Goldstone's Theorem

Suppose Q is a conserved charge

$$[Q, H] = 0$$

arising from a conserved current j^μ

$$Q = \int d^3x \, j^0(x,t) .$$

So $\dot{Q} = 0$ since $\partial_\mu j^\mu = 0$
and we drop surface terms.

Q generates a symmetry if $Q|0\rangle = 0$
where $|0\rangle$ is the vacuum state. For then

$$\langle 0| A(x_1) A(x_2) \cdots A(x_n) |0\rangle$$

$$= \langle 0| e^{-iQ\theta} e^{iQ\theta} A(x_1) e^{-iQ\theta} e^{iQ\theta} A(x_2) e^{-iQ\theta} \cdots$$

$$e^{iQ\theta} A(x_n) e^{-iQ\theta} e^{iQ\theta} |0\rangle$$

$$= \langle 0| e^{iQ\theta} A(x_1) e^{-iQ\theta} \cdots e^{iQ\theta} A(x_n) e^{-iQ\theta} |0\rangle$$

$$= \langle 0| A_\theta(x_1) \cdots A_\theta(x_n) |0\rangle .$$

Add a constant c to H so that

$$H|0\rangle = 0.$$

What happens if $Q|0\rangle \neq 0$? We say the symmetry is <u>spontaneously</u> <u>broken</u>. The state $Q|0\rangle$ must have zero energy for

$$H\,Q|0\rangle = [H,Q]|0\rangle = 0$$

So $\quad H\left(Q|0\rangle\right) = 0.$

Now, consider the state

$$|\vec{k}\rangle \equiv \int d^3x\, e^{-i\vec{k}\cdot\vec{x}}\, j^0(x,t)|0\rangle$$

which tends to $Q|0\rangle$ as $\vec{k} \to 0$

$$\lim_{\vec{k}\to 0} |\vec{k}\rangle = \int d^3x\, j^0(x,t)|0\rangle = Q|0\rangle.$$

But $|\vec{k}\rangle$ is a state of momentum $\vec{k}$ because

$$P^i|\vec{k}\rangle = \int d^3x\, e^{-i\vec{k}\cdot x}\,[P^i, j^0(x,t)]|0\rangle = \int d^3x\, e^{-i\vec{k}\cdot x}(-i)\,\partial_i j^0(x,t)|0\rangle$$

$$= i\int d^3x\, j^0(x,t)\,\partial_i e^{-ik_i x_i}|0\rangle = k_i|\vec{k}\rangle.$$

5. if $Q|0\rangle \neq 0$, then there's a state $Q|0\rangle$ of zero energy and a family of states $|\vec{k}\rangle$ of momentum $\vec{k}$ which have energy zero as $\vec{k} \to 0$, and $|\vec{k}\rangle \to Q|0\rangle$ as $\vec{k} \to 0$. This is what we mean by a state of zero mass.

When Q is a bosonic operator, the state $|\vec{k}\rangle$ is called a Goldstone boson or a Nambu-Goldstone boson.

Example: Let $\phi = (\phi_1, \phi_2)$ and

$$\mathcal{L} = \frac{1}{2}\partial\phi^\dagger\partial\phi + \frac{\mu^2}{2}\phi^\dagger\phi - \frac{\lambda}{4}(\phi^\dagger\phi)^2 .$$

Note μ^2 has the "wrong sign."

$\mathcal{L}$ is invariant under $O(2) \simeq U(1)$. More simply, let $\varphi = (\phi_1 + i\phi_2)/\sqrt{2}$. Then

$$\mathcal{L} = \partial\varphi^*\partial\varphi + \mu^2\varphi^*\varphi - \lambda(\varphi^*\varphi)^2 .$$

Let $\varphi(x) = \rho(x)e^{i\theta(x)}$, so $\partial_\mu\varphi = (\partial_\mu\rho + i\rho\partial_\mu\theta)e^{i\theta}$ and

$$\mathcal{L} = \rho^2(\partial\theta)^2 + (\partial\rho)^2 + \mu^2\rho^2 - \lambda\rho^4 .$$

The minimum is a circle in the (ϕ_1, ϕ_2) plane with radius

$$v = \frac{\mu}{\sqrt{2\lambda}}.$$

Let $\rho = v + \chi$. Then

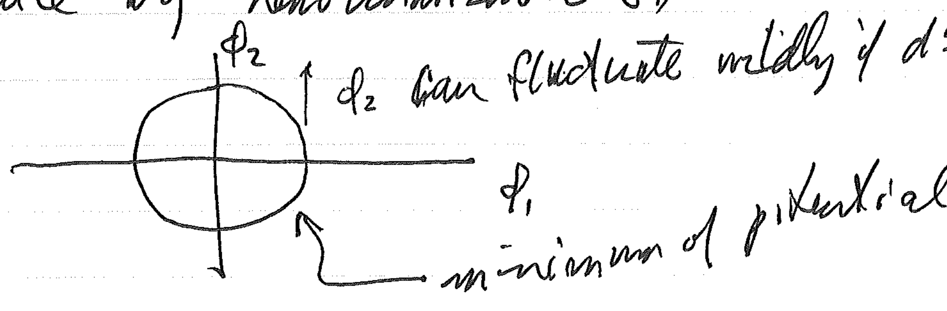
$$\mathcal{L} = v^2 (\partial\theta)^2 + \left[(\partial\chi)^2 - 2\mu^2\chi^2 - 4\sqrt{\frac{\mu^2\lambda}{2}}\,\chi^3 - \lambda\chi^4 \right]$$
$$+ \left(\sqrt{\frac{2\mu^2}{\lambda}}\,\chi + \chi^2 \right) (\partial\theta)^2 \qquad - \lambda v^4 .$$

The field χ has mass μ, but the field $\theta(x)$ is massless. It is the N-G boson.

This happens when the number of space-time dimensions is $d > 2$. But if $d \leq 2$, then

$$\langle T(\phi_2(x)\,\phi_2(0))\rangle = \int \frac{d^d x}{(2\pi)^d} \frac{e^{ikx}}{k^2}$$

has an infra-red divergence. (We cancel the U.V. divergence by renormalization.)



The number of N-G bosons is the number of conserved charges that don't annihilate the vacuum. If $\mathcal{L}$ is invariant under a group G with $n(G)$ generators, but the vacuum is left invariant only by a subgroup H of G with $n(H)$ generators, then there are $n(G)-n(H)$ Nambu-Goldstone bosons. They live in the coset space G/H.

Incidentally, the field ϕ that got a mean value in the vacuum was elementary. Had it been a dynamically generated field — like a Cooper-pair field in superconductivity — we'd have called the symmetry breaking <u>dynamical</u>.