

where τ^{ν}_{μ} is the non-tensor

$$\tau^{\nu}_{\mu} \equiv T^{\nu}_{\mu} + \frac{1}{8\pi G} \left(R^{\nu}_{\mu} - \frac{1}{2} \delta^{\nu}_{\mu} R \right)_{\text{NONLINEAR}},$$

analogous to $\mathcal{J}_{\alpha}^{\nu}$. Like $\mathcal{J}_{\alpha}^{\nu}$, τ^{ν}_{μ} is conserved in the ordinary sense

$$\partial_{\nu} \tau^{\nu}_{\mu} = 0$$

and may be regarded as the current of energy and momentum:

$$P_{\mu} = \int \tau^0_{\mu} d^3x.$$

It contains a purely gravitational term, because gravitational fields carry energy and momentum; without this term, τ^{ν}_{μ} could not be conserved. Similarly, $\mathcal{J}_{\alpha}^{\nu}$ contains a gauge-field term (the first term on the right in Eq. (15.3.3)) because for non-Abelian groups (those with $C^{\gamma}_{\alpha\beta} \neq 0$) the gauge fields carry the quantum numbers with which they interact. Because $\mathcal{J}_{\alpha}^{\nu}$ is conserved in the ordinary sense, it can be regarded as the current of these quantum numbers, with the symmetry generators given by the time-independent quantities

$$T_{\alpha} = \int \mathcal{J}_{\alpha}^0 d^3x. \quad (15.3.10)$$

(Also, the homogeneous equations (15.3.9) involve covariant derivatives, just as do the Bianchi identities of general relativity.) In contrast, none of these complications arises in quantum electrodynamics, because photons do not carry the quantum number, electric charge, with which they interact.

15.4 Quantization

We now proceed to quantize the gauge theories described in the previous two sections. The Lagrangian density is taken in the form (15.3.1):

$$\mathcal{L} = -\frac{1}{4} F_{\alpha\mu\nu} F^{\mu\nu}_{\alpha} + \mathcal{L}_M(\psi, D_{\mu}\psi), \quad (15.4.1)$$

with

$$\begin{aligned} F_{\alpha\mu\nu} &\equiv \partial_{\mu} A_{\alpha\nu} - \partial_{\nu} A_{\alpha\mu} + C_{\alpha\beta\gamma} A_{\beta\mu} A_{\gamma\nu}, \\ D_{\mu}\psi &\equiv \partial_{\mu}\psi - it_{\alpha} A_{\alpha\mu}\psi. \end{aligned}$$

We cannot immediately quantize this theory by setting commutators equal to i times the corresponding Poisson brackets. The problem is one of constraints. In the terminology of Dirac, described in Section 7.6, there is

a primary constraint that

$$\Pi_{\alpha 0} \equiv \frac{\partial \mathcal{L}}{\partial(\partial_0 A_\alpha^0)} = 0 \quad (15.4.2)$$

and a secondary constraint provided by the field equation for A_α^0 :

$$\begin{aligned} -\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_{\alpha 0})} + \frac{\partial \mathcal{L}}{\partial A_{\alpha 0}} &= \partial_\mu F_\alpha^{\mu 0} + F_\gamma^{\mu 0} C_{\gamma\alpha\beta} A_{\beta\mu} + J_\alpha^0 \\ &= \partial_k \Pi_\alpha^k + \Pi_\gamma^k C_{\gamma\alpha\beta} A_{\beta k} + J_\alpha^0 = 0, \end{aligned} \quad (15.4.3)$$

where $\Pi_\alpha^k \equiv \partial \mathcal{L} / \partial(\partial_0 A_{\alpha k}) = F_\alpha^{k0}$ is the ‘momentum’ conjugate to $A_{\alpha k}$, with k running over the values 1, 2, 3. The Poisson brackets of $\Pi_{\alpha 0}$ and $\partial_k \Pi_\alpha^k + \Pi_\gamma^k C_{\gamma\alpha\beta} A_{\beta k} + J_\alpha^0$ vanish (because the latter quantity is independent of A_α^0), so these are first class constraints, which cannot be dealt with by replacing Poisson brackets with Dirac brackets.

As in the case of electrodynamics, we deal with these constraints by choosing a gauge. The Coulomb gauge adopted for electrodynamics would lead to painful complications here,* so instead we will work in what is known as *axial gauge*, based on the condition

$$A_{\alpha 3} = 0. \quad (15.4.4)$$

The canonical variables of the gauge field are then $A_{\alpha i}$, with i now running over the values 1 and 2, together with their canonical conjugates

$$\Pi_{\alpha i} \equiv \frac{\partial \mathcal{L}}{\partial(\partial_0 A_{\alpha i})} = -F_\alpha^{0i} = \partial_0 A_{\alpha i} - \partial_i A_{\alpha 0} + C_{\alpha\beta\gamma} A_{\beta 0} A_{\gamma i}. \quad (15.4.5)$$

The field $A_{\alpha 0}$ is not an independent canonical variable, but rather is defined in terms of the other variables by the constraint (15.4.3). To see this, note that the ‘electric’ field strengths $F_\alpha^{\mu 0}$ are

$$F_\alpha^{i0} = \Pi_{\alpha i}, \quad F_\alpha^{30} = \partial_3 A_\alpha^0, \quad (15.4.6)$$

so the constraint (15.4.3) reads

$$-(\partial_3)^2 A_\alpha^0 = \partial_i \Pi_{\alpha i} + \Pi_{\gamma i} C_{\gamma\alpha\beta} A_{\beta i} + J_\alpha^0, \quad (15.4.7)$$

which can easily be solved (with reasonable boundary conditions) to give A_α^0 as a functional of $\Pi_{\gamma i}$, $A_{\beta i}$, and J_α^0 . (We are using a summation

* In addition to purely algebraic complications, Coulomb gauge (like many other gauges) has a problem known as the Gribov ambiguity:⁹ even with the condition that \mathbf{A}_α vanishes at spatial infinity, for each solution of the Coulomb gauge condition $\nabla \cdot \mathbf{A}_\alpha = 0$ there are other solutions that differ by finite gauge transformations. The Gribov ambiguity will not bother us here, because we quantize in axial gauge where it is absent, and we shall use other gauges like Lorentz gauge only to generate a perturbation series.

convention, with indices i, j , etc. summed over the values 1 and 2.) It should be noted that the canonical conjugate to the matter field ψ_ℓ is

$$\pi_\ell = \frac{\partial \mathcal{L}}{\partial(\partial_0 \psi_\ell)} = \frac{\partial \mathcal{L}_M}{\partial(D_0 \psi_\ell)}, \quad (15.4.8)$$

so the time component of the matter current can be expressed in terms of the canonical variables of the matter fields *alone*

$$J_\alpha^0 = -i \frac{\partial \mathcal{L}_M}{\partial D_0 \psi_\ell} (t_\alpha)_{\ell m} \psi_m = -i \pi_\ell (t_\alpha)_{\ell m} \psi_m. \quad (15.4.9)$$

Hence Eq. (15.4.7) defines A_α^0 at a given time as a functional of the canonical variables $\Pi_{\gamma i}$, $A_{\beta i}$, π_ℓ , and ψ_m at the same time.

Now that we have identified the canonical variables in this gauge, we can proceed to the construction of a Hamiltonian. The Hamiltonian density is

$$\begin{aligned} \mathcal{H} &= \Pi_{\alpha i} \partial_0 A_{\alpha i} + \pi_\ell \partial_0 \psi_\ell - \mathcal{L} \\ &= \Pi_{\alpha i} (F_{\alpha 0 i} + \partial_i A_{\alpha 0} - C_{\alpha \beta \gamma} A_{\beta 0} A_{\gamma i}) + \pi_\ell \partial_0 \psi_\ell \\ &\quad - \frac{1}{2} F_{\alpha 0 i} F_{\alpha 0 i} + \frac{1}{2} F_{\alpha i j} F_{\alpha i j} + \frac{1}{2} F_{\alpha i 3} F_{\alpha i 3} \\ &\quad - \frac{1}{2} F_{\alpha 0 3} F_{\alpha 0 3} - \mathcal{L}_M. \end{aligned} \quad (15.4.10)$$

Using Eqs. (15.4.4) and (15.4.6), this is

$$\begin{aligned} \mathcal{H} &= \mathcal{H}_M + \Pi_{\alpha i} (\partial_i A_{\alpha 0} - C_{\alpha \beta \gamma} A_{\beta 0} A_{\gamma i}) + \frac{1}{2} \Pi_{\alpha i} \Pi_{\alpha i} \\ &\quad + \frac{1}{2} F_{\alpha i j} F_{\alpha i j} + \frac{1}{2} \partial_3 A_{\alpha i} \partial_3 A_{\alpha i} - \frac{1}{2} \partial_3 A_{\alpha 0} \partial_3 A_{\alpha 0}, \end{aligned} \quad (15.4.11)$$

where \mathcal{H}_M is the matter Hamiltonian density:

$$\mathcal{H}_M \equiv \pi_\ell \partial_0 \psi_\ell - \mathcal{L}_M. \quad (15.4.12)$$

Following the general rules derived in Section 9.2, we can now use this Hamiltonian density to calculate matrix elements as path integrals over $A_{\alpha i}$, $\Pi_{\alpha i}$, ψ_ℓ , and π_ℓ , with weighting factor $\exp(iI)$, where

$$I = \int d^4x \left[\Pi_{\alpha i} \partial_0 A_{\alpha i} + \pi_\ell \partial_0 \psi_\ell - \mathcal{H} + \epsilon \text{ terms} \right], \quad (15.4.13)$$

in which the ' ϵ terms' serve only to supply the correct imaginary infinitesimal terms in propagator denominators. (See Section 9.2.) We note that Eqs. (15.4.7) and (15.4.9) give A_α^0 as a functional of the canonical variables, linear in $\Pi_{\alpha i}$ and π_ℓ . Inspection of Eq. (15.4.11) shows then (assuming \mathcal{L}_M to be no more than quadratic in $D_\mu \psi$) that the integrand of the complete action (15.4.13) is no more than quadratic in $\Pi_{\alpha i}$ and π_α . We could therefore carry out the path integral over these canonical 'momenta' by the usual rules of Gaussian integration. The trouble with this procedure is that the coefficients of the terms in Eq. (15.4.13) of second order

in $\Pi_{\alpha i}$ are functions of the $A_{\alpha i}$, so the Gaussian integral would yield an awkward field-dependent determinant factor. Also, the whole formalism at this point looks hopelessly non-Lorentz-invariant.

Instead of proceeding in this way, we will apply a trick like that used in the path integral formulation of electrodynamics in Section 9.6. Note that if for a moment we think of $A_{\alpha 0}$ as an independent variable, then the action (15.4.13) is evidently quadratic in $A_{\alpha 0}$, with the coefficient of the second-order term $A_{\alpha 0}(x)A_{\beta 0}(y)$ equal to the field-independent kernel $(\partial_3)^2\delta^4(x-y)$. As we saw in the appendix to Chapter 9, the integral of such a Gaussian over $A_{\alpha 0}(x)$ is, up to a constant factor, equal to the value of the integrand at the stationary 'point' of the argument of the exponential. But the variational derivative of the action here is

$$\frac{\delta I}{\delta A_{\alpha 0}} = -\frac{\partial \mathcal{H}}{\partial A_{\alpha 0}} = J_\alpha^0 + \partial_i \Pi_{\alpha i} + C_{\beta\alpha\gamma} \Pi_{\beta i} A_{\gamma i} - \partial_3^2 A_{\alpha 0},$$

so the stationary 'point' of the action is the solution of the constraint equation (15.4.7). Hence, instead of using for $A_{\alpha 0}$ the solution of Eq. (15.4.7), we can just as well treat it as an independent variable of integration.

With $A_{\alpha 0}$ now regarded as an independent variable, the Hamiltonian $\int d^3x \mathcal{H}$ is evidently quadratic in $\Pi_{\alpha i}$, with the coefficient of the second-order term $\Pi_{\alpha i}(x)\Pi_{\beta j}(y)$ given by the field-independent kernel $\frac{1}{2}\delta^4(x-y)\delta_{ij}$. Assuming that the same is true for the matter variable π_ℓ , we can evaluate path integrals over π_ℓ and $\Pi_{\alpha i}$ up to a constant factor by simply setting π_ℓ and $\Pi_{\alpha i}$ at the stationary 'points' of the action corresponding to Eq. (15.4.1):

$$\begin{aligned} 0 &= \frac{\delta I}{\delta \pi_\ell} = \partial_0 \psi_\ell - \frac{\partial \mathcal{H}_M}{\partial \pi_\ell}, \\ 0 &= \frac{\delta I}{\delta \Pi_{\alpha i}} = \partial_0 A_{\alpha i} - \Pi_{\alpha i} - \partial_i A_{\alpha 0} + C_{\alpha\beta\gamma} A_{\beta 0} A_{\gamma i} = F_{\alpha 0 i} - \Pi_{\alpha i}. \end{aligned}$$

Inserting these back into Eq. (15.4.13) gives

$$\begin{aligned} I &= \int d^4x \left[\mathcal{L}_M + \frac{1}{2} F_{\alpha 0 i} F_{\alpha 0 i} \right. \\ &\quad \left. - \frac{1}{2} F_{\alpha i j} F_{\alpha i j} - \frac{1}{2} \partial_3 A_{\alpha i} \partial_3 A_{\alpha i} + \frac{1}{2} (\partial_3 A_{\alpha 0})^2 \right] \\ &= \int d^4x \mathcal{L}, \end{aligned} \tag{15.4.14}$$

where \mathcal{L} is the Lagrangian (15.3.1) with which we started! In other words, we are to do path integrals over $\psi_\ell(x)$ and all four components of $A_{\alpha\mu}(x)$, with a manifestly covariant weighting factor $\exp(iI)$ given by Eqs. (15.4.14) and (15.3.1), but with the axial-gauge condition enforced by inserting a

factor

$$\prod_{x,\alpha} \delta(A_{\alpha 3}(x)) . \quad (15.4.15)$$

As long as $\mathcal{O}_A, \mathcal{O}_B \cdots$ are gauge-invariant, we have

$$\begin{aligned} \langle T\{\mathcal{O}_A \mathcal{O}_B \cdots\} \rangle_{\text{VACUUM}} &\propto \int \left[\prod_{\ell, x} d\psi_\ell(x) \right] \left[\prod_{\alpha, \mu, x} dA_{\alpha\mu}(x) \right] \\ &\times \mathcal{O}_A \mathcal{O}_B \cdots \exp\{iI + \epsilon \text{ terms}\} \prod_{x,\alpha} \delta(A_{\alpha 3}(x)) , \end{aligned} \quad (15.4.16)$$

with Lorentz- and gauge-invariant action I given by Eq. (15.4.14).

* * *

For future reference, we note that the volume element $\prod_{\alpha, \mu, x} dA_{\alpha\mu}(x)$ for the integration over gauge fields in (15.4.16) is gauge-invariant, in the sense that

$$\prod_{\alpha, \mu, x} dA_{\Lambda \alpha\mu}(x) = \prod_{\alpha, \mu, x} dA_{\alpha\mu}(x) , \quad (15.4.17)$$

where $A_{\Lambda \alpha\mu}(x)$ is the result of acting on $A_{\alpha\mu}(x)$ with a gauge transformation having transformation parameters $\Lambda_\alpha(x)$. It will be enough to show that this is true for transformations near the identity, say with infinitesimal transformation parameters $\lambda_\alpha(x)$. In this case,

$$A_{\lambda \alpha}^\mu = A_\alpha^\mu + \partial^\mu \lambda_\alpha + C_{\alpha\beta\gamma} A_\beta^\mu \lambda_\gamma ,$$

so the volume elements are related by

$$\prod_{\alpha, \mu, x} dA_{\lambda \alpha\mu}(x) = \text{Det}(\mathcal{N}) \prod_{\alpha, \mu, x} dA_{\alpha\mu}(x) ,$$

where \mathcal{N} is the 'matrix':

$$\mathcal{N}_{\alpha\mu x, \beta\nu y} = \frac{\delta A_{\lambda \alpha\mu}(x)}{\delta A_{\beta\nu}(y)} = \delta^4(x-y) \delta_\mu^\nu \left[\delta_{\alpha\beta} + C_{\alpha\beta\gamma} \lambda_\gamma(x) \right] .$$

The determinant of \mathcal{N} is unity to first order in λ_γ because the trace $C_{\alpha\alpha\gamma}$ vanishes.

In this chapter we shall assume that the volume element $\prod_{n,x} d\psi_n(x)$ for the integration over matter fields is also gauge-invariant. There are important subtleties here, to which we shall return in Chapter 22, but as shown there this assumption turns out to be valid in our present non-Abelian gauge theories of strong and electroweak interactions.

15.5 The De Witt–Faddeev–Popov Method

Our formula (15.4.16) for the path integral was derived in a gauge that is convenient for canonical quantization, but the Feynman rules that would be derived from this formula would hide the underlying rotational and Lorentz invariance of the theory. In order to derive manifestly Lorentz-invariant Feynman rules, we need to change the gauge.

We first note that Eq. (15.4.16) is (up to an unimportant constant factor) a special case of a general class of functional integrals, of the form:

$$\mathcal{I} = \int \left[\prod_{n,x} d\phi_n(x) \right] \mathcal{G}[\phi] B[f[\phi]] \text{Det } \mathcal{F}[\phi], \quad (15.5.1)$$

where $\phi_n(x)$ are a set of gauge and matter fields; $\prod_{n,x} d\phi_n(x)$ is a volume element; and $\mathcal{G}[\phi]$ is a functional of the $\phi_n(x)$, satisfying the gauge-invariance condition:

$$\mathcal{G}[\phi_\lambda] \prod_{n,x} d\phi_{\lambda n}(x) = \mathcal{G}[\phi] \prod_{n,x} d\phi_n(x), \quad (15.5.2)$$

where $\phi_{\lambda n}(x)$ is the result of operating on ϕ with a gauge transformation having parameters $\lambda_\alpha(x)$. (Usually when this is satisfied both the functional \mathcal{G} and the volume element are separately invariant, but Eq. (15.5.2) is all we need here.) Also, $f_\alpha[\phi; x]$ is a non-gauge-invariant ‘gauge-fixing functional’ of these fields that also depends on x and α ; $B[f]$ is some numerical functional defined for general functions $f_\alpha(x)$ of x and α ; and \mathcal{F} is the ‘matrix’:

$$\mathcal{F}_{\alpha x, \beta y}[\phi] \equiv \left. \frac{\delta f_\alpha[\phi_\lambda; x]}{\delta \lambda_\beta(y)} \right|_{\lambda=0}. \quad (15.5.3)$$

(In accordance with our usual notation for functionals of functions or of functionals, $B[f[\phi]]$ is understood to depend on the values taken by $f_\alpha[\phi; x]$ for all values of the undisplayed variables α and x , with the displayed variable, the function $\phi_n(x)$, held fixed.) Eq. (15.5.1) does not represent the widest possible generalization of Eq. (15.4.16); we will see in Section 15.7 that there is a further generalization that is needed for some purposes. We start here with Eq. (15.5.1) because it will help to motivate the formalism of Section 15.7, and it is adequate for dealing with non-Abelian gauge theories in the most convenient gauges.

We now must check that the path integral (15.4.16) is in fact a special case of Eq. (15.5.1). In Eq. (15.4.16) the fields $\phi_n(x)$ consist of both $A_{\alpha\mu}(x)$ and matter fields $\psi_\ell(x)$, and

$$f_\alpha[A, \psi; x] = A_{\alpha 3}(x), \quad (15.5.4)$$

$$B[f] = \prod_{x,\alpha} \delta(f_\alpha(x)), \quad (15.5.5)$$

$$\mathcal{G}[A, \psi] = \exp\{iI + \epsilon \text{ terms}\} \mathcal{O}_A \mathcal{O}_B \cdots, \quad (15.5.6)$$

$$\prod_{n,x} d\phi_n(x) = \left[\prod_{\ell,x} d\psi_\ell(x) \right] \left[\prod_{\alpha,\mu,x} dA_\alpha^\mu(x) \right]. \quad (15.5.7)$$

(We are now dropping the distinction between upper and lower indices α, β, \dots .) Comparison of Eq. (15.4.16) with Eqs. (15.5.1)–(15.5.3) shows that these path integrals are indeed the same, aside from the factor $\text{Det } \mathcal{F}[\phi]$. For the particular gauge-fixing functional (15.5.4), this factor is field-independent: if $A_\alpha^3(x) = 0$, then the change in $A_\alpha^3(x)$ under a gauge transformation with parameters $\lambda_\alpha(x)$ is

$$A_{\lambda_\alpha}^3(x) = \partial_3 \lambda_\alpha(x) = \int d^4y \lambda_\alpha(y) \partial_3 \delta^4(x-y),$$

so that here Eq. (15.5.3) is the field-independent ‘matrix’

$$\mathcal{F}_{\alpha x, \beta y}[\phi] = \delta_{\alpha\beta} \partial_3 \delta^4(x-y).$$

The determinant in Eq. (15.5.1) is therefore also field-independent in this gauge. As discussed in Chapter 9, field-independent factors in the functional integral affect only the vacuum-fluctuation part of expectation values and S -matrix elements, and so are irrelevant to the calculation of the connected parts of the S -matrix.

The point of recognizing the functional integral (15.4.16) for non-Abelian gauge theories as a special case of the general path integral (15.5.1) is that in this form we may freely change the gauge. Specifically, we have a theorem, that *the integral (15.5.1) is actually independent (within broad limits) of the gauge-fixing functional $f_\alpha[\phi; x]$, and depends on the choice of the functional $B[f]$ only through an irrelevant constant factor.*

Proof: Replace the integration variable ϕ everywhere in Eq. (15.5.1) with a new integration variable ϕ_Λ , with $\Lambda^\alpha(x)$ any arbitrary (but fixed) set of gauge transformation parameters:

$$\mathcal{I} = \int \left[\prod_{n,x} d\phi_{\Lambda n}(x) \right] \mathcal{G}[\phi_\Lambda] B[f[\phi_\Lambda]] \text{Det } \mathcal{F}[\phi_\Lambda]. \quad (15.5.8)$$

(This step is a mathematical triviality, like changing an integral $\int_{-\infty}^{\infty} f(x) dx$ to read $\int_{-\infty}^{\infty} f(y) dy$, and does not yet make use of our assumptions regarding gauge invariance.) Now use the assumed gauge invariance (15.5.2) of the measure $\Pi d\phi$ times the functional $\mathcal{G}[\phi]$ to rewrite this as

$$\mathcal{I} = \int \left[\prod_{n,x} d\phi_n(x) \right] \mathcal{G}[\phi] B[f[\phi_\Lambda]] \text{Det } \mathcal{F}[\phi_\Lambda]. \quad (15.5.9)$$

Since $\Lambda^\alpha(x)$ was arbitrary, the left-hand side here cannot depend on it. Integrating over $\Lambda^\alpha(x)$ with some suitable weight-functional $\rho[\Lambda]$ (to be

chosen below) thus gives

$$\mathcal{J} \int \left[\prod_{\alpha, x} d\Lambda^\alpha(x) \right] \rho[\Lambda] = \int \left[\prod_{n, x} d\phi_n(x) \right] \mathcal{G}[\phi] C[\phi], \quad (15.5.10)$$

where

$$C[\phi] \equiv \int \left[\prod_{\alpha, x} d\Lambda^\alpha(x) \right] \rho[\Lambda] B[f[\phi_\Lambda]] \text{Det } \mathcal{F}[\phi_\Lambda]. \quad (15.5.11)$$

Now, Eq. (15.5.3) gives

$$\mathcal{F}_{\alpha x, \beta y}[\phi_\Lambda] = \frac{\delta f_\alpha[(\phi_\Lambda)_\lambda; x]}{\delta \lambda^\beta(y)} \Big|_{\lambda=0}. \quad (15.5.12)$$

We are assuming that these transformations form a group; that is, we may write the result of performing the gauge transformation with parameters $\Lambda^\alpha(x)$ followed by the gauge transformation with parameters $\lambda^\alpha(x)$ as the action of a single ‘product’ gauge transformation with parameters $\tilde{\Lambda}^\alpha(x; \Lambda, \lambda)$,

$$(\phi_\Lambda)_\lambda = \phi_{\tilde{\Lambda}(\Lambda, \lambda)}. \quad (15.5.13)$$

Using the chain rule of partial (functional) differentiation, we have then

$$\mathcal{F}_{\alpha x, \beta y}[\phi_\Lambda] = \int \mathcal{J}_{\alpha x, \gamma z}[\phi, \Lambda] \mathcal{R}^{\gamma z}_{\beta y}[\Lambda] d^4 z, \quad (15.5.14)$$

where

$$\mathcal{J}_{\alpha x, \gamma z}[\phi, \Lambda] \equiv \frac{\delta f_\alpha[\phi_{\tilde{\Lambda}}; x]}{\delta \tilde{\Lambda}^\gamma(z)} \Big|_{\tilde{\Lambda}=\Lambda} = \frac{\delta f_\alpha[\phi_\Lambda; x]}{\delta \Lambda^\gamma(z)} \quad (15.5.15)$$

and

$$\mathcal{R}^{\gamma z}_{\beta y}[\Lambda] = \frac{\delta \tilde{\Lambda}^\gamma(z; \Lambda, \lambda)}{\delta \lambda^\beta(y)} \Big|_{\lambda=0}. \quad (15.5.16)$$

It follows that

$$\text{Det } \mathcal{F}[\phi_\Lambda] = \text{Det } \mathcal{J}[\phi, \Lambda] \text{Det } \mathcal{R}[\Lambda]. \quad (15.5.17)$$

We note that $\text{Det } \mathcal{J}[\phi, \Lambda]$ is nothing but the Jacobian of the transformation of integration variables from the $\Lambda^\alpha(x)$ to (for a fixed ϕ) the $f_\alpha[\phi_\Lambda; x]$. Hence, if we choose the weight-function $\rho(\Lambda)$ as

$$\rho(\Lambda) = 1 / \text{Det } \mathcal{R}[\Lambda] \quad (15.5.18)$$

then

$$\begin{aligned} C[\phi] &= \int \left[\prod_{\alpha, x} d\Lambda^\alpha(x) \right] \text{Det } \mathcal{J}[\phi, \Lambda] B[f[\phi_\Lambda]] \\ &= \int \left[\prod_{\alpha, x} df_\alpha(x) \right] B[f] \equiv C, \end{aligned} \quad (15.5.19)$$

which is clearly independent of ϕ . (Eq. (15.5.18) may be recognized by the reader as giving the invariant (Haar) measure on the space of group parameters.) We have then at last

$$\mathcal{I} = \frac{C \int \left[\prod_{n,x} d\phi_n(x) \right] \mathcal{G}[\phi]}{\int \left[\prod_{\alpha,x} d\Lambda^\alpha(x) \right] \rho[\Lambda]}. \quad (15.5.20)$$

This is clearly independent of our choice of $f_\alpha[\phi; x]$, which has been reduced to a mere variable of integration, and it depends on $B[f]$ only through the constant C , as was to be proved.

Before proceeding with the applications of this theorem, we should pause to note a tricky point in the derivation. The integrals in the numerator and denominator of Eq. (15.5.20) are both ill-defined for the same reason. Since $\mathcal{G}[\phi]$ is assumed to be gauge-invariant, its integral over ϕ cannot possibly converge; the integrand is constant along all 'orbits,' obtained by sending ϕ into ϕ_λ with all possible $\lambda^\alpha(x)$. Likewise, the integrand in the denominator is divergent, because $\rho(\Lambda) \prod d\Lambda$ is nothing but the usual invariant volume element for integrating over the group, and this is also constant along 'orbits' $\Lambda \rightarrow \tilde{\Lambda}(\Lambda, \lambda)$. This divergence can be eliminated in both the numerator and denominator of Eq. (15.5.20) by formulating the theory on a finite spacetime lattice, in which case the volume of the gauge group is just the volume of the global Lie group itself times the number of lattice sites. Because the gauge-fixing factor $B[f]$ eliminates this divergence in the original definition (15.5.1) of the left-hand side of Eq. (15.5.20), we may presume that, as the number of lattice sites goes to infinity, it cancels between the numerator and denominator of the right-hand side of Eq. (15.5.20).

Now to the point. We have seen that the vacuum expectation value (15.4.16) in axial gauge is given by a functional integral of the general form (15.5.1). Armed with the above theorem, we conclude then that

$$\begin{aligned} \langle T \{ \mathcal{O}_A \mathcal{O}_B \cdots \} \rangle_V &\propto \int \left[\prod_{\ell,x} d\psi_\ell(x) \right] \left[\prod_{\alpha,\mu,x} dA^\mu_\alpha(x) \right] \\ &\times \mathcal{O}_A \mathcal{O}_B \cdots \exp\{iI + \epsilon \text{ terms}\} B[f[A, \psi]] \text{Det } \mathcal{F}[A, \psi] \end{aligned} \quad (15.5.21)$$

for (almost) any choice of $f_\alpha[A, \psi; x]$ and $B[f]$. We are now therefore free to use Eq. (15.5.21) to derive the Feynman rules in a more convenient gauge.

The path integrals that we understand how to calculate are of Gaussians times polynomials, so we will generally take

$$B[f] = \exp \left(-\frac{i}{2\xi} \int d^4x f_\alpha(x) f_\alpha(x) \right) \quad (15.5.22)$$

with arbitrary real parameter ξ . With this choice, the effect of the factor B in Eq. (15.5.21) is just to add a term to the effective Lagrangian

$$\mathcal{L}_{\text{EFF}} = \mathcal{L} - \frac{1}{2\xi} f_\alpha f_\alpha. \quad (15.5.23)$$

The simplest Lorentz-invariant choice of the gauge-fixing function f_α is the same as in electrodynamics:

$$f_\alpha = \partial_\mu A_\alpha^\mu. \quad (15.5.24)$$

The bare gauge-field propagator can then be calculated just as in quantum electrodynamics. The free-vector-boson part of the effective action can be written

$$\begin{aligned} I_{0A} &= - \int d^4x \left[\frac{1}{4} (\partial_\mu A_{\alpha\nu} - \partial_\nu A_{\alpha\mu}) (\partial^\mu A_\alpha^\nu - \partial^\nu A_\alpha^\mu) \right. \\ &\quad \left. + \frac{1}{2\xi} (\partial_\mu A_\alpha^\mu) (\partial_\nu A_\alpha^\nu) + \epsilon \text{ terms} \right] \\ &= - \frac{1}{2} \int d^4x \mathcal{D}_{\alpha\mu x, \beta\nu y} A_\alpha^\mu(x) A_\beta^\nu(y), \end{aligned}$$

where

$$\begin{aligned} \mathcal{D}_{\alpha\mu x, \beta\nu y} &= \eta_{\mu\nu} \frac{\partial^2}{\partial x^\lambda \partial y^\lambda} \delta^4(x-y) \\ &\quad - \left(1 - \frac{1}{\xi}\right) \frac{\partial^2}{\partial x^\mu \partial y^\nu} \delta^4(x-y) + \epsilon \text{ terms} \\ &= (2\pi)^{-4} \int d^4p \left[\eta_{\mu\nu} (p^2 - i\epsilon) - \left(1 - \frac{1}{\xi}\right) p_\mu p_\nu \right] e^{ip \cdot (x-y)}. \end{aligned}$$

Taking the reciprocal of the matrix in square brackets, we find the propagator:

$$\begin{aligned} \Delta_{\alpha\mu, \beta\nu}(x, y) &= (\mathcal{D}^{-1})_{\alpha\mu x, \beta\nu y} \\ &= (2\pi)^{-4} \int d^4p \left[\eta_{\mu\nu} + (\xi - 1) \frac{p_\mu p_\nu}{p^2} \right] \frac{e^{ip \cdot (x-y)}}{p^2 - i\epsilon}. \end{aligned} \quad (15.5.25)$$

This is a generalization of both Landau and Feynman gauges, which are recovered by taking $\xi = 0$ and $\xi = 1$, respectively. For $\xi \rightarrow 0$, the functional (15.5.22) oscillates very rapidly except near $f_\alpha = 0$, so this functional acts like a delta-function imposing the Landau gauge condition $\partial_\mu A^\mu = 0$, leading naturally to a propagator satisfying the corresponding condition $\partial^\mu \Delta_{\alpha\mu, \beta\nu} = 0$. For non-zero values of ξ the functional $B[f]$ does not pick out gauge fields satisfying any specific gauge condition on the field $A_{\alpha\mu}$, but it is common to refer to the propagator (15.5.25) as being in a ‘generalized Feynman gauge’ or ‘generalized ξ -gauge’. It is often a good

strategy to calculate physical amplitudes with ξ left arbitrary, and then at the end of the calculation check that the results are ξ -independent.

With one qualification, the Feynman rules are now obvious: the contributions of vertices are to be read off from the interaction terms in the original Lagrangian \mathcal{L} , with gauge-field propagators given by Eq. (15.5.25), and matter-field propagators calculated as before. To be specific, the trilinear interaction term in \mathcal{L}

$$-\frac{1}{2} C_{\alpha\beta\gamma} (\partial_\mu A_{\alpha\nu} - \partial_\nu A_{\alpha\mu}) A_\beta^\mu A_\gamma^\nu$$

corresponds to a vertex to which are attached three vector boson lines. If these lines carry (incoming) momenta p, q, k and Lorentz and gauge-field indices $\mu\alpha, \nu\beta, \rho\gamma$, then according to the momentum-space Feynman rules, the contribution of such a vertex to the integrand is

$$i(2\pi)^4 \delta^4(p+q+k) [-i C_{\alpha\beta\gamma}] [p_\nu \eta_{\mu\lambda} - p_\lambda \eta_{\mu\nu} + q_\lambda \eta_{\nu\mu} - q_\mu \eta_{\nu\lambda} + k_\mu \eta_{\lambda\nu} - k_\nu \eta_{\lambda\mu}]. \quad (15.5.26)$$

Also, the A^4 interaction term in \mathcal{L} ,

$$-\frac{1}{4} C_{\epsilon\alpha\beta} C_{\epsilon\gamma\delta} A_{\alpha\mu} A_{\beta\nu} A_\gamma^\mu A_\delta^\nu,$$

corresponds to a vertex to which are attached four vector boson lines. If these lines carry (incoming) momenta p, q, k, ℓ , and Lorentz and gauge indices $\mu\alpha, \nu\beta, \rho\gamma$, and $\sigma\delta$, then the contribution of such a vertex to the integrand is

$$i(2\pi)^4 \delta^4(p+q+k+\ell) \times \left[-C_{\epsilon\alpha\beta} C_{\epsilon\gamma\delta} (\eta_{\mu\rho} \eta_{\nu\sigma} - \eta_{\mu\sigma} \eta_{\nu\rho}) \right. \\ \left. - C_{\epsilon\alpha\gamma} C_{\epsilon\delta\beta} (\eta_{\mu\sigma} \eta_{\rho\nu} - \eta_{\mu\nu} \eta_{\sigma\rho}) - C_{\epsilon\alpha\delta} C_{\epsilon\beta\gamma} (\eta_{\mu\nu} \eta_{\rho\sigma} - \eta_{\mu\rho} \eta_{\sigma\nu}) \right]. \quad (15.5.27)$$

(Recall that the structure constants $C_{\alpha\beta\gamma}$ contain coupling constant factors, so the factors (15.5.26) and (15.5.27) are respectively of first and second order in coupling constants.)

The one complication in the Feynman rules with which we have not yet dealt is the presence in Eq. (15.5.21) of the factor $\text{Det } \mathcal{F}$, which for general gauges is *not* a constant. We now turn to a consideration of this factor.

15.6 Ghosts

21

We now consider the effect of the factor $\text{Det } \mathcal{F}$ in Eq. (15.5.22) on the Feynman rules for a non-Abelian gauge theory. In order to be able to treat this effect as a modification of the Feynman rules, recall that as shown in

Section 9.5, the determinant of any matrix $\mathcal{F}_{\alpha x, \beta y}$ may be expressed as a path integral

$$\text{Det } \mathcal{F} \propto \int \left[\prod_{\alpha, x} d\omega_{\alpha}^*(x) \right] \left[\prod_{\alpha, x} d\omega_{\alpha}(x) \right] \exp(iI_{GH}), \quad (15.6.1)$$

where

$$I_{GH} \equiv \int d^4x d^4y \omega_{\alpha}^*(x) \omega_{\beta}(y) \mathcal{F}_{\alpha x, \beta y}. \quad (15.6.2)$$

Here ω_{α}^* and ω_{α} are a set of independent anticommuting classical variables, and the constant of proportionality is field-independent. (We have to choose the ω_{α} and ω_{α}^* field variables to be fermionic in order to reproduce the factor $\text{Det } \mathcal{F}$; had we chosen these field variables to be bosonic, the path integral (15.6.1) would have been proportional to $(\text{Det } \mathcal{F})^{-1}$.) The fields ω_{α}^* and ω_{α} are not necessarily related by complex conjugation; indeed, in Section 15.7 we shall see that for some purposes we need to assume that ω_{α}^* and ω_{α} are independent *real* variables. The whole effect of the factor $\text{Det } \mathcal{F}$ is the same as that of including $I_{GH}(\omega, \omega^*)$ in the full effective action, and integrating over ‘fields’ ω and ω^* . That is, for arbitrary gauge-fixing functionals $f_{\alpha}(x)$,

$$\begin{aligned} \langle T\{\mathcal{O}_A \cdots\} \rangle_V &\propto \int \left[\prod_{n, x} d\psi_n(x) \right] \left[\prod_{\alpha, \mu, x} dA_{\alpha\mu}(x) \right] \\ &\times \left[\prod_{\alpha, x} d\omega_{\alpha}(x) d\omega_{\alpha}^*(x) \right] \exp\left(iI_{\text{MOD}}[\psi, A, \omega, \omega^*]\right) \mathcal{O}_A \cdots, \end{aligned} \quad (15.6.3)$$

where I_{MOD} is a modified action

$$I_{\text{MOD}} = \int d^4x \left[\mathcal{L} - \frac{1}{2\xi} f_{\alpha} f_{\alpha} \right] + I_{GH}. \quad (15.6.4)$$

The fields ω_{α} and ω_{α}^* are Lorentz scalars (at least in covariant gauges) but satisfy Fermi statistics. The connection between spin and statistics is not really violated here, because there are no particles described by these fields that can appear in initial or final states. For that reason, ω_{α} and ω_{α}^* are called the fields of ‘ghost’ and ‘antighost’ particles. Inspection of Eq. (15.6.2) shows that the action respects the conservation of a quantity known as ‘ghost number,’ equal to +1 for ω_{α} , -1 for ω_{α}^* , and zero for all other fields.

The Feynman rules for the ghosts are simplest in the case in which the ‘matrix’ \mathcal{F} may be expressed as

$$\mathcal{F} = \mathcal{F}_0 + \mathcal{F}_1, \quad (15.6.5)$$

where \mathcal{F}_0 is field-independent and of zeroth order in coupling constants, while \mathcal{F}_1 is field-dependent and proportional to one or more coupling

constant factors. In this case, the ghost propagator is just

$$\Delta_{\alpha\beta}(x, y) = (\mathcal{F}_0^{-1})_{\alpha\beta, xy} \quad (15.6.6)$$

and the ghost vertices are to be read off from the interaction term

$$I'_{GH} = \int d^4x d^4y \omega_\alpha^*(x) \omega_\beta(y) (\mathcal{F}_1)_{\alpha\beta, xy} . \quad (15.6.7)$$

For instance, in the generalized ξ -gauge discussed in the previous section, we have

$$f_\alpha = \partial_\mu A_\alpha^\mu \quad (15.6.8)$$

and for infinitesimal gauge parameters λ_α , Eq. (15.1.9) gives:

$$A_{\alpha\lambda}^\mu = A_\alpha^\mu + \partial^\mu \lambda_\alpha + C_{\alpha\gamma\beta} \lambda_\beta A_\gamma^\mu$$

so that

$$\begin{aligned} \mathcal{F}_{\alpha\beta, xy} &= \left. \frac{\delta \partial_\mu A_{\alpha\lambda}^\mu(x)}{\delta \lambda_\beta(y)} \right|_{\lambda=0} \\ &= \square \delta^4(x-y) + C_{\alpha\gamma\beta} \frac{\partial}{\partial x^\mu} \left[A_\gamma^\mu(x) \delta^4(x-y) \right] . \end{aligned} \quad (15.6.9)$$

This is of the form (15.6.5), with

$$(\mathcal{F}_0)_{\alpha\beta, xy} = \square \delta^4(x-y) \delta_{\alpha\beta} , \quad (15.6.10)$$

$$(\mathcal{F}_1)_{\alpha\beta, xy} = -C_{\alpha\beta\gamma} \frac{\partial}{\partial x^\mu} \left[A_\gamma^\mu(x) \delta^4(x-y) \right] . \quad (15.6.11)$$

From Eqs. (15.6.6) and (15.6.10), we see that the ghost propagator is

$$\Delta_{\alpha\beta}(x, y) = \delta_{\alpha\beta} (2\pi)^{-4} \int d^4p (p^2 - i\epsilon)^{-1} e^{ip \cdot (x-y)} , \quad (15.6.12)$$

so in this gauge the ghosts behave like spinless fermions of zero mass, transforming according to the adjoint representation of the gauge group. Using Eqs. (15.6.7) and (15.6.11) and integrating by parts, we find that the ghost interaction term in the action is now

$$I'_{GH} = \int d^4x C_{\alpha\beta\gamma} \frac{\partial \omega_\alpha^*}{\partial x^\mu} A_\gamma^\mu \omega_\beta . \quad (15.6.13)$$

This interaction corresponds to vertices to which are attached one outgoing ghost line, one incoming ghost line, and one vector boson line. If these lines carry (incoming) momenta p, q, k respectively and gauge group indices α, β, γ respectively, and the gauge field carries a vector index μ , then the contribution of such a vertex to the integrand is given by the momentum-space Feynman rules as

$$i(2\pi)^4 \delta^4(p+q+k) \times ip_\mu C_{\alpha\beta\gamma} . \quad (15.6.14)$$

The ghosts propagate around loops, with single vector boson lines attached at each vertex along the loops, and with an extra minus sign supplied for each loop as is usual for fermionic field variables.

The extra minus sign for ghost loops suggests that each ghost field ω_α together with the associated antighost field ω_α^* represents something like a negative degree of freedom. These negative degrees of freedom are necessary because in using covariant gauge field propagators we are really over-counting; the physical degrees of freedom are the components of $A_\alpha^\mu(x)$, less the parameters $\Lambda_\alpha(x)$ needed to describe a gauge transformation.

In summary, the modified action (15.6.4) may be written in generalized ξ -gauge as

$$I_{\text{MOD}} = \int d^4x \mathcal{L}_{\text{MOD}} \quad (15.6.15)$$

with a modified Lagrangian density:

$$\begin{aligned} \mathcal{L}_{\text{MOD}} = \mathcal{L}_M - \frac{1}{4} F_\alpha^{\mu\nu} F_{\alpha\mu\nu} - \frac{1}{2\xi} (\partial_\mu A_\alpha^\mu)(\partial_\nu A_\alpha^\nu) \\ - \partial_\mu \omega_\alpha^* \partial^\mu \omega_\alpha + C_{\alpha\beta\gamma} (\partial_\mu \omega_\alpha^*) A_\gamma^\mu \omega_\beta. \end{aligned} \quad (15.6.16)$$

It is important that this Lagrangian is *renormalizable* (if the matter Lagrangian \mathcal{L}_M is), in the elementary sense that its terms involve products of fields and their derivatives of total dimensionality (in powers of mass) four or less. (The kinematic term $-\partial_\mu \omega_\alpha^* \partial^\mu \omega_\alpha$ in Eq. (15.6.16) fixes the dimensionality of the fields ω and ω^* to be mass to the power unity, just like ordinary scalar and gauge fields.) However, there is more to renormalizability than power counting; it is necessary also that there be a counterterm to absorb every divergence. In the next section we shall consider a remarkable symmetry that will be used in Section 17.2 to show that non-Abelian gauge theories are indeed renormalizable in this sense, and that can even take the place of the Faddeev–Popov–De Witt approach that we have been following.

15.7 BRST Symmetry

Although the Faddeev–Popov–De Witt method described in the previous two sections makes the Lorentz invariance of the theory manifest, it still rests on a choice of gauge, and hence naturally it hides the underlying gauge invariance of the theory. This is a serious problem in trying to prove the renormalizability of the theory — gauge invariance restricts the form of the terms in the Lagrangian that are available as counterterms to absorb ultraviolet divergences, but once we choose a gauge, how do we