The Anderson-Higgs Mechanism

Add a $U(1)$ gauge field to the previous theory. Now

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + (D\phi)^{\dagger} D\phi + \mu^2 \phi^{\dagger}\phi - \lambda(\phi^{\dagger}\phi)^2$$

where $D_\mu \phi = (\partial_\mu - ieA_\mu)\phi$.

So if $\phi = \rho e^{i\theta}$, then

$$D_\mu \phi = [\partial_\mu \rho + i\rho(\partial_\mu \theta - eA_\mu)] e^{i\theta}$$

and so

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \rho^2 (\partial_\mu \theta - eA_\mu)^2$$
$$+ (\partial\rho)^2 + \mu^2 \rho^2 - \lambda \rho^4$$

Let $B_\mu = A_\mu - \frac{1}{e} \partial_\mu \theta$. Then B_μ is gauge invariant

$$B_\mu' = A_\mu' - \frac{1}{e} \partial_\mu \theta' = A_\mu + \frac{1}{e} \partial_\mu \alpha - \frac{1}{e} \partial_\mu (\theta + \alpha)$$

$$= A_\mu - \frac{1}{e} \partial_\mu \theta = B_\mu.$$ And so is

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = \partial_\mu B_\nu - \partial_\nu B_\mu.$$

So we have a $U(1)$ gauge symmetry, but the minimum of the potential again is at

$$\rho = \frac{1}{\sqrt{2}}(v + \chi) \quad \text{with} \quad v = \frac{\mu}{\sqrt{\lambda}}$$

and

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}M^2 B_\mu^2 + e^2 v \chi B_\mu^2$$
$$+ \frac{1}{2}e^2\chi^2 B_\mu^2 + \frac{1}{2}(\partial\chi)^2 - \mu^2\chi^2$$
$$-\sqrt{\lambda}\,\mu\,\chi^3 - \frac{\lambda}{4}\chi^4 + \frac{\mu^4}{4\lambda}.$$

The massless vector field A_μ has become a massive vector field $B_\mu = A_\mu - \frac{1}{e}\partial_\mu\theta$ with mass

$$\frac{1}{2}M^2 = \frac{e^2 v^2}{2} \quad \text{so} \quad M = e v.$$

In $O(3)$ gauge theory, with $\phi = (\phi^1, \phi^2, \phi^3)$, the kinetic term is

$$\frac{1}{2}(D_\mu\phi^a)^2 = \frac{1}{2}\left(\partial_\mu\phi^a + g\epsilon^{abc}A_\mu^b\phi^c\right)^2.$$

Let $\langle\phi^a\rangle = v\,\delta^{a3}$. Set $\phi^3 = v$. Then

$$\frac{1}{2}(D_\mu\phi^a)^2 \longrightarrow \frac{1}{2}(gv)^2\left(A_\mu^1 A^{\mu 1} + A_\mu^2 A^{\mu 2}\right).$$

So A_μ^1 & A_μ^2 now have mass $M = gv$, but A^3_μ remains massless.

$n(G) - n(H)$ now is the number of massive vector bosons in the theory.

$SU(5)$: Here the ϕ is a 5×5 matrix $\phi^\dagger = \phi$. and $\mathrm{tr}\,\phi = 0$. So ϕ has 24 components. It goes via the adjoint rep:

$$\phi \to \phi' + i\theta^a [T^a, \phi] \quad \text{and so}$$

$$D_\mu \phi = \partial_\mu \phi - ig A^a_\mu [T^a, \phi] \qquad a = 1 \cdots 24$$

$SU(5)$ has 24 generators. Make $\langle \phi \rangle$ diagonal.

$$\langle \phi_{ij} \rangle = v_j \delta_{ij}, \quad i,j = 1 \cdots 5. \qquad \Sigma v_j = 0.$$

In the $\mathcal{L}$ we have a term whose vev is

$$\langle \mathrm{tr}\, D_\mu \phi D^\mu \phi \rangle = g^2\, \mathrm{tr}\left([T^a, \langle\phi\rangle][\langle\phi\rangle, T^b]\right) A^a_\mu A^{b\mu}$$

The gauge boson masses are the eigenvalues of

$$M^2 = g^2\, \mathrm{tr}\left([T^a, \langle\phi\rangle][\langle\phi\rangle, T^b]\right)$$

where e.g. $\langle \phi \rangle = \begin{pmatrix} 2 & & & & \\ & 2 & & & \\ & & 2 & & \\ & & & -3 & \\ & & & & -3 \end{pmatrix}$.

So if $T^a = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$ A 3×3, then

$[T^a, \langle \phi \rangle] = 0$ and A^a_μ is massless.

Also $T^b = \begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix}$ B 2×2

commutes with $\langle \phi \rangle$

$$[T^b, \langle \phi \rangle] = 0,$$

so A^b_μ is massless.

The 8 traceless 3×3 A's generate $SU(3)$

The 3 traceless σ matrices generate $SU(2)$.

The matrix $T^0 = \begin{pmatrix} 2 & & & & \\ & 2 & & & \\ & & 2 & & \\ & & & -3 & \\ & & & & -3 \end{pmatrix}$

generates $U(1)$. So the gauge bosons of $SU(3) \times SU(2) \times U(1)$ remain massless at this high-energy scale of spontaneous symmetry breaking. So $8+3+1$ are massless. So 12 gauge bosons get masses of the order of 10^{15} GeV or so.

More generally, we can suppose that the Higgs field goes as

$$\phi' = \exp(i\theta^a T^a)\phi.$$

Then $D_\mu \phi = \partial_\mu \phi + g A^a_\mu T^a \phi$

Suppose $\langle \phi \rangle = v$ then

$$\langle D_\mu \phi \rangle = g A^a_\mu T^a v$$

so

$$\left\langle \frac{1}{2} D^\mu \phi D_\mu \phi \right\rangle = \frac{g^2}{2} (T^a v) \cdot (T^b v) A^{\mu a} A_\mu^b$$

$$= \frac{1}{2} A^{\mu a} \mu^{2ab} A_\mu^b$$

Here

$$(\mu^2)^{ab} = g^2 (T^a v) \cdot (T^b v)$$

is the mass-squared matrix. Its eigenvalues are the squared masses of the gauge bosons after spontaneous symmetry breaking.