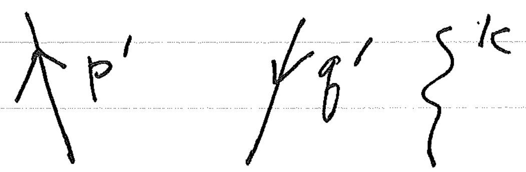
Feynman's Rules for QED

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}\left[i\gamma^\mu(\partial_\mu + ieA_\mu) - m\right]\psi$$

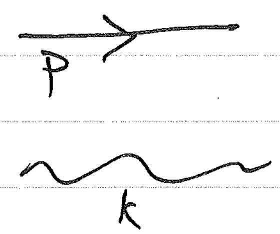
$$J^\mu = -e\,\bar{\psi}\gamma^\mu\psi \qquad (e>0)$$

Out



$\bar{u}(p',s')$, $v(q',s')$

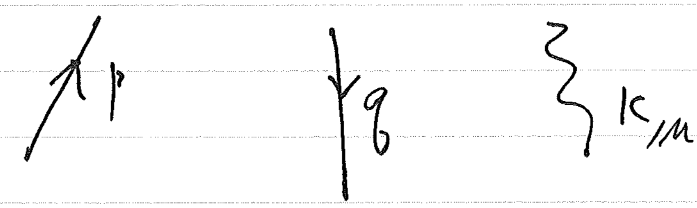
$\epsilon_\mu^*(k)$



$$\frac{i(\not{p}+m)}{p^2-m^2+i\epsilon}$$

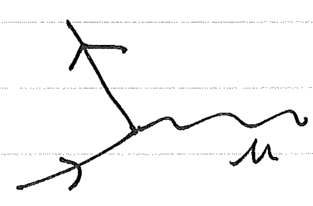
$$\frac{-i\eta_{\mu\nu}}{k^2+i\epsilon}$$

in



$u(p,s)$, $\bar{v}(q,s)$

$\epsilon_\mu(k)$



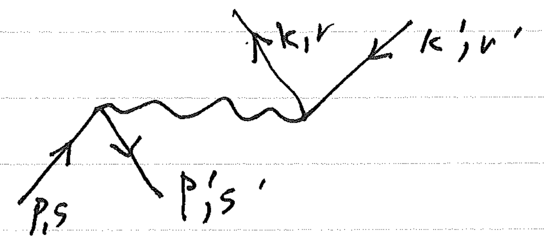
$$-ie\gamma^\mu$$

These give $i\mathcal{M}$; $S = i\mathcal{M}\,\delta^{(4)}(\Sigma p' - \Sigma p)\,(2\pi)^4$.
But watch out for minus signs due to ψ and $\bar{\psi}$.

Now we apply these rules to $e^+e^- \to \mu^+\mu^-$.

$e^+ e^- \to \mu^+ \mu^-$

$$\mathcal{H}_I(x) \equiv V(x) = -e \bar{\Psi}_e \gamma^\mu \Psi_e A_\mu - e \bar{\Psi}_\mu \gamma^\nu \Psi_\mu A_\nu$$



Apply Feynman rules: The amplitude $i\mathcal{M}$ is

$$\bar{v}(p',s')(-ie\gamma^\mu)\, u(p,s) \frac{(-i\eta_{\mu\nu})}{(p+p')^2 - i\epsilon} \bar{u}(k,r)(-ie\gamma^\nu) v(k',r')$$

$$= ie^2 \bar{v}(p',s')\gamma^\mu \frac{u(p,s)\,\eta_{\mu\nu}\,\bar{u}(k,r)}{(p+p')^2}\gamma^\nu v(k',r')$$

$$= ie^2 \bar{v}(p',s')\gamma^\mu \frac{u(p,s)\; \bar{u}(k,r)\gamma_\mu}{(p+p')^2} v(k',r')$$

$$S = ie^2 \frac{\bar{v}\;\gamma^\mu u\, \bar{u}'\gamma_\mu v'}{(p+p')^2} (2\pi)^4 \delta^{(4)}(k+k'-p-p')$$

$$\equiv i\mathcal{M}\, (2\pi)^4 \delta^{(4)}(k+k'-p-p').$$

We'll need $|M|^2$.

$$(\bar{v}\gamma^\mu u)^* = u^\dagger \gamma^{\mu\dagger} \gamma^{0\dagger} v$$

$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ so $\gamma^{0\dagger} = +\gamma^0$

$\gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma_i & 0 \end{pmatrix}$ so $\gamma^{i\dagger} = -\gamma^i$.

So

$$(\bar{v}\gamma^\mu u)^* = u^\dagger \begin{Bmatrix} \gamma^0\gamma^0 \\ -\gamma^i\gamma^0 \end{Bmatrix} v = u^\dagger \begin{Bmatrix} \gamma^0\gamma^0 \\ \gamma^0\gamma^i \end{Bmatrix} v = \bar{u}\gamma^\mu v$$

since $\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}$.

So

$|\bar{v}\gamma^\mu u|^2 = \bar{u}\gamma^\mu v \, \bar{v}\gamma^\mu u$, but we actually have

$$(\bar{v}\gamma^\mu u \, \bar{u}'\gamma_\mu v')^* (\bar{v}\gamma^\nu u \, \bar{u}'\gamma_\nu v')$$

$$= \bar{u}\gamma^\mu v \, \bar{v}'\gamma_\mu u' \, \bar{v}\gamma^\nu u \, \bar{u}'\gamma_\nu v'$$

$$= \bar{u}\gamma^\mu v \, \bar{v}\gamma^\nu u \quad \bar{v}'\gamma_\mu u' \, \bar{u}'\gamma_\nu v'.$$

We now sum over the spins of the μ's and average over the spins of the e's. So we sum over all the spins and divide by 4.

$$\sum_{s,s'} \bar{u}_\alpha(p,s)\,\gamma^\mu_{\alpha\beta}\,v_\beta(p',s')\,\bar{v}_a(p',s')\,\gamma^\nu_{ab}\,u_b(p,s)$$

$$= \sum_{ss'} \gamma^\mu_{\alpha\beta}\,v_\beta(p',s')\,\bar{v}_a(p',s')\,\gamma^\nu_{ab}\,u_b(p,s)\,\bar{u}_\alpha(p,s)$$

$$= \gamma^\mu_{\alpha\beta}\,(\not{p}'-m)_{\beta a}\,\gamma^\nu_{ab}\,(\not{p}+m)_{b\alpha}$$

$$= \mathrm{tr}\left[\gamma^\mu(\not{p}'-m_e)\gamma^\nu(\not{p}+m_e)\right].$$

So we've "done" the incoming spinors.

The outgoing ones are

$$\sum_{rr'} \bar{v}(k',r')\,\gamma_\mu\,u(k,r)\,\bar{u}(k,r)\,\gamma_\nu\,v(k',r')$$

$$= \mathrm{tr}\left[\gamma_\mu(\not{k}+m_\mu)\gamma_\nu(\not{k}'-m_\mu)\right]$$

So

$$S = (2\pi)^4\,\delta^4(k+k'-p-p')\,i\,M \quad \text{and}$$

$$\frac{1}{4}\sum_{ss'}\sum_{rr'} |M|^2 = \frac{e^4}{4(p+p')^4}\,\mathrm{tr}\left[(\not{p}'-m_e)\gamma^\nu(\not{p}+m_e)\gamma^\mu\right]$$

$$\times\,\mathrm{tr}\left[(\not{k}+m_\mu)\gamma_\nu(\not{k}'-m_\mu)\gamma_\mu\right].$$

We'll now drop the m_e terms since $m_e/m_\mu \sim 1/200$.

So the electron trace is

$$\mathrm{tr}\,[\gamma^\mu \not{p}' \gamma^\nu \not{p}] = p'_\rho\, p_\sigma\, \mathrm{tr}\,[\gamma^\mu \gamma^\rho \gamma^\nu \gamma^\sigma].$$

First $\mathrm{tr}\,\gamma^\mu = 0$. Next

$$\mathrm{tr}\,\gamma^\mu \gamma^\nu = \mathrm{tr}\,\gamma^\nu\gamma^\mu = \tfrac{1}{2}\mathrm{tr}\{\gamma^\mu, \gamma^\nu\} = \mathrm{tr}\,\eta^{\mu\nu} I$$

$$= \eta^{\mu\nu} 4 = 4\eta^{\mu\nu}.$$

$\mathrm{tr}\,\gamma^a\gamma^b\gamma^c$: Well, $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ and

$$\gamma^{5^2} = -\gamma^0\gamma^1\gamma^2\gamma^3\gamma^0\gamma^1\gamma^2\gamma^3$$
$$= \gamma^{0^2}\gamma^1\gamma^2\gamma^3\gamma^1\gamma^2\gamma^3 = \gamma^1\gamma^2\gamma^3\gamma^1\gamma^2\gamma^3$$
$$= \gamma^{1^2}\gamma^2\gamma^3\gamma^2\gamma^3 = -\gamma^2\gamma^3\gamma^2\gamma^3 = \gamma^{2^2}\gamma^{3^2}$$
$$= (-1)^2 = I. \quad \text{Also}$$

$$\gamma^a\gamma^5 = -\gamma^5\gamma^a \quad \text{if } a = 0,1,2, \text{ or } 3.$$

So

$$\mathrm{tr}\,\gamma^a\gamma^b\gamma^c = \mathrm{tr}\,\gamma^a\gamma^b\gamma^c\gamma^{5^2} = -\mathrm{tr}\,\gamma^5\gamma^a\gamma^b\gamma^c\gamma^5$$
$$= -\mathrm{tr}\,\gamma^a\gamma^b\gamma^c\gamma^{5^2} = -\mathrm{tr}\,\gamma^a\gamma^b\gamma^c$$

So $2\,\mathrm{tr}\,\gamma^a\gamma^b\gamma^c = 0$.

$$\mathrm{tr}\,\gamma^a\gamma^b\gamma^c\gamma^d = \mathrm{tr}\left(2\eta^{ab}\gamma^c\gamma^d - \gamma^b\gamma^a\gamma^c\gamma^d\right)$$

$$= \mathrm{tr}\left(2\eta^{ab}\gamma^c\gamma^d - \gamma^b(2\eta^{ac} - \gamma^c\gamma^a)\gamma^d\right)$$

$$= \mathrm{tr}\left(2\eta^{ab}\gamma^c\gamma^d - 2\eta^{ac}\gamma^b\gamma^d + \gamma^b\gamma^c\gamma^a\gamma^d\right)$$

$$= \mathrm{tr}\,(2\eta^{ab}\gamma^c\gamma^d - 2\eta^{ac}\gamma^b\gamma^d + 2\eta^{ad}\gamma^b\gamma^c - \gamma^b\gamma^c\gamma^d\gamma^a)$$

So

$$\mathrm{tr}\,\gamma^a\gamma^b\gamma^c\gamma^d = 4\eta^{ab}\eta^{cd} - 4\eta^{ac}\eta^{bd} + 4\eta^{ad}\eta^{bc}.$$

There exist computer programs to do such things.

$\mathrm{tr}\,(\gamma^5)^{2n+1} = 0$. Also $\mathrm{tr}\,\gamma^a\gamma^b\gamma^5 = 0$.

And $\mathrm{tr}\,\gamma^a\gamma^b\gamma^c\gamma^d\gamma^5 = -4i\,\epsilon^{abcd}$ where

$\epsilon^{0123} = 1$, and ϵ is totally antisymmetric.

So the e-trace is for $m_e = 0$

$$\mathrm{tr}\,[\gamma^\mu \not{p}' \gamma^\nu \not{p}] = p'_\rho\, p_\sigma\, \mathrm{tr}\,\gamma^\mu\gamma^\rho\gamma^\nu\gamma^\sigma]$$

$$= 4\, p'_\rho p_\sigma \left(\eta^{\mu\rho}\eta^{\nu\sigma} - \eta^{\mu\nu}\eta^{\rho\sigma} + \eta^{\mu\sigma}\eta^{\rho\nu}\right)$$

$$= 4\,(p'^\mu p^\nu - \eta^{\mu\nu} p' p + p^\mu p'^\nu).$$

The muon trace is

$$\mathrm{tr}\,[(\not{k} + m_\mu)\gamma_\nu(\not{k}' - m_\mu)\gamma_\mu] = \mathrm{tr}\,\not{k}\gamma_\nu\not{k}'\gamma_\mu - m_\mu^2\,\mathrm{tr}\,\gamma_\nu\gamma_\mu$$

$$= 4\,(k'_\mu k_\nu + k_\mu k'_\nu - \eta_{\mu\nu}(k k' + m_\mu^2)).$$

So

$$\frac{1}{4}\sum_{\substack{ss'\\ rr'}} |M|^2 = \frac{4e^4}{(p+p')^4}\left(p'^{\mu}p^{\nu} + p^{\mu}p'^{\nu} - \eta^{\mu\nu}pp'\right)\left(k'_{\mu}k_{\nu} + k_{\mu}k'_{\nu} - \eta_{\mu\nu}(kk'+m_\mu^2)\right)$$

$$= \frac{4e^4}{(p+p')^4}\Big[p'k'pk + p'kpk' - p'p(k'k+m_\mu^2)$$

$$+ pk'p'k + pkp'k' - pp'(kk'+m_\mu^2)$$

$$- kk'pp' - kk'pp' + 4\,pp'(kk'+m_\mu^2)\Big]$$

$$= \frac{8e^4}{(p+p')^4}\left[(pk)(p'k') + (pk')(p'k) + m_\mu^2\, pp'\right]$$

Go to c.o.m. frame $p = E(1,\hat{z})$ $p' = E(1,-\hat{z})$

$k = (E, k)$ $k' = (E, -k)$ $k\cdot\hat{z} = k\cos\theta$

$k \equiv |\vec{k}|.$

$(p+p')^2 = 4E^2$ $p\cdot p' = 2E^2$

$p\cdot k = p'\cdot k' = E^2 - Ek\cos\theta$ $p\cdot k' = p'\cdot k = E^2 + Ek\cos\theta$

then

$$\frac{1}{4}\sum_{\substack{ss'\\ rr'}} |M|^2 = \frac{8e^4}{16E^4}\left[E^2(E-k\cos\theta)^2 + E^2(E+k\cos\theta)^2 + 2m_\mu^2E^2\right]$$

$$= e^4\left[\left(1+\frac{m_\mu^2}{E^2}\right) + \left(1-\frac{m_\mu^2}{E^2}\right)\cos^2\theta\right].$$

$$S \to \frac{d\sigma}{d\Omega}$$

$$S = i M (2\pi)^4 \delta^4(k+k'-p-p')$$

$$|p\rangle = \sqrt{2E_p}\, a^\dagger(p,s)|0\rangle \qquad \{a, a'^\dagger\} = (2\pi)^3 \delta(p-p')$$

$$\delta^3(p-p') = \int \frac{d^3x}{(2\pi)^3} e^{i(p-p')\cdot x} \qquad \text{So} \quad \delta^{(3)}(0) = \frac{V}{(2\pi)^3}$$

So

$$\| |p\rangle \|^2 = \langle p|p\rangle = 2E_p (2\pi)^3 \delta^3(0) = 2E_p V.$$

So for states of unit norm, the probability is

$$P_r = \frac{|S|^2}{(2EV)^4} = \frac{|M|^2 (2\pi)^4 \delta(k+k'-p-p') VT}{(2V)^4\, E_k E_{k'} E_p E_{p'}}$$

the rate is

$$R = \frac{P_r}{T} = \frac{|M|^2 (2\pi)^4 \delta^4(k+k'-p-p')\, V}{(2E's)\; V^4} = \tilde{\sigma} f$$

where $f = \text{flux} = \frac{v}{V}$. So

$$\tilde{\sigma} = \frac{R}{f} = \frac{|M|^2 (2\pi)^4 \delta^{(4)}(k+k'-p-p')\, V}{(2E's)\; V^3\; v}$$

$$= \frac{|M|^2 (2\pi)^4 \delta^{(4)}(k+k'-p-p')}{(2E's)\; V^2\; v}$$

The number of final states of k and k' is their phase-space volume divided by h^6. So

$$d\sigma = \tilde{\sigma}\, \frac{d^3k\, V}{(2\pi)^3} \frac{d^3k'\, V}{(2\pi)^3}$$

$$= \frac{\frac{1}{4}\sum |M|^2 (2\pi)^4\, \delta^{(4)}(k+k'-p-p')\, d^3k\, d^3k'}{(2E's)\, v\, (2\pi)^6} \qquad (4.79)$$

Now here $E_k = E_{k'} = E$

$$\int \frac{d^3k\, d^3k'\, (2\pi)^4\, \delta^4(k+k'-p-p')}{(2\pi)^3(2\pi)^3\, 2E_k\, 2E_{k'}} = \int \frac{dk\, k^2 d\Omega\, \delta(2E-2E_k)}{(2\pi)^2 (2E_k)^2}$$

$$= \frac{k^2 d\Omega}{16\pi^2 E^2} \frac{1}{\frac{2k}{E}} = \frac{k\, d\Omega}{32\pi^2 E} = \frac{k\, d\Omega}{16\pi^2 E_{tot}}$$

So

$$\left(\frac{d\sigma}{d\Omega}\right)_{cm} = \frac{1}{2E_p\, 2E_q\, v\, (2\pi)^2\, 4E_{cm}}\, k \sum \frac{1}{4}|M|^2$$

$E_p = E_q = \frac{1}{2}E_{tot} = \frac{1}{2}E_{cm}$

$v = 2$

$\frac{e^2}{4\pi} \equiv \alpha \approx \frac{1}{137}$

$$= \frac{1}{2E_{cm}^2} \frac{k}{16\pi^2 E_{cm}} \frac{1}{4} \sum |M|^2$$

$$= \frac{\alpha^2}{4E_{cm}^2} \sqrt{1 - \frac{m_\mu^2}{E^2}} \left[\left(1 + \frac{m_\mu^2}{E^2}\right) + \left(1 - \frac{m_\mu^2}{E^2}\right)\cos^2\theta\right]$$

$$\alpha = \frac{e^2}{4\pi \hbar c}$$

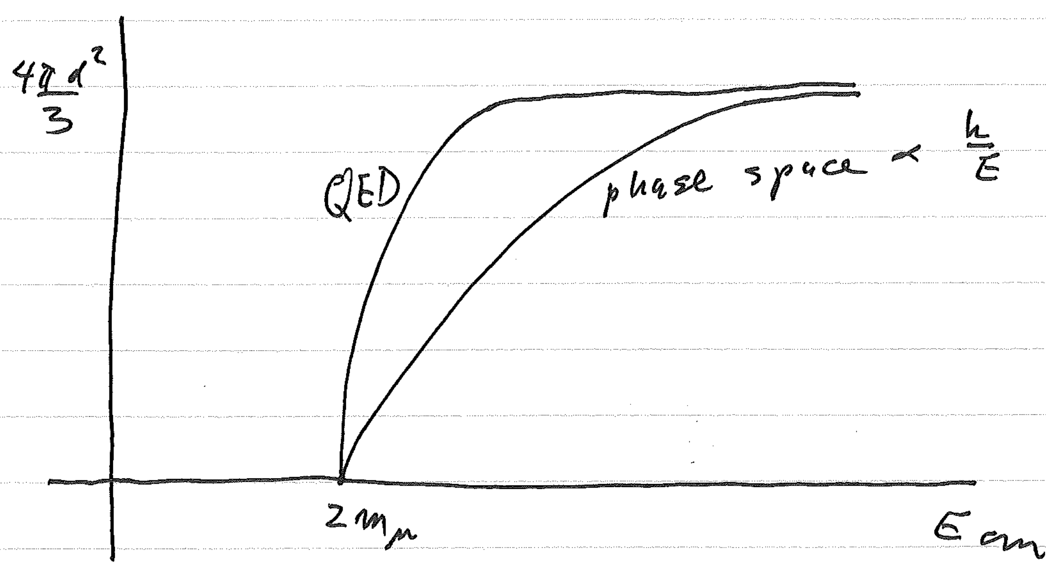
$$\int d\Omega = 4\pi; \quad \int \cos^2\theta \, d\Omega = 2\pi \int_0^\pi \sin\theta \cos^2\theta \, d\theta$$

$$= 2\pi \int_{-1}^{1} x^2 dx = 2\pi \frac{2}{3} = \frac{4\pi}{3}$$

So

$$\sigma = \int \frac{d\sigma}{d\Omega} d\Omega = \frac{\alpha^2}{4 E_{cm}^2} \sqrt{1 - \frac{m_\mu^2}{E^2}} \left[\left(1 + \frac{m_\mu^2}{E^2}\right) 4\pi + \left(1 - \frac{m_\mu^2}{E^2}\right) \frac{4\pi}{3} \right]$$

$$= \frac{4\pi \alpha^2}{3 E_{cm}^2} \sqrt{1 - \frac{m_\mu^2}{E^2}} \left(1 + \frac{1}{2} \frac{m_\mu^2}{E^2} \right).$$



For $E^2 \gg m_\mu^2$, $\quad \frac{d\sigma}{d\Omega} \to \frac{\alpha^2}{4 E_{cm}^2} (1 + \cos^2\theta)$

$$\sigma \to \frac{4\pi\alpha^2}{3 E_{cm}^2} \left(1 - \frac{3}{8} \left(\frac{m_\mu}{E}\right)^4 \cdots \right).$$

At high energy,
E_{cm} is the only relevant dimensionful quantity.

$e^+ e^- \to q^+ q^-$

The quarks u, c, and t have charge $\frac{2}{3}|e|$, the d, s, and b quarks have charge $-\frac{1}{3}|e|$.

The strong (QCD) interaction dresses the quarks with q-$\bar{q}$ pairs and gluons, so one sees colorless hadrons in the final state.

There are 3 colors, so in the high-energy limit

$$\sigma_{u\bar{u}} = \frac{4\pi\alpha^2}{3E_{cm}^2} \times 3 \times \left(\frac{2}{3}\right)^2$$

$$\sigma_{d\bar{d}} = \frac{4\pi\alpha^2}{3E_{cm}^2} \times 3 \times \left(\frac{1}{3}\right)^2 .$$

The unit $\quad R \equiv \dfrac{4\pi\alpha^2}{3E_{cm}^2} = \dfrac{86.8 \text{ nbarns}}{E_{cm}^2}$.

So once E_{cm} is well passed the threshold for $q\bar{q}$ production

$$\sigma(e^+e^- \to \text{hadrons}) \to 3 \sum Q_i^2 \, R \qquad \text{as } E_{cm} \to \infty$$