Warning: these notes have been translated from the normal metric to the Peskin metric $(+,-,-,-)$.

Example: **Feynman's propagator** for a spinless quantum field $\phi(x)$ of mass m is

$$\triangle_F(x) = \int \frac{\exp(-ikx)}{-k^2+m^2-i\epsilon} \frac{d^4k}{(2\pi)^4} \tag{0.1}$$

where

$$kx \equiv -\boldsymbol{k}\cdot\boldsymbol{x} + k^0x^0 \tag{0.2}$$

$x^0 = ct$, and all physical quantities are in **natural units** ($c = \hbar = 1$). The tiny imaginary term $-i\epsilon$ makes $\triangle_F(x-y)$ proportional to the mean-value in the vacuum state $|0\rangle$ of the **time-ordered product**

$$\mathcal{T}\{\phi(x)\phi(y)\} \equiv \theta(x^0-y^0)\phi(x)\phi(y) + \theta(y^0-x^0)\phi(y)\phi(x) \tag{0.3}$$

of the fields $\phi(x)$ and $\phi(y)$ in which $\theta(a) = (a+|a|)/(2|a|)$ is the Heaviside step function. The exact formula is

$$\langle 0|\mathcal{T}\{\phi(x)\phi(y)\}|0\rangle = -i\,\triangle_F(x-y). \tag{0.4}$$

P&S's D_F is $\langle 0|\,\mathcal{T}\{\phi(x)\phi(y)\}\,|0\rangle$

$$\begin{aligned} D_F(x-y) &= -i\,\triangle_F(x-y) = \langle 0|\,\mathcal{T}\{\phi(x)\phi(y)\}\,|0\rangle \\ &= \int e^{-ikx}\,\frac{i}{k^2-m^2+i\epsilon}\,\frac{d^4k}{(2\pi)^4}. \end{aligned} \tag{0.5}$$

***Example*—The Feynman Propagator:** Adding $\pm i\epsilon$ to the denominator of a pole term of an integral formula for a function $f(x)$ can slightly shift the pole into the upper or lower half plane, causing the pole to contribute if a ghost contour goes around the UHP or the LHP. The choice of ghost contour often is influenced by the argument x of the function $f(x)$. Such $i\epsilon$'s impose boundary conditions on Green's functions.

The Feynman propagator $\Delta_F(x)$ is a Green's function for the Klein-Gordon differential operator (Weinberg, 1995, pp. 274–280)

$$(\Box + m^2)\Delta_F(x) = \delta^4(x) \tag{0.6}$$

in which $x = (x^0, \boldsymbol{x})$ and

$$\Box = \frac{\partial^2}{\partial t^2} - \triangle = \frac{\partial^2}{\partial (x^0)^2} - \triangle \tag{0.7}$$

is the four-dimensional version of the laplacian $\triangle \equiv \nabla\cdot\nabla$. Here $\delta^4(x)$ is the four-dimensional version of Dirac's delta function

$$\delta^4(x) = \int \frac{d^4q}{(2\pi)^4} \exp[\pm i(\boldsymbol{q}\cdot\boldsymbol{x} - q^0x^0)] = \int \frac{d^4q}{(2\pi)^4}\, e^{\pm iqx} \tag{0.8}$$

in which $qx = q^0x^0 - \boldsymbol{q}\cdot\boldsymbol{x}$ is the Lorentz-invariant inner product of the 4-vectors q and x. There are many Green's functions that satisfy Eq.(0.6).

Feynman's propagator $\Delta_F(x)$ is the one that satisfies certain boundary conditions which will become evident when we analyze the effect of its $i\epsilon$

$$\Delta_F(x) = \int \frac{d^4q}{(2\pi)^4} \frac{\exp(-iqx)}{-q^2+m^2-i\epsilon}. \tag{0.9}$$

The quantity $E_{\mathbf{q}} = \sqrt{\boldsymbol{q}^2+m^2}$ is the energy of a particle of mass m and momentum $\boldsymbol{q}$ in natural units with the speed of light $c = 1$. Using this abbreviation and setting $\epsilon' = \epsilon/(2E_q)$, we may write the denominator as

$$-q^2+m^2-i\epsilon = \boldsymbol{q}\cdot\boldsymbol{q} - \left(q^0\right)^2 + m^2 - i\epsilon = \left(E_q - i\epsilon' - q^0\right)\left(E_q - i\epsilon' + q^0\right) + \epsilon'^2 \tag{0.10}$$

in which ϵ'^2 is negligible. We now drop the prime on the ϵ and do the q^0 integral

$$I(\boldsymbol{q}) = -\int_{-\infty}^{\infty} \frac{dq^0}{2\pi}\, e^{-iq^0x^0}\, \frac{1}{[q^0-(E_{\mathbf{q}}-i\epsilon)]\,[q^0-(-E_{\mathbf{q}}+i\epsilon)]}. \tag{0.11}$$

The function

$$f(q^0) = e^{-iq^0x^0}\, \frac{1}{[q^0-(E_{\mathbf{q}}-i\epsilon)]\,[q^0-(-E_{\mathbf{q}}+i\epsilon)]} \tag{0.12}$$

has poles at $E_{\mathbf{q}} - i\epsilon$ and at $-E_{\mathbf{q}} + i\epsilon$, as shown in Fig. 0.1. If $x^0 > 0$, then we can add a ghost contour that goes cw around the LHP, and we get

$$I(\boldsymbol{q}) = ie^{-iE_{\mathbf{q}}x^0}\, \frac{1}{2E_{\mathbf{q}}} \quad x^0 > 0. \tag{0.13}$$

If $x^0 < 0$, we add a ghost contour that goes ccw around the UHP, and we get

$$I(\boldsymbol{q}) = ie^{iE_{\mathbf{q}}x^0}\, \frac{1}{2E_{\mathbf{q}}} \quad x^0 < 0. \tag{0.14}$$

Using Heaviside's step function

$$\theta(x) = \frac{x+|x|}{2}, \tag{0.15}$$

we may combine the last two equations into

$$-iI(\boldsymbol{q}) = \frac{1}{2E_{\mathbf{q}}}\left[\theta(x^0)\, e^{-iE_{\mathbf{q}}x^0} + \theta(-x^0)\, e^{iE_{\mathbf{q}}x^0}\right]. \tag{0.16}$$

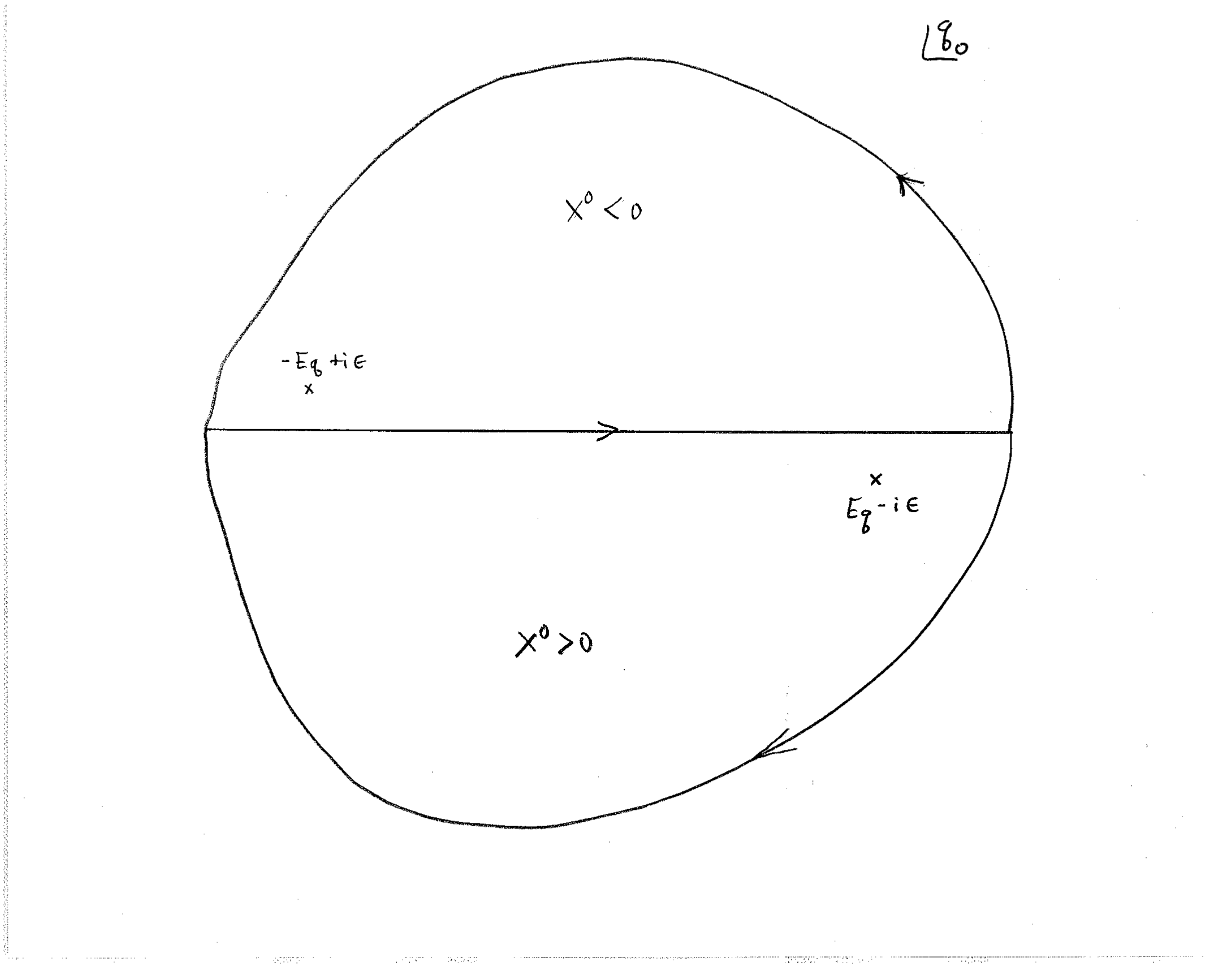

Figure 0.1 In Eq. (0.12), the function $f(q^0)$ has poles at $\pm(E_{\mathbf{q}} - i\epsilon)$, and the function $\exp(-iq^0x^0)$ is exponentially suppressed in the LHP if $x^0 > 0$ and in the UHP if $x^0 < 0$. So we can add a ghost contour in the LHP if $x^0 > 0$ and in the UHP if $x^0 < 0$.

In terms of the Lorentz-invariant function

$$\Delta_+(x) = \frac{1}{(2\pi)^3} \int \frac{d^3q}{2E_{\mathbf{q}}} \exp[i(\boldsymbol{q} \cdot \boldsymbol{x} - E_{\mathbf{q}} x^0)] \tag{0.17}$$

and with a factor of $-i$, the Feynman propagator is

$$-i\Delta_F(x) = \theta(x^0)\,\Delta_+(x) + \theta(-x^0)\,\Delta_+(\mathbf{x}, -x^0). \tag{0.18}$$

But the integral (0.17) defining $\Delta_+(x)$ is insensitive to the sign of $\boldsymbol{q}$, and so

$$\begin{aligned}\Delta_+(-x) &= \frac{1}{(2\pi)^3}\int \frac{d^3q}{2E_{\mathbf{q}}}\,\exp[i(-\boldsymbol{q}\cdot\boldsymbol{x}+E_{\mathbf{q}}x^0)] \\ &= \frac{1}{(2\pi)^3}\int \frac{d^3q}{2E_{\mathbf{q}}}\,\exp[i(\boldsymbol{q}\cdot\boldsymbol{x}+E_{\mathbf{q}}x^0)] = \Delta_+(\mathbf{x},-x^0).\end{aligned} \tag{0.19}$$

Thus we arrive at the standard form of the Feynman propagator

$$-\,i\Delta_F(x) = \theta(x^0)\,\Delta_+(x) + \theta(-x^0)\,\Delta_+(-x). \tag{0.20}$$

The Lorentz-invariant function $\Delta_+(x-y)$ is the commutator of the positive-frequency part

$$\phi^+(x) = \int \frac{d^3p}{(2\pi)^3\sqrt{2p^0}}\,\exp[i(\boldsymbol{p}\cdot\boldsymbol{x}-p^0x^0)]\,a(\boldsymbol{p}) \tag{0.21}$$

of a scalar field $\phi = \phi^+ + \phi^-$ with its negative-frequency part

$$\phi^-(y) = \int \frac{d^3q}{(2\pi)^3\sqrt{2q^0}}\,\exp[-i(\boldsymbol{q}\cdot\boldsymbol{y}-q^0y^0)]\,a^\dagger(\boldsymbol{q}) \tag{0.22}$$

where $p^0 = E_{\mathbf{p}} = \sqrt{\mathbf{p}^2+m^2}$ and $q^0 = E_{\mathbf{q}}$. For since the annihilation operators $a(\boldsymbol{q})$ and the creation operators $a^\dagger(\boldsymbol{p})$ satisfy the commutation relation

$$[a(\boldsymbol{q}), a^\dagger(\boldsymbol{p})] = (2\pi)^3\,\delta^3(\boldsymbol{q}-\boldsymbol{p}) \tag{0.23}$$

we have

$$\begin{aligned}[\phi^+(x), \phi^-(y)] &= \int \frac{d^3p\,d^3q}{(2\pi)^6\sqrt{2q^0 2p^0}}\,e^{-ipx+iqy}\,[a(\mathbf{p}), a^\dagger(\mathbf{q})] \\ &= \int \frac{d^3p}{(2\pi)^3 2p^0}\,e^{-ip(x-y)} = \Delta_+(x-y)\end{aligned} \tag{0.24}$$

in which $px = p^0x^0 - \mathbf{p}\cdot\mathbf{x}$, *etc.*

Incidentally, at points x that are space-like

$$x^2 = (x^0)^2 - \mathbf{x}^2 \equiv -\,r^2 < 0 \tag{0.25}$$

the Lorentz-invariant function $\Delta_+(x)$ depends only upon $r = +\sqrt{-\,x^2}$ and has the value (Weinberg, 1995, p. 202)

$$\Delta_+(x) = \frac{m}{4\pi^2 r}\,K_1(mr) \tag{0.26}$$

in which the Hankel function K_1 is

$$K_1(z) = -\frac{\pi}{2}\,[J_1(iz) + iN_1(iz)] = \frac{1}{z} + \frac{z}{2j+2}\left[\ln\left(\frac{z}{2}\right) + \gamma - \frac{1}{2j+2}\right] + \dots \tag{0.27}$$

where J_1 is the first Bessel function, N_1 is the first Neumann function, and $\gamma = 0.57721\dots$ is the Euler-Mascheroni constant.

The Feynman propagator arises most simply as the mean value in the vacuum of the **time-ordered product** of the fields $\phi(x)$ and $\phi(y)$

$$\mathcal{T}\{\phi(x)\phi(y)\} \equiv \theta(x^0 - y^0)\phi(x)\phi(y) + \theta(y^0 - x^0)\phi(y)\phi(x). \tag{0.28}$$

Since the operators $a(\mathbf{p})$ and $a^\dagger(\boldsymbol{p})$ respectively annihilate the vacuum ket $a(\boldsymbol{p})|0\rangle = 0$ and bra $\langle 0|a^\dagger(\boldsymbol{p}) = 0$, the same is true of the positive- and negative-frequency parts of the field: $\phi^+(z)|0\rangle = 0$ and $\langle 0|\phi^-(z) = 0$. Thus, the mean value in the vacuum of the time-ordered product is proportional to the Feynman propagator $-i\Delta_F(x-y)$

$$\begin{aligned}
\langle 0|\mathcal{T}\{\phi(x)\phi(y)\}|0\rangle &= \langle 0|\theta(x^0 - y^0)\phi(x)\phi(y) + \theta(y^0 - x^0)\phi(y)\phi(x)|0\rangle \\
&= \langle 0|\theta(x^0 - y^0)\phi^+(x)\phi^-(y) + \theta(y^0 - x^0)\phi^+(y)\phi^-(x)|0\rangle \\
&= \langle 0|\theta(x^0 - y^0)[\phi^+(x), \phi^-(y)] \\
&\qquad + \theta(y^0 - x^0)[\phi^+(y), \phi^-(x)]|0\rangle \\
&= \theta(x^0 - y^0)\Delta_+(x-y) + \theta(y^0 - x^0)\Delta_+(y-x) \\
&= -i\Delta_F(x-y)
\end{aligned} \tag{0.29}$$

in the last step of which we used (0.20). Feynman put $i\epsilon$ in the denominator of the Fourier transform of his propagator to get this result.

References

Weinberg, S. 1995. *The Quantum Theory of Fields*. Vol. I Foundations. Cambridge, UK: Cambridge University Press.