

Quantum Hall Fluids

Consider a single spinless particle of charge e in a plane thru which a magnetic field passes perpendicularly

$$-[(\partial_x - ieA_x)^2 + (\partial_y - ieA_y)^2] \psi = 2mE \psi.$$

This is a problem in $(2+1)$ dimensions.

Landau solved this problem. The energy levels are

$$E_n = (n + \frac{1}{2}) \frac{eB}{m} \text{ for } n = 0, 1, 2, \dots$$

If A is the area, then the degeneracy is the same for each n :

$$d = \frac{eBA}{h} = \frac{eBA}{2\pi\hbar} = \frac{eBA}{2\pi}.$$

The jump in energy $E_{n+1} - E_n = \frac{eB}{m}$.

The filling factor ν is

$$\nu = \frac{N_e}{(eBA/2\pi)} \quad \text{where } N_e \text{ is the}$$

number of electrons. For $\nu < 1$, one can add e 's without needing to go to the next level.

But for integral values of ν , all the levels are full or empty. This is the integer

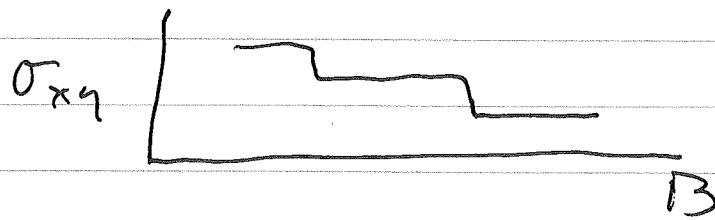
Hall effect: fluids with integral filling factors are incompressible.

An electric field E_y produces a current

$$J_x = \sigma_{xy} E_y$$

with $\sigma_{xy} = v e^2 / 2\pi = \frac{e^2 N_e 2\pi}{2\pi e B A} = \frac{e N_e}{B A}$.

But due to impurities



Fractional Hall Effect

Amazingly, a Hall fluid is also incompressible when

$$\nu = \frac{1}{2m+1} = \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \text{ etc.}$$

The number of flux quanta is

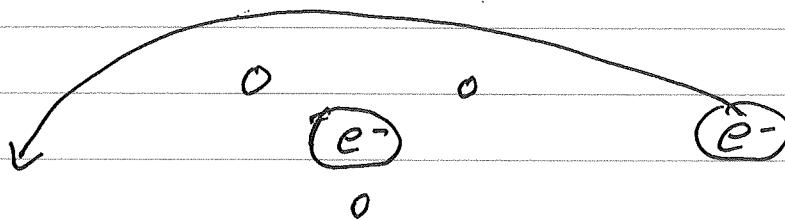
$$N_\phi = \frac{e B A}{2\pi}$$

Now $\frac{N_\phi}{N_e} = \frac{e B A}{2\pi e B A} \nu = \frac{1}{\nu}$.

So the quantum Hall fluid is incompressible when

$$\frac{N\phi}{Ne} = \frac{1}{V} = 2m+1.$$

That is, when there are $2m+1$ flux quanta for each electron.



$$\begin{aligned} \text{Phase change is } e^{i\pi + ig\Phi/2} &= (-1)e^{i\frac{g\Phi}{2}} \\ &= (-1)e^{i\frac{e(2m+1)\Phi_0}{2}} = (-1)e^{i\frac{e(2m+1)2\pi}{e}} \\ &= (-1)e^{i\frac{(2m+1)\pi}{2}} = +1. \end{aligned}$$

So the electrons become bosons and condense.

Kivelson, Hansson, and Zeeberg
med

$$L = \psi^+ i(\partial_0 - ieA_0) \psi + \frac{1}{2m} \psi^+ (\partial_i - ieA_i)^2 \psi$$

$$+ V(\psi^+ \psi). \quad (*)$$

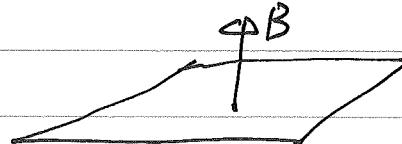
We saw that with a Chern-Simons term added to \mathcal{L}_0 so that

$$\mathcal{L} = \mathcal{L}_0 + \gamma \epsilon^{ijk} a_i \partial_j a_k + a_k j^k$$

we can change statistics by $\theta = 1/4\pi$.

The effective field theory of a Hall fluid — that is, the theory that works at large distances — is a Chern-Simons theory.

- 1) Electrons in a plane



form a $(2+1)$ -D system.

- 2) The electric current j_μ is conserved $\partial_\mu j^\mu = 0$
- 3) We want to use a local action density, a polynomial in fields and their derivatives at one point integrated over the point.
- 4) We are interested in big distances and long times — low \vec{k} and long w .
→
- 5) B breaks P and T.

The coupling constant g_{np}

of a term in d dimensions

$$\int d^d x g_{np} \partial^p \phi^n$$

wavers with L — the stretch — as

$$d - n(d-2)/2 = p$$

(See section 18.3 of my online notes.)

So what terms can we have in our long-range theory of a Hall fluid?

Well, $a^m a_m$ is ruled out by gauge invariance.

The Chern-Simons term $g_{2,1} \epsilon^{mnp} a_m \partial_n a_p$

goes as $d - n(d-2)/2 = p$

$$g_{2,1}(L) = L^{3 - 2(3-2)/2 - 1} g_{2,1}$$

$$= L^{3-1-1} g_{2,1} = L g_{2,1}$$

$$= L g_{2,1} = L g_{2,1}$$

so it gets strong as $L \rightarrow \infty$.

The Maxwell term is $g_{2,2}$ and

$$3 - 2(2-2)/2 - 2$$

$$g_{2,2}(L) = L g_{2,2} = g_{2,2}.$$

5. the Maxwell term is less important as $L \rightarrow \infty$ than the Chern-Simons term.

We now add to the C-S gauge field a_μ an electromagnetic field A_μ ; so \mathcal{L} is

$$\mathcal{L} = \frac{\hbar}{4\pi} \epsilon^{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda - \frac{1}{2\pi} \epsilon^{\mu\nu\lambda} a_\mu \partial_\nu A_\lambda$$

$$= \frac{\hbar}{4\pi} \epsilon^{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda - \frac{1}{2\pi} \epsilon^{\mu\nu\lambda} a_\mu \partial_\nu A_\lambda.$$

A_λ is not the one whose curl is \vec{B} ; that's already in \mathbf{k} . Note the integration by parts

$$\epsilon^{\mu\nu\lambda} A_\mu \partial_\nu a_\lambda \rightarrow -\epsilon^{\mu\nu\lambda} a_\lambda \partial_\nu A_\mu = -\epsilon^{\lambda\mu\nu} a_\lambda \partial_\nu A_\mu$$

$$= -\epsilon^{\mu\nu\lambda} a_\mu \partial_\nu A_\lambda = \epsilon^{\mu\nu\lambda} a_\mu \partial_\nu A_\lambda.$$

Adding in the current j^m of quasiparticles

$$\mathcal{L} = \frac{\hbar}{4\pi} \epsilon^{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda + a_m j^m - \frac{1}{2\pi} \epsilon^{\mu\nu\lambda} a_\mu \partial_\nu A_\lambda.$$

Define by \tilde{j}^m what couples to a_m :

$$\mathcal{L} = \frac{\hbar}{4\pi} \epsilon^{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda + a_m \tilde{j}^m \quad \text{with}$$

$$\tilde{j}^m = j^m - \frac{1}{2\pi} \epsilon^{\mu\nu\lambda} \partial_\nu A_\lambda.$$

If we integrate out the C-S a^μ , we get

$$\mathcal{L} = \frac{\pi}{h} j_\mu \frac{\epsilon^{\mu\nu\lambda}}{\partial^2} j_\lambda.$$

Here, as before, we're using $\int D\phi e^{-\frac{i}{2}\phi \cdot K \cdot \phi + J \cdot \phi} = e^{\frac{i}{2}J \cdot K^{-1} \cdot J}$

So \mathcal{L} now is

$$\mathcal{L} = \frac{\pi}{h} \left(j^\mu - \frac{1}{2\pi} \epsilon^{\mu\nu\lambda} \partial_\nu A_\lambda \right) \frac{\epsilon_{\mu\nu\lambda} \partial^\lambda}{\partial^2} \left(j^\rho - \frac{1}{2\pi} \epsilon^{\rho\sigma\tau} \partial_\sigma A_\tau \right)$$

We have jj , jA , and AA terms here.

The AA part is, since $\epsilon \partial \epsilon \partial \sim \partial^2$

$$A (\epsilon \partial \epsilon \partial \partial / \partial^2) A \sim A \epsilon \partial A$$

so

$$\mathcal{L}_{AA} = \frac{1}{4\pi h} A_\mu \epsilon^{\mu\nu\lambda} \partial_\nu A_\lambda$$

and so the electromagnetic current is

$$j_{EM}^\mu = \frac{1}{4\pi h} \epsilon^{\mu\nu\lambda} \partial_\nu A_\lambda.$$

QM 8

$$S_0 j_{EM}^0 = \frac{1}{4\pi h} e^{i\omega t} \partial_r A_2$$

$$= \frac{1}{4\pi h} (\partial_1 A_2 - \partial_2 A_1) = \frac{Bz}{4\pi k}.$$

A fluctuation: δB implies an excess of

$$S_m = \frac{1}{2\pi h} \delta B \quad \text{electrons.}$$

So the filling factor is $\nu = \frac{1}{k}$.

Next,

$$j_{EM}^i = \frac{1}{4\pi h} e^{i\omega t} \partial_r A_2$$

$$j_{EM}^1 = \frac{1}{4\pi h} e^{i\omega t} \partial_r A_1 = \frac{1}{4\pi h} (\partial_2 A_0 - \partial_0 A_2)$$

$$= \frac{E_2}{4\pi h} \quad \text{So} \quad \sigma_{12} = \frac{1}{k} = \nu.$$

The A_i term is $A(\epsilon \partial \epsilon \partial / \partial^2) j_i$, so

$$L_{A_i} = \frac{1}{k} A_i j^i$$

so the quasiparticle j^i carries charge $q = \frac{1}{k} = \nu$.

Finally, the j^m of the quasi-particles has action

$$L_{jj} = \frac{\pi}{\hbar} j^1 \frac{\epsilon_{mnj}}{\partial^2} j^2 j^3$$

So the quasi-particles obey "fractional" statistics with

$$\frac{\theta}{\pi} = \frac{1}{l_e} = v.$$

But why should v be an odd integer?

The action

$$L_{Aj} = \frac{1}{\hbar} A_m j^m$$

tells us that a composite of h quasi-particles has charge 1. Is this the hole? (the hole). If so, h must be an integer.

If one quasi-particle moves by π about another, the phase change is $\frac{\theta}{\pi} = \frac{1}{h}$.

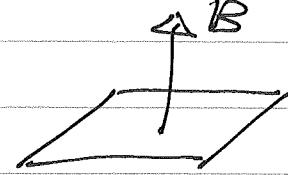
But if we move a composite of h q.p's about another composite of h q.p's, then

$$\frac{\theta}{\pi} = \frac{1}{h} h^2 = h \quad \text{i.e. } \frac{\theta}{\pi} = h.$$

So if the composite of n g.p.'s is to be a fermion, we need $\frac{q}{\pi} = 2n+1$.

$$\text{So } q = 2n+1 = \frac{1}{v}.$$

If Electrons in plane



have $v = \frac{1}{3}$, then each e is like 3 pieces of charge $\frac{1}{3}$ and fractional statistics $\frac{1}{3}$.

People call this "topological order" because the C-S theory depends upon the topology, not the metric, of the manifold.

The short-distance properties of a Hall fluid can depend upon other terms such as F^2 etc. and upon impurities.

An effective field theory like that of K-H-Z in eq.(*) says the quasiparticle is a vortex with electrons whirling around it. This fits Wilczek's models.

PRL 48 (1982) 1144.

Recall we dropped a surface term when we showed that the C-S action was gauge invariant. But real Hall fluids do have quite finite edges.

There are physical degrees of freedom on the edges whose action cancels the surface term we dropped.

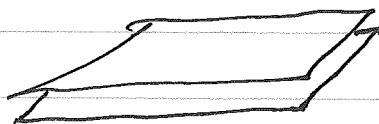
An incompressible fluid has edge excitations like waves on its boundary.

The current j^m has dimension L^{-2} so as to make $\partial_\mu j^m$ have dimension L^{-3} or m^3 . So $j^m j^m \sim (\text{mass})^4$ which is the same as the Maxwell term F^2 .

We can't make \mathcal{L} of m^3 out of j^m alone. That's why we need $\mathcal{I}(1/6\alpha)\mathcal{J}$, which is the Hopf term, which is nonlocal.

We use gauge fields to make the theory local.

Double-layered Hall systems



allow tunneling. $J_I^m = \frac{1}{2\pi} \epsilon^{mn} \partial_n A_I$.

$$\text{Now } \mathcal{L} = \sum_{I,J} \frac{K_{IJ}}{4\pi} \vec{a}_I^\dagger \vec{\epsilon}_{IJ} \vec{a}_J + \dots$$

If K has a zero eigenvalue, then the Maxwell term F^2 becomes important and a super fluid appears. Seen in experiments -

Some use a gas of C_S monopoles and C_S antimonopoles to describe the transfer of the electrons from layer I to layer J.