Landau - Ginzburg Theory of Critical Phenomena

Ferromagnets have a magnetization $M(\vec{x})$ that vanishes as $T \to T_c$ from below as

$$|M| \sim (T_c - T)^{\beta}$$

where $\beta \simeq 0.37$. This critical exponent arises from nonanalytic behavior which arises because the infinite sum of a series of analytic functions isn't necessarily analytic.

L & G took the free energy to be

$$G = V \left[a M^2 + b (M^2)^2 + \cdots \right]$$

and let $\qquad a = a_1 (T - T_c) + \cdots$

So for $T < T_c$, $a < 0$, and G is minimized at

$$0 = a + 2bM^2 \quad \text{or} \quad M^2 = -\frac{a}{2b}$$

or $\quad |M| = \sqrt{-\frac{a}{2b}} \sim (T_c - T)^{\frac{1}{2}}.$

Well, $\frac{1}{2} \neq 0.37$, but this is so simple!

Let

$$G = \int d^3x \left[\nabla M \cdot \nabla M + a M^2 + b (M^2)^2 + \cdots \right]$$

$\frac{1}{\sqrt{a}}$ sets the length scale.

Add in a small external field $\vec{H}(x)$.

$$G = \int d^3x \left[\nabla M^2 + a M^2 + b(M^2)^2 - H \cdot M + \cdots \right]$$

get $\quad 0 = \delta M (a M - H - \nabla^2 M)$

so

$$-\nabla^2 M = H - aM$$

or

$$(-\nabla^2 + a) M = H$$

So

$$\vec{M}(x) = \int d^3y \int \frac{d^3k}{(2\pi)^3} \frac{e^{ik(x-y)} \vec{H}(y)}{k^2 + a}$$

$$= \int d^3y \frac{1}{4\pi |\vec{x - y}|} e^{-\sqrt{a}|\vec{x-y}|} \vec{H}(y).$$

So

$$\langle M(x) M(0) \rangle \sim e^{-|x|/\xi} \qquad \xi \sim \frac{1}{\sqrt{a}} \quad \text{for } T > T_c.$$

Correlation length $\xi \sim \frac{1}{\sqrt{T - T_c}}$.