eugene Wigner, 1902–1995, and Carl Eckart, 1902–1973).

9.8 Characters

Suppose the $n \times n$ matrices $D_{ij}(g)$ form a representation of a group $G \ni g$. The **character** $\chi_D(g)$ of the matrix $D(g)$ is the trace

$$\chi_D(g) = \mathrm{Tr} D(g) = \sum_{i=1}^{n} D_{ii}(g). \tag{9.26}$$

Traces are cyclic, that is, $\mathrm{Tr}ABC = \mathrm{Tr}BCA = \mathrm{Tr}CAB$. So if two representations D and D' are equivalent, so that $D'(g) = S^{-1}D(g)S$, then they have the same characters because

$$\chi_{D'}(g) = \mathrm{Tr}D'(g) = \mathrm{Tr}\left(S^{-1}D(g)S\right) = \mathrm{Tr}\left(D(g)SS^{-1}\right) = \mathrm{Tr}D(g) = \chi_D(g). \tag{9.27}$$

If two group elements g_1 and g_2 are in the same conjugacy class, that is, if $g_2 = g g_1 g^{-1}$ for some $g \in G$, then they have the same character in a given representation $D(g)$ because

$$\begin{aligned}\chi_D(g_2) = \mathrm{Tr}D(g_2) &= \mathrm{Tr}D(g g_1 g^{-1}) = \mathrm{Tr}\left(D(g)D(g_1)D(g^{-1})\right)\\ &= \mathrm{Tr}\left(D(g_1)D^{-1}(g)D(g)\right) = \mathrm{Tr}D(g_1) = \chi_D(g_1).\end{aligned} \tag{9.28}$$