Compton Scattering

$$\bar{u}(p',s')(-ie\gamma^\mu \epsilon^*_\mu(k',\epsilon'))\,\frac{i(\not p+\not k+m)}{(p+k)^2-m^2}\,(-ie\gamma^\nu \epsilon_\nu(k,\epsilon))\,u(p,s)$$

$$+\ \bar{u}(p',s')(-ie\gamma^\mu \epsilon_\mu(k,\epsilon))\,\frac{i(\not p-\not k'+m)}{(p-k')^2-m^2}\,(-ie\gamma^\nu \epsilon^*_\nu(k',\epsilon'))\,u(p,s)$$

No minus sign because the Fermi parts of the two diagrams are the same. Probably good to interchange μ & ν in second term. Then

$$i\mathcal{M} = -ie^2\,\epsilon^*_\mu(k',\epsilon')\,\epsilon_\nu(k,\epsilon)\,\bar{u}(p',s')\left[\frac{\gamma^\mu(\not p+\not k+m)\gamma^\nu}{(p+k)^2-m^2}+\frac{\gamma^\nu(\not p-\not k'+m)\gamma^\mu}{(p-k')^2-m^2}\right]$$

$$\times\, u(p,s)$$

Now

$$(p+k)^2-m^2 = p^2+2p\cdot k+k^2-m^2 = 2\,p\cdot k$$

$$(p-k')^2-m^2 = p^2-2\,p\cdot k'+k'^2-m^2 = -2\,p\cdot k'$$

since $p^2=m^2$ and $k^2=k'^2=0$.

The Dirac equation

$$(i\not\partial - m)\psi = 0$$

in momentum space is, since $i\partial_\mu e^{-ipx} = p_\mu e^{-ipx}$,

$$(\not p - m)u(p) = 0 \qquad \left(\text{and} \quad (\not p + m)v(p) = 0\right).$$

So

$$(\not p + m)\gamma^\nu u(p,s) = (\gamma^\mu\gamma^\nu p_\mu + m\gamma^\nu)u(p,s)$$

$$= (2\eta^{\mu\nu}p_\mu - \gamma^\nu\gamma^\mu p_\mu + m\gamma^\nu)u(p,s)$$

$$= (2p^\nu - \gamma^\nu(\not p - m))u(p,s) = 2p^\nu u(p,s).$$

So

$$iM = -ie^2\epsilon^*_\mu(k',\epsilon')\epsilon_\nu(k)\bar u(p',s')\left[\frac{\gamma^\mu \not k\gamma^\nu + 2\gamma^\mu p^\nu}{2p\cdot k} + \frac{-\gamma^\nu \not k'\gamma^\mu + 2\gamma^\nu p^\mu}{-2p\cdot k'}\right]u(p,s)$$

or

$$iM = -ie^2\bar u(p',s')\left[\frac{\not\epsilon'^* \not k \not\epsilon + 2\not\epsilon'^* p\cdot\epsilon}{2p\cdot k} + \frac{2\not\epsilon\, p\cdot\epsilon'^* - \not\epsilon \not k' \not\epsilon'^*}{-2p\cdot k'}\right]u(p,s)$$

$$= -ie^2\bar u(p',s')\left[\frac{\not\epsilon'^* \not k \not\epsilon + 2\not\epsilon'^* p\cdot\epsilon}{2p\cdot k} + \frac{\not\epsilon \not k' \not\epsilon'^* - 2\not\epsilon\, p\cdot\epsilon'^*}{2p\cdot k'}\right]u(p,s),$$

where $\not\epsilon'^* = \epsilon'^{*\mu}\gamma_\mu = \epsilon'^*_\mu\gamma^\mu$ not $(\not\epsilon')^*$.

The complex conjugate of $\bar{u}'[\ ]u$ is

$$\left(u'^{\dagger}\gamma^0[\ ]u\right)^* = u^{\dagger}[\ ]^{\dagger}\gamma^{0\dagger}u' = u^{\dagger}\gamma^0\gamma^0[\ ]^{\dagger}\gamma^0u'$$

$$= \bar{u}\gamma^0[\ ]^{\dagger}\gamma^0u'.$$

We've seen that because $\gamma^{i\dagger} = -\gamma^i$ and $\gamma^{0\dagger} = \gamma^0$

$$\gamma^0\gamma^{a\dagger}\gamma^0 = \gamma^a$$

It's also true that

$$\gamma^0(\gamma^a\gamma^b)^{\dagger}\gamma^0 = \gamma^0\gamma^{b\dagger}\gamma^{a\dagger}\gamma^0 = \gamma^0\gamma^{b\dagger}\gamma^0\gamma^0\gamma^{a\dagger}\gamma^0$$

$$= \gamma^b\gamma^a.$$ So it's also true that

$$\gamma^0(\gamma^a\gamma^b\ldots\gamma^z)^{\dagger}\gamma^0 = \gamma^z\gamma^y\ldots\gamma^b\gamma^a.$$

So

$$\left(u'^{\dagger}\gamma^0[\text{gammas}]u\right)^* = \left(\bar{u}'[\text{gammas}]u\right)^*$$

$$= \bar{u}\,[\text{reversed gammas}]\,u'.$$

But we must complex-conjugate the ϵ and ϵ^*'s.

So

$$|M|^2 = e^4\, \bar{u}'_\alpha [\quad]_{\alpha\beta} u_\beta\, \bar{u}_a \left[\frac{\not{\epsilon}^* \not{k} \not{\epsilon}' + 2 \not{\epsilon}' p\cdot\epsilon^*}{2\, p\cdot k} + \frac{\not{\epsilon}' \not{k}' \not{\epsilon}^* - 2 \not{\epsilon}^* p\cdot\epsilon'}{2\, p\cdot k'} \right]_{ab} u'_b$$

in which $\epsilon' = \epsilon(k', t')$ and $\epsilon = \epsilon(k, t)$.

Now $\sum_s u_\beta \bar{u}_a = (\not{p}+m)_{\beta\alpha}$ and $\sum_{s'} u'_b \bar{u}'_\alpha = (\not{p}'+m)_{b\alpha}$.

So

$$\frac{1}{2}\sum_{ss'} |M|^2 = \frac{e^4}{8} \operatorname{tr}\left\{ \left[\frac{\not{\epsilon}'^* \not{k} \not{\epsilon} + 2 \not{\epsilon}'^* p\cdot\epsilon}{p\cdot k} + \frac{\not{\epsilon} \not{k}' \not{\epsilon}'^* - 2 \not{\epsilon}\, p\cdot\epsilon'^*}{p\cdot k'} \right] (\not{p}+m) \right.$$
$$\left. \times \left[\frac{\not{\epsilon}^* \not{k} \not{\epsilon}' + 2 \not{\epsilon}' p\cdot\epsilon^*}{p\cdot k} + \frac{\not{\epsilon}' \not{k}' \not{\epsilon}^* - 2 \not{\epsilon}^* p\cdot\epsilon'}{p\cdot k'} \right] (\not{p}'+m) \right\}$$

in which some of the traces involve 8 γ's!
At least we know that the trace of an odd number of γ's vanishes.

We now go to Coulomb's gauge in the laboratory frame in which $p = (m, \vec{0})$. Here we have

$$\epsilon\cdot p = \epsilon^*\cdot p = \epsilon'\cdot p = \epsilon'^*\cdot p = 0.$$

So

$$\frac{1}{2}\sum_{ss'}|M|^2 = \frac{e^4}{4}\,\mathrm{tr}\left\{(\not{p}'+m)\left[\frac{\not{\epsilon}'^{*}\not{k}\not{\epsilon}}{p\cdot k} + \frac{\not{\epsilon}\not{k}'\not{\epsilon}'^{*}}{p\cdot k'}\right](\not{p}+m)\left[\frac{\not{\epsilon}^{*}\not{k}\not{\epsilon}'}{p\cdot k} + \frac{\not{\epsilon}'\not{k}'\not{\epsilon}^{*}}{p\cdot k'}\right]\right\}.$$

We now choose to use real ϵ's and we drop terms with $2n+1$ γ's :

$$\frac{1}{2}\sum_{ss'}|M|^2 = \frac{e^4}{4}\left[\frac{T_1}{(p\cdot k)^2} + \frac{T_2}{p\cdot k\; p\cdot k'} + \frac{T_3}{p\cdot k\; p\cdot k'} + \frac{T_4}{(p\cdot k')^2}\right.$$

$$\left. + \, m^2\left(\frac{t_1}{(p\cdot k)^2} + \frac{t_2}{p\cdot k\; p\cdot k'} + \frac{t_3}{p\cdot k\, p\cdot k'} + \frac{t_4}{(p\cdot k')^2}\right)\right]$$

where

$$T_1 = \mathrm{tr}\left[\not{p}'\not{\epsilon}'\not{k}\not{\epsilon}\not{p}\not{\epsilon}\not{k}\not{\epsilon}'\right]$$

$$T_2 = \mathrm{tr}\left[\not{p}'\not{\epsilon}'\not{k}\not{\epsilon}\not{p}\not{\epsilon}'\not{k}'\not{\epsilon}\right]$$

$$T_3 = \mathrm{tr}\left[\not{p}'\not{\epsilon}\not{k}'\not{\epsilon}'\not{p}\not{\epsilon}\not{k}\not{\epsilon}'\right]$$

$$T_4 = \mathrm{tr}\left[\not{p}'\not{\epsilon}\not{k}'\not{\epsilon}'\not{p}\not{\epsilon}'\not{k}'\not{\epsilon}\right]$$

$$t_1 = \mathrm{tr}\left[\not{\epsilon}'\not{k}\not{\epsilon}\not{\epsilon}\not{k}\not{\epsilon}'\right], \qquad t_2 = \mathrm{tr}\left[\not{\epsilon}'\not{k}\not{\epsilon}\not{\epsilon}'\not{k}'\not{\epsilon}\right]$$

$$t_3 = \mathrm{tr}\left[\not{\epsilon}\not{k}'\not{\epsilon}'\not{\epsilon}\not{k}\not{\epsilon}'\right], \qquad t_4 = \mathrm{tr}\left[\not{\epsilon}\not{k}'\not{\epsilon}'\not{\epsilon}'\not{k}'\not{\epsilon}\right].$$

Inside T_1 we have since $p\cdot\epsilon = 0$ and $\epsilon_\mu \epsilon^\mu = -1$

$$\not\epsilon \not p \not\epsilon = -\not p \not\epsilon \not\epsilon = +\not p, \quad \text{so}$$

$$T_1 = \mathrm{tr}\,[\not p' \not\epsilon' \not k \not p \not k \not\epsilon'] . \qquad \text{Now } k^2 = 0, \text{ so}$$

$$\not k \not p \not k = -\not k \not k \not p + 2 \not k\, p\cdot k = 2 \not k\, p\cdot k \quad \text{so}$$

$$T_1 = 2\,\mathrm{tr}\,[\not p' \not\epsilon' \not k \not\epsilon']\, p\cdot k$$

$$= 2\, p\cdot k \left(p'\cdot\epsilon'\, k\cdot\epsilon' - p'\cdot k\, \epsilon'^2 + p'\cdot\epsilon'\, \epsilon'\cdot k \right) 4$$

$$= 8\, p\cdot k \left(2\, p'\cdot\epsilon'\, k\cdot\epsilon' + p'\cdot k \right)$$

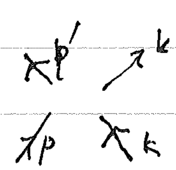
$$\epsilon'\cdot p' = \epsilon'\cdot(p + k - k') = \epsilon'\cdot k$$

$$k\cdot p' = -\tfrac{1}{2}(p' - k)^2 + \tfrac{1}{2} m^2 = -\tfrac{1}{2}(p - k')^2 + \tfrac{1}{2} m^2 = k'\cdot p$$

So

$$T_1 = 16\, p\cdot k\, (\epsilon'\cdot k)^2 + 8\, p\cdot k\, p\cdot k' .$$

P&S use Mandelstam's variables:



$$s = (p+k)^2 = 2\, p\cdot k + m^2 = 2\, p'\cdot k' + m^2$$

$$t = (p'-p)^2 = -2\, p\cdot p' + 2m^2 = -2\, k\cdot k'$$

$$u = (k'-p)^2 = -2\, k'\cdot p + m^2 = -2\, k\cdot p' + m^2 .$$

Note that

$$s+t+u = (p+k)^2 + (p'-p)^2 + (k'-p)^2$$

$$= p^2 + p'^2 + k^2 + k'^2 + \frac{1}{2}(p+k-p'-k')^2$$

$$= \sum m^2 = 2m^2 \quad \text{since } k^2 = k'^2 = 0.$$

The sum $s+t+u$ is always the sum of the squares of the four p^2 of any $2 \to 2$ process.

In the lab. frame the flux is $\frac{v}{V} = \frac{1}{V}$.

$$\frac{1}{2}\sum_{ss'} |S|^2 = (2\pi)^4 V T \, \delta(p'+k'-p-k) \frac{1}{2}\sum_{ss'} |M|^2$$

$$P_n = \frac{(2\pi)^4 \delta V T \frac{1}{2}\sum |M|^2}{(2EV\text{'s})}$$

$$R = \frac{(2\pi)^4 \delta V \frac{1}{2}\sum |M|^2}{(2EV\text{'s})}$$

$$d\sigma = \frac{R}{f} \frac{V d^3p'}{(2\pi)^3} \frac{V d^3k'}{(2\pi)^3} = \frac{1}{(2\pi)^2} \frac{1}{(2E\text{'s})} \frac{1}{2}\sum |M|^2 \delta \, d^3p' d^3k'$$

$$= \frac{1}{(2\pi)^2} \frac{1}{(2E\text{'s})} \frac{1}{2}\sum |M|^2 \delta(p^{0\prime} + k' - m - k) \, d^3k'$$

$$= \frac{1}{(2\pi)^2} \frac{1}{(2E\text{'s})} \frac{1}{2}\sum |M|^2 k'^2 \, d\Omega'$$

So we gets, summing and averaging over the electron spins

$$\frac{1}{2}\sum_{ss'} \frac{d\sigma}{d\Omega} = \left[\frac{\omega}{\omega'} + \frac{\omega'}{\omega} - 2 + 4(\epsilon\cdot\epsilon')^2 \right] \frac{e^4 \omega'^2}{64\pi^2 m^2 \omega^2}$$

as found by Klein and Nishina in 1929.

If we now average over the initial polarizations ϵ, then we get

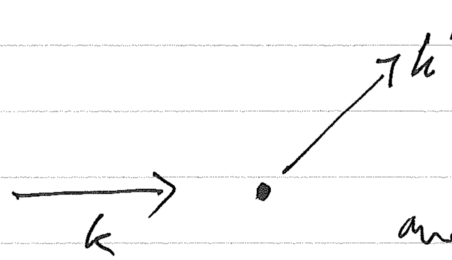
$$\frac{1}{4}\sum_{\epsilon s s'} \frac{d\sigma}{d\Omega} = \frac{e^4 \omega'^2}{64\pi^2 m^2 \omega^2} \left[\frac{\omega}{\omega'} + \frac{\omega'}{\omega} - 2(\hat{k}\cdot\epsilon')^2 \right]$$

because

$$\frac{1}{2}\sum_{\epsilon} 4\, \epsilon'\cdot\epsilon\, \epsilon\cdot\epsilon' = 2\, \epsilon'^i \left(\delta^{ij} - \hat{k}^i \hat{k}^j \right) \epsilon'^j$$

$$= 2 - 2\,(\epsilon'\cdot\hat{k})^2 .$$

The scattered photon is partially polarized in a direction $\perp$ to $\vec{k}$ as well as to $\vec{k}'$. So ϵ' tends to be $\perp$ to the $(\vec{k}, \vec{k}')$ plane of scattering.



$\epsilon'\cdot k' = 0$

and $\frac{d\sigma}{d\Omega}$ biggest when $\epsilon'\cdot\vec{k} = 0$ too.

If further we sum over ϵ', then

$$\frac{1}{4}\sum_{\epsilon\epsilon' s s'} \frac{d\sigma}{d\Omega} = \frac{e^4 \omega'^2}{32\pi^2 m^2 \omega^2}\left(\frac{\omega}{\omega'} + \frac{\omega'}{\omega} - 1 + \cos^2\theta\right).$$

because

$$-\frac{2}{64}\sum_{\epsilon'} \hat{k}\cdot\epsilon' = -\frac{1}{32}\hat{k}^i\left(\delta^{ij} - \hat{k}'^i\hat{k}'^j\right)\hat{k}^j$$

$$= -\frac{1}{32}\left(\hat{k}^{i2} - (\hat{k}^i\hat{k}'^i)^2\right)$$

$$= -\frac{1}{32}\left(1 - \cos^2\theta\right).$$

In the non-relativistic limit, when $\omega \ll m$, we have

$$\frac{1}{4}\sum \frac{d\sigma}{d\Omega} = \frac{e^4}{32\pi^2 m^2}\left(1 + \cos^2\theta\right)$$

since then $\omega' = \dfrac{\omega}{1 + \frac{\omega}{m}(1 - \cos\theta)} \to \omega$. (5.88)

$$\int (1 + \cos^2\theta)\, d\Omega = 2\pi \int_{-1}^{1}(1 + x^2)\, dx = 2\pi\left(2 + \frac{2}{3}\right) = \frac{16\pi}{3}$$

and $\sigma \to \sigma_T = \dfrac{e^4}{6\pi m^2} = \dfrac{8\pi r_0^2}{3}$; $r_0 = \dfrac{e^2}{4\pi m} = 3 \cdot 10^{-13}$ cm is classical radius of e.