5

Quantum Fields and Antiparticles

We now have all the pieces needed to motivate the introduction of quantum fields.[1] In the course of this construction, we shall encounter some of the most remarkable and universal consequences of the union of relativity with quantum mechanics: the connection between spin and statistics, the existence of antiparticles, and various relationships between particles and antiparticles, including the celebrated CPT theorem.

5.1 Free Fields

We have seen in Chapter 3 that the S-matrix will be Lorentz-invariant if the interaction can be written as

$$V(t) = \int d^3x\; \mathscr{H}(\mathbf{x}, t)\,, \tag{5.1.1}$$

where $\mathscr{H}$ is a scalar, in the sense that

$$U_0(\Lambda, a)\mathscr{H}(x)U_0^{-1}(\Lambda, a) = \mathscr{H}(\Lambda x + a)\,, \tag{5.1.2}$$

and satisfies the additional condition:

$$[\mathscr{H}(x), \mathscr{H}(x')] = 0 \quad \text{for} \quad (x - x')^2 \geq 0\,. \tag{5.1.3}$$

As we shall see, there are more general possibilities, but none of them are very different from this. (For the present we are leaving it as an open question whether Λ here is restricted to a proper orthochronous Lorentz transformation, or can also include space inversions.) In order to facilitate also satisfying the cluster decomposition principle we are going to construct $\mathscr{H}(x)$ out of creation and annihilation operators, but here we face a problem: as shown by Eq. (4.2.12), under Lorentz transformations each such operator is multiplied by a matrix that depends on the momentum carried by that operator. How can we couple such operators together to make a scalar? The solution is to build $\mathscr{H}(x)$ out of *fields* — both annihilation fields $\psi_\ell^+(x)$ and creation fields $\psi_\ell^-(x)$:

$$\psi_\ell^+(x) = \sum_{\sigma n} \int d^3p \; u_\ell(x; \mathbf{p}, \sigma, n) a(\mathbf{p}, \sigma, n) \,, \tag{5.1.4}$$

$$\psi_\ell^-(x) = \sum_{\sigma n} \int d^3p \; v_\ell(x; \mathbf{p}, \sigma, n) a^\dagger(\mathbf{p}, \sigma, n). \tag{5.1.5}$$

with coefficients* $u_\ell(x; \mathbf{p}, \sigma, n)$ and $v_\ell(x; \mathbf{p}, \sigma, n)$ chosen so that under Lorentz transformations each field is multiplied with a position-independent matrix:

$$U_0(\Lambda, a)\psi_\ell^+(x)U_0^{-1}(\Lambda, a) = \sum_{\bar{\ell}} D_{\ell\bar{\ell}}(\Lambda^{-1})\psi_{\bar{\ell}}^+(\Lambda x + a) \,, \tag{5.1.6}$$

$$U_0(\Lambda, a)\psi_\ell^-(x)U_0^{-1}(\Lambda, a) = \sum_{\bar{\ell}} D_{\ell\bar{\ell}}(\Lambda^{-1})\psi_{\bar{\ell}}^-(\Lambda x + a) \,. \tag{5.1.7}$$

(We might, in principle, have different transformation matrices $D^\pm$ for the annihilation and creation fields, but as we shall see, it is always possible to choose the fields so that these matrices are the same.) By applying a second Lorentz transformation $\bar{\Lambda}$, we find that

$$D(\Lambda^{-1})D(\bar{\Lambda}^{-1}) = D((\bar{\Lambda}\Lambda)^{-1}) \,,$$

so taking $\Lambda_1 = (\Lambda)^{-1}$ and $\Lambda_2 = (\bar{\Lambda})^{-1}$, we see that the D-matrices furnish a *representation* of the homogeneous Lorentz group:

$$D(\Lambda_1)D(\Lambda_2) = D(\Lambda_1\Lambda_2) \,. \tag{5.1.8}$$

There are many such representations, including the scalar $D(\Lambda) = 1$, the vector $D(\Lambda)^\mu{}_\nu = \Lambda^\mu{}_\nu$, and a host of tensor and spinor representations. These particular representations are irreducible, in the sense that it is not possible by a choice of basis to reduce all $D(\Lambda)$ to the same block-diagonal form, with two or more blocks. However, we do not require at this point that $D(\Lambda)$ be irreducible; in general it is a block-diagonal matrix with an arbitrary array of irreducible representations in the blocks. That is, the index ℓ here includes a label that runs over the types of particle described and the irreducible representations in the different blocks, as well as another that runs over the components of the individual irreducible representations. Later we will separate these fields into irreducible fields that each describe only a single particle species (and its antiparticle) and transform irreducibly under the Lorentz group.

* A reminder: the labels n and σ run over all different particle species and spin z-components, respectively.

Once we have learned how to construct fields satisfying the Lorentz transformation rules (5.1.6) and (5.1.7), we will be able to construct the interaction density as

$$\mathscr{H}(x) = \sum_{NM} \sum_{\ell'_1\cdots\ell'_N} \sum_{\ell_1\cdots\ell_M} g_{\ell'_1\cdots\ell'_N,\,\ell_1\cdots\ell_M} \times \psi^-_{\ell'_1}(x)\cdots\psi^-_{\ell'_N}(x)\,\psi^+_{\ell_1}(x)\cdots\psi^+_{\ell_M}(x) \tag{5.1.9}$$

and this will be a scalar in the sense of Eq. (5.1.2) if the constant coefficients $g_{\ell'_1\cdots\ell'_N,\,\ell_1\cdots\ell_M}$ are chosen to be Lorentz covariant, in the sense that for all Λ:

$$\sum_{\ell'_1\cdots\ell'_N} \sum_{\ell_1\cdots\ell_M} D_{\ell'_1\bar{\ell}'_1}(\Lambda^{-1})\cdots D_{\ell'_N\bar{\ell}'_N}(\Lambda^{-1}) D_{\ell_1\bar{\ell}_1}(\Lambda^{-1})\cdots D_{\ell_M\bar{\ell}_M}(\Lambda^{-1}) \times g_{\ell'_1\cdots\ell'_N\ell_1\cdots\ell_M} = g_{\bar{\ell}'_1\cdots\bar{\ell}'_N\bar{\ell}_1\cdots\bar{\ell}_M}. \tag{5.1.10}$$

(Note that we do not include derivatives here, because we regard the derivatives of components of these fields as just additional sorts of field components.) The task of finding coefficients $g_{\ell'_1\cdots\ell'_N,\,\ell_1\cdots\ell_M}$ that satisfy Eq. (5.1.10) is no different in principle (and not much more difficult in practice) than that of using Clebsch–Gordan coefficients to couple together various representations of the three-dimensional rotation group to form rotational scalars. Later we will be able to combine creation and annihilation fields so that this density also commutes with itself at space-like separations.

Now, what shall we take as the coefficient functions $u_\ell(x;\mathbf{p},\sigma,n)$ and $v_\ell(x;\mathbf{p},\sigma,n)$? Eq. (4.2.12) and its adjoint give the transformation rules** for the annihilation and creation operators

$$U_0(\Lambda,b)a(\mathbf{p},\sigma,n)U_0^{-1}(\Lambda,b) = \exp\Big(i(\Lambda p)\cdot b\Big)\sqrt{(\Lambda p)^0/p^0} \times \sum_{\bar\sigma} D^{(j_n)}_{\sigma\bar\sigma}\Big(W^{-1}(\Lambda,p)\Big)a(\mathbf{p}_\Lambda,\bar\sigma,n)\,, \tag{5.1.11}$$

$$U_0(\Lambda,b)a^\dagger(\mathbf{p},\sigma,n)U_0^{-1}(\Lambda,b) = \exp\Big(-i(\Lambda p)\cdot b\Big)\sqrt{(\Lambda p)^0/p^0} \times \sum_{\bar\sigma} D^{(j_n)*}_{\sigma\bar\sigma}\Big(W^{-1}(\Lambda,p)\Big)a^\dagger(\mathbf{p}_\Lambda,\bar\sigma,n) \tag{5.1.12}$$

where j_n is the spin of particles of species n, and $\mathbf{p}_\Lambda$ is the three-vector part of Λp. (We have used the unitarity of the rotation matrices $D^{(j_n)}_{\sigma\bar\sigma}$ to put both Eqs. (5.1.11) and (5.1.12) in the form shown here.) Also, as we saw in Section 2.5 the volume element d^3p/p^0 is Lorentz-invariant, so we

** This is for massive particles. The case of zero mass will be taken up in Section 5.9.

can replace d^3p in Eqs. (5.1.4) and (5.1.5) with $d^3(\Lambda p)p^0/(\Lambda p)^0$. Putting this all together, we find

$$U_0(\Lambda, b)\psi^+_\ell(x)U_0^{-1}(\Lambda, b) = \sum_{\sigma\bar\sigma n}\int d^3(\Lambda p)\; u_\ell(x;\mathbf{p},\sigma,n)$$
$$\times\;\exp\Big(i(\Lambda p)\cdot b\Big)D^{(j_n)}_{\sigma\bar\sigma}\Big(W^{-1}(\Lambda,p)\Big)\sqrt{p^0/(\Lambda p)^0}\;a(\mathbf{p}_\Lambda,\bar\sigma,n)$$

and

$$U_0(\Lambda, b)\psi^-_\ell(x)U_0^{-1}(\Lambda, b) = \sum_{\sigma\bar\sigma n}\int d^3(\Lambda p)\; v_\ell(x;\mathbf{p},\sigma,n)$$
$$\times\;\exp\Big(-i(\Lambda p)\cdot b\Big)D^{(j_n)*}_{\sigma\bar\sigma}\Big(W^{-1}(\Lambda,p)\Big)\sqrt{p^0/(\Lambda p)^0}\;a^\dagger(\mathbf{p}_\Lambda,\bar\sigma,n)\,.$$

We see that in order for the fields to satisfy the Lorentz transformation rules (5.1.6) and (5.1.7), it is necessary and sufficient that

$$\sum_{\bar\ell} D_{\ell\bar\ell}(\Lambda^{-1})u_{\bar\ell}(\Lambda x+b;\mathbf{p}_\Lambda,\sigma,n) = \sqrt{p^0\big/(\Lambda p)^0}$$
$$\times\sum_{\bar\sigma} D^{(j_n)}_{\sigma\bar\sigma}\Big(W^{-1}(\Lambda,p)\Big)\exp\Big(+i(\Lambda p)\cdot b\Big)u_\ell(x;\mathbf{p},\bar\sigma,n)$$

$D^{(jn)}_{\bar\sigma\sigma}$

and

$$\sum_{\bar\ell} D_{\ell\bar\ell}(\Lambda^{-1})v_{\bar\ell}(\Lambda x+b;\mathbf{p}_\Lambda,\sigma,n) = \sqrt{p^0\big/(\Lambda p)^0}$$
$$\times\sum_{\bar\sigma} D^{(j_n)*}_{\sigma\bar\sigma}\Big(W^{-1}(\Lambda,p)\Big)\exp\Big(-i(\Lambda p)\cdot b\Big)v_\ell(x;\mathbf{p},\bar\sigma,n)$$

$D^{(j_n)*}_{\bar\sigma\sigma}$

or somewhat more conveniently

$$\sum_{\bar\sigma} u_{\bar\ell}(\Lambda x+b;\mathbf{p}_\Lambda,\bar\sigma,n)D^{(j_n)}_{\bar\sigma\sigma}\Big(W(\Lambda,p)\Big) = \sqrt{p^0\big/(\Lambda p)^0}$$
$$\times\sum_{\ell} D_{\bar\ell\ell}(\Lambda)\exp\Big(i(\Lambda p)\cdot b\Big)u_\ell(x;\mathbf{p},\sigma,n) \tag{5.1.13}$$

and

$$\sum_{\bar\sigma} v_{\bar\ell}(\Lambda x+b;\mathbf{p}_\Lambda,\bar\sigma,n)D^{(j_n)*}_{\bar\sigma\sigma}\Big(W(\Lambda,p)\Big) = \sqrt{p^0\big/(\Lambda p)^0}$$
$$\times\sum_{\ell} D_{\bar\ell\ell}(\Lambda)\exp\Big(-i(\Lambda p)\cdot b\Big)v_\ell(x;\mathbf{p},\sigma,n)\,. \tag{5.1.14}$$

These are the fundamental requirements that will allow us to calculate the u_ℓ and v_ℓ coefficient functions in terms of a finite number of free parameters.

We will use Eqs. (5.1.13) and (5.1.14) in three steps, considering in turn three different types of proper orthochronous Lorentz transformation:

Translations

First we consider Eqs. (5.1.13) and (5.1.14) with $\Lambda = 1$ and b arbitrary. We see immediately that $u_\ell(x;\mathbf{p},\sigma,n)$ and $v_\ell(x;\mathbf{p},\sigma,n)$ must take the form

$$u_\ell(x;\mathbf{p},\sigma,n) = (2\pi)^{-3/2} e^{ip\cdot x} u_\ell(\mathbf{p},\sigma,n)\,, \tag{5.1.15}$$

$$v_\ell(x;\mathbf{p},\sigma,n) = (2\pi)^{-3/2} e^{-ip\cdot x} v_\ell(\mathbf{p},\sigma,n)\,, \tag{5.1.16}$$

so the fields are Fourier transforms:

$$\psi_\ell^+(x) = \sum_{\sigma,n} (2\pi)^{-3/2} \int d^3p\; u_\ell(\mathbf{p},\sigma,n) e^{ip\cdot x} a(\mathbf{p},\sigma,n)\,, \tag{5.1.17}$$

and

$$\psi_\ell^-(x) = \sum_{\sigma,n} (2\pi)^{-3/2} \int d^3p\; v_\ell(\mathbf{p},\sigma,n) e^{-ip\cdot x} a^\dagger(\mathbf{p},\sigma,n)\,. \tag{5.1.18}$$

(The factors $(2\pi)^{-3/2}$ could be absorbed into the definition of u_ℓ and v_ℓ, but it is conventional to show them explicitly in these Fourier integrals.) Using Eqs. (5.1.15) and (5.1.16), we see that Eqs. (5.1.13) and (5.1.14) are satisfied if and only if

$$\sum_{\bar\sigma} u_{\bar\ell}(\mathbf{p}_\Lambda,\bar\sigma,n) D^{(j_n)}_{\bar\sigma\sigma}\Big(W(\Lambda,p)\Big) = \sqrt{\frac{p^0}{(\Lambda p)^0}} \sum_\ell D_{\bar\ell\ell}(\Lambda) u_\ell(\mathbf{p},\sigma,n) \tag{5.1.19}$$

and

$$\sum_{\bar\sigma} v_{\bar\ell}(\mathbf{p}_\Lambda,\bar\sigma,n) D^{(j_n)*}_{\bar\sigma\sigma}\Big(W(\Lambda,p)\Big) = \sqrt{\frac{p^0}{(\Lambda p)^0}} \sum_\ell D_{\bar\ell\ell}(\Lambda) v_\ell(\mathbf{p},\sigma,n)\,. \tag{5.1.20}$$

for arbitrary homogeneous Lorentz transformations Λ.

Boosts

Next take $\mathbf{p} = 0$ in Eqs. (5.1.19) and (5.1.20), and let Λ be the standard boost $L(q)$ that takes a particle of mass m from rest to some four-momentum q^μ. Then $L(p) = 1$, and

$$W(\Lambda,p) \equiv L^{-1}(\Lambda p)\Lambda L(p) = L^{-1}(q)L(q) = 1\,.$$

Hence in this special case, Eqs. (5.1.19) and (5.1.20) give

$$u_{\bar\ell}(\mathbf{q},\sigma,n) = (m/q^0)^{1/2} \sum_\ell D_{\bar\ell\ell}(L(q))\, u_\ell(0,\sigma,n) \tag{5.1.21}$$

We will use Eqs. (5.1.13) and (5.1.14) in three steps, considering in turn three different types of proper orthochronous Lorentz transformation:

Translations

First we consider Eqs. (5.1.13) and (5.1.14) with $\Lambda = 1$ and b arbitrary. We see immediately that $u_\ell(x;\mathbf{p},\sigma,n)$ and $v_\ell(x;\mathbf{p},\sigma,n)$ must take the form

$$u_\ell(x;\mathbf{p},\sigma,n) = (2\pi)^{-3/2} e^{ip\cdot x} u_\ell(\mathbf{p},\sigma,n)\,, \tag{5.1.15}$$

$$v_\ell(x;\mathbf{p},\sigma,n) = (2\pi)^{-3/2} e^{-ip\cdot x} v_\ell(\mathbf{p},\sigma,n)\,, \tag{5.1.16}$$

so the fields are Fourier transforms:

$$\psi_\ell^+(x) = \sum_{\sigma,n} (2\pi)^{-3/2} \int d^3p\; u_\ell(\mathbf{p},\sigma,n) e^{ip\cdot x} a(\mathbf{p},\sigma,n)\,, \tag{5.1.17}$$

and

$$\psi_\ell^-(x) = \sum_{\sigma,n} (2\pi)^{-3/2} \int d^3p\; v_\ell(\mathbf{p},\sigma,n) e^{-ip\cdot x} a^\dagger(\mathbf{p},\sigma,n)\,. \tag{5.1.18}$$

(The factors $(2\pi)^{-3/2}$ could be absorbed into the definition of u_ℓ and v_ℓ, but it is conventional to show them explicitly in these Fourier integrals.) Using Eqs. (5.1.15) and (5.1.16), we see that Eqs. (5.1.13) and (5.1.14) are satisfied if and only if

$$\sum_{\bar\sigma} u_{\bar\ell}(\mathbf{p}_\Lambda,\bar\sigma,n) D^{(j_n)}_{\bar\sigma\sigma}\Big(W(\Lambda,p)\Big) = \sqrt{\frac{p^0}{(\Lambda p)^0}} \sum_\ell D_{\bar\ell\ell}(\Lambda) u_\ell(\mathbf{p},\sigma,n) \tag{5.1.19}$$

and

$$\sum_{\bar\sigma} v_{\bar\ell}(\mathbf{p}_\Lambda,\bar\sigma,n) D^{(j_n)*}_{\bar\sigma\sigma}\Big(W(\Lambda,p)\Big) = \sqrt{\frac{p^0}{(\Lambda p)^0}} \sum_\ell D_{\bar\ell\ell}(\Lambda) v_\ell(\mathbf{p},\sigma,n)\,. \tag{5.1.20}$$

for arbitrary homogeneous Lorentz transformations Λ.

Boosts

Next take $\mathbf{p} = 0$ in Eqs. (5.1.19) and (5.1.20), and let Λ be the standard boost $L(q)$ that takes a particle of mass m from rest to some four-momentum q^μ. Then $L(p) = 1$, and

$$W(\Lambda,p) \equiv L^{-1}(\Lambda p)\Lambda L(p) = L^{-1}(q)L(q) = 1\,.$$

Hence in this special case, Eqs. (5.1.19) and (5.1.20) give

$$u_{\bar\ell}(\mathbf{q},\sigma,n) = (m/q^0)^{1/2} \sum_\ell D_{\bar\ell\ell}(L(q))\, u_\ell(0,\sigma,n) \tag{5.1.21}$$

and

$$v_{\bar{\ell}}(\mathbf{q},\sigma,n) = (m/q^0)^{1/2} \sum_{\ell} D_{\bar{\ell}\ell}(L(q))\, v_\ell(0,\sigma,n)\,. \tag{5.1.22}$$

In other words, if we know the quantities $u_\ell(0,\sigma,n)$ and $v_\ell(0,\sigma,n)$ for zero momentum, then for a given representation $D(\Lambda)$ of the homogeneous Lorentz group, we know the functions $u_\ell(\mathbf{p},\sigma,n)$ and $v_\ell(\mathbf{p},\sigma,n)$ for all $\mathbf{p}$. (Explicit formulas for the matrices $D_{\bar{\ell}\ell}(L(q))$ will be given for arbitrary representations of the homogeneous Lorentz group in Section 5.7.)

Rotations

Next, take $\mathbf{p} = 0$, but this time let Λ be a Lorentz transformation with $\mathbf{p}_\Lambda = 0$; that is, take Λ as a rotation R. Here obviously $W(\Lambda, p) = R$, and so Eqs. (5.1.19) and (5.1.20) read

$$\sum_{\bar{\sigma}} u_{\bar{\ell}}(0,\bar{\sigma},n)\, D^{(j_n)}_{\bar{\sigma}\sigma}(R) = \sum_{\ell} D_{\bar{\ell}\ell}(R) u_\ell(0,\sigma,n) \tag{5.1.23}$$

and

$$\sum_{\bar{\sigma}} v_{\bar{\ell}}(0,\bar{\sigma},n)\, D^{(j_n)*}_{\bar{\sigma}\sigma}(R) = \sum_{\ell} D_{\bar{\ell}\ell}(R) v_\ell(0,\sigma,n)\,, \tag{5.1.24}$$

or equivalently

$$\sum_{\bar{\sigma}} u_{\bar{\ell}}(0,\bar{\sigma},n) \mathbf{J}^{(j_n)}_{\bar{\sigma}\sigma} = \sum_{\ell} \boldsymbol{\mathcal{J}}_{\bar{\ell}\ell} u_\ell(0,\sigma,n) \tag{5.1.25}$$

and

$$\sum_{\bar{\sigma}} v_{\bar{\ell}}(0,\bar{\sigma},n) \mathbf{J}^{(j_n)*}_{\bar{\sigma}\sigma} = -\sum_{\ell} \boldsymbol{\mathcal{J}}_{\bar{\ell}\ell} v_\ell(0,\sigma,n), \tag{5.1.26}$$

where $\mathbf{J}^{(j)}$ and $\boldsymbol{\mathcal{J}}$ are the angular-momentum matrices in the representations $D^{(j)}(R)$ and $D(R)$, respectively. Any representation $D(\Lambda)$ of the homogeneous Lorentz group obviously yields a representation of the rotation group when Λ is restricted to rotations R; Eqs. (5.1.25) and (5.1.26) tell us that if the field $\psi^{\pm}_\ell(x)$ is to describe particles of some particular spin j, then this representation $D(R)$ must contain among its irreducible components the spin-j representation $D^{(j)}(R)$, with the coefficients $u_\ell(0,\sigma,n)$ and $v_\ell(0,\sigma,n)$ simply describing how the spin-j representation of the rotation group is embedded in $D(R)$. We shall see in Section 5.6 that each *irreducible* representation of the proper orthochronous Lorentz group contains any given irreducible representation of the rotation group at most once, so that if the fields $\psi^+_\ell(x)$ and $\psi^-_\ell(x)$ transform irreducibly, then they are unique up to overall scale. More generally, the number of free parameters in the annihilation or creation fields (including their overall scales) is equal to the number of irreducible representations in the field.

It is straightforward to show that coefficient functions $u_\ell(\mathbf{p},\sigma,n)$ and $v_\ell(\mathbf{p},\sigma,n)$ given by Eqs. (5.1.21) and (5.1.22), with $u_\ell(0,\sigma,n)$ and $v_\ell(0,\sigma,n)$ satisfying Eqs. (5.1.23) and (5.1.24), will automatically satisfy the more general requirements (5.1.19) and (5.1.20). This is left as an exercise for the reader.

Let us now return to the cluster decomposition principle. Inserting Eqs. (5.1.17) and (5.1.18) in Eq. (5.1.9) and integrating over $\mathbf{x}$, the interaction Hamiltonian is

$$\begin{aligned} V = \sum_{NM} \int d^3\mathbf{p}'_1 \cdots d^3\mathbf{p}'_N\, d^3\mathbf{p}_1 \cdots d^3\mathbf{p}_M \sum_{\sigma'_1\cdots\sigma'_N} \sum_{\sigma_1\cdots\sigma_M} \sum_{n'_1\cdots n'_N} \sum_{n_1\cdots n_M} \\ \times\, a^\dagger(\mathbf{p}'_1\,\sigma'_1\,n'_1)\cdots a^\dagger(\mathbf{p}'_N\,\sigma'_N\,n'_N)\, a(\mathbf{p}_M\sigma_M n_M)\cdots a(\mathbf{p}_1\sigma_1 n_1) \\ \times\, \mathscr{V}_{NM}(\mathbf{p}'_1\,\sigma'_1\,n'_1\cdots\mathbf{p}'_N\,\sigma'_N\,n'_N\,,\ \mathbf{p}_1\sigma_1 n_1\cdots\mathbf{p}_M\sigma_M n_M) \end{aligned} \tag{5.1.27}$$

with coefficient functions given by

$$\begin{aligned} \mathscr{V}_{NM}(\mathbf{p}'_1\,\sigma'_1\,n'_1\cdots,\mathbf{p}_1\sigma_1 n_1\cdots) = \delta^3(\mathbf{p}'_1+\cdots-\mathbf{p}_1-\cdots) \\ \times\, \tilde{\mathscr{V}}_{NM}(\mathbf{p}'_1\,\sigma'_1\,n'_1\cdots,\mathbf{p}_1\sigma_1 n_1\cdots)\,, \end{aligned} \tag{5.1.28}$$

where

$$\begin{aligned} \tilde{\mathscr{V}}_{NM}(\mathbf{p}'_1\,\sigma'_1\,n'_1\cdots\mathbf{p}'_N\,\sigma'_N\,n'_N\,,\ \mathbf{p}_1\sigma_1 n_1\cdots\mathbf{p}_M\sigma_M n_M) = (2\pi)^{3-3N/2-3M/2} \\ \times \sum_{\ell'_1\cdots\ell'_N} \sum_{\ell_1\cdots\ell_M} g_{\ell'_1\cdots\ell'_N,\ell_1\cdots\ell_M}\, v_{\ell'_1}(\mathbf{p}'_1\,\sigma'_1\,n'_1)\cdots v_{\ell'_N}(\mathbf{p}'_N\,\sigma'_N\,n'_N) \\ \times\, u_{\ell_1}(\mathbf{p}_1\sigma_1 n_1)\cdots u_{\ell_M}(\mathbf{p}_M\sigma_M n_M)\,. \end{aligned} \tag{5.1.29}$$

This interaction is manifestly of the form that will guarantee that the S-matrix satisfies the cluster decomposition principle: $\mathscr{V}_{NM}$ has a single delta function factor, with a coefficient $\tilde{\mathscr{V}}_{NM}$ that (at least for a finite number of field types) has at most branch point singularities at zero particle momenta. In fact, we could turn this argument around; any operator can be written as in Eq. (5.1.27), and the cluster decomposition principle requires that the coefficient $\mathscr{V}_{NM}$ may be written as in Eq. (5.1.28) as the product of a single momentum-conservation delta function times a smooth coefficient function. Any sufficiently smooth function (but *not* one containing additional delta functions) can be expressed as in Eq. (5.1.29).† *The cluster decomposition principle together with Lorentz invariance thus makes it natural that the interaction density should be constructed out of the annihilation and creation fields.*

† For general functions the indices ℓ and ℓ' may have to run over an infinite range. The reasons for restricting ℓ and ℓ' to a finite range have to do with the principle of renormalizability, discussed in Chapter 12.