QED

The interaction hamiltonian is

$$H_I = \int e \bar{\psi} \gamma_\mu \psi A^\mu d^3x .$$

So the current is

$$j^\mu = \bar{\psi} \gamma^\mu \psi.$$

Is it conserved?

$$\partial_\mu j^\mu = \partial_\mu (\bar{\psi} \gamma^\mu \psi)$$

$$= \bar{\psi} \partial_\mu \gamma^\mu \psi + (\partial_\mu \bar{\psi} \gamma^\mu) \psi .$$

We've seen that ψ satisfies Dirac's equation

$$i \gamma^\mu \partial_\mu \psi = m \psi.$$

So

$$-i \partial_\mu \psi^\dagger \gamma^{\mu\dagger} = m \psi^\dagger .$$

But $\gamma^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$ and $\vec{\gamma} = \begin{pmatrix} 0 & \sigma \\ -\sigma & 0 \end{pmatrix}$

and $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$, and $\bar{\psi} = \psi^\dagger \gamma^0$.

So

$$\gamma^{\mu\dagger} \gamma^0 = \gamma^0 \gamma^\mu \quad \text{and so}$$

$$-i \partial_\mu \psi^\dagger \gamma^{\mu\dagger} \gamma^0 = -i \partial_\mu \psi^\dagger \gamma^0 \gamma^\mu = m \psi^\dagger \gamma^0 \quad \text{or}$$

$$-i \partial_\mu \bar{\psi} \gamma^\mu = m \bar{\psi} .$$

So

$$\partial_\mu j^\mu = \bar{\psi}\,\partial_\mu \gamma^\mu \psi + (\partial_\mu \bar{\psi})\gamma^\mu \psi$$

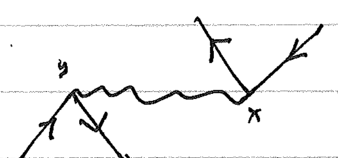
$$= -im\,\bar{\psi}\psi + im\,\bar{\psi}\psi = 0.$$

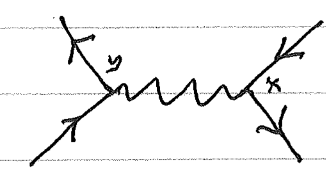
So the current $j^\mu = \bar{\psi}\gamma^\mu\psi$ is conserved.

So we can use the nice form of the photon propagator

(P) $$\langle 0|T(A^\mu(x)A^\nu(y))|0\rangle = \int \frac{d^4k}{(2\pi)^4}\,\frac{-i\eta^{\mu\nu}e^{-ik(x-y)}}{k^2+i\epsilon}.$$

Let's consider the process $e^+e^- \to e^+e^-$. The diagrams are



and


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This process is Bhabha scattering. The amplitude is

$$S = \langle p'q'|Te^{-i\int \mathcal{H}_I(x)\,d^4x}|pq\rangle$$

$$= \sqrt{2E's}\,\langle 0|b'a'\left(\frac{-i}{2}\right)^2\int T(\mathcal{H}_I(x)\,\mathcal{H}_I(y))\,a^\dagger b^\dagger|0\rangle\,d^4x\,d^4y.$$

Stop $a^\dagger$ at y, get z :

$$S = -e^2\sqrt{2E_s}\,\langle 0|b'a'\int T\left(\bar{\psi}(x)\gamma_\mu\psi(x)A^\mu(x)\bar{\psi}(y)\gamma_\nu\psi^{(+)}(y)A^\nu(y)\right) \times a^\dagger b^\dagger|0\rangle\, d^4x\, d^4y .$$

Recall

$$\psi_\alpha(x) = \int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\sum_s\left(a(p,s)u_\alpha(p,s)e^{-ipx} + b^\dagger(p,s)v_\alpha(p,s)e^{ipx}\right)$$

$$\bar{\psi}_\alpha(x) = \int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\sum_s\left(b(p,s)\bar{v}_\alpha(p,s)e^{-ipx} + a^\dagger(p,s)\bar{u}_\alpha(p,s)e^{ipx}\right).$$

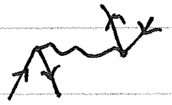
So

$$S = -\frac{e^2\sqrt{2E_s}}{\sqrt{2E_p}}\langle 0|b'a'\int T\left(\bar{\psi}(x)\gamma_\mu\psi^{(-)}(x)A^\mu(x)\bar{\psi}(y)\gamma_\nu A^\nu(y)\,u(p,s)\,b^\dagger|0\rangle d^4x\, d^4y\right) \times e^{-ipy}$$

since $b^{\dagger\prime}$ is made at x

$$S = -\frac{e^2\sqrt{2E_s}}{\sqrt{2E_p 2E_{q'}}}\langle 0|a'\int T\left(\bar{\psi}(x)\gamma_\mu v(q',t')e^{iq'x}A^\mu(x)\bar{\psi}(y)\gamma_\nu u(p,s)A^\nu(y)\right) \times e^{-ipy}\, b^\dagger|0\rangle\, d^4x\, d^4y$$

in which the T adds a minus sign when $y^0 > x^0$.

In  $b^\dagger$ is stopped at y, so

$$S_1 = -\frac{e^2\sqrt{2E_s}}{\sqrt{2E_p 2E_{q'} 2E_q}}\langle 0|a'\int T\left(\bar{\psi}(x)\gamma_\mu v' e^{iq'x}A^\mu(x)\, e^{-iqy}\,\bar{v}\,\gamma_\nu u\, A^\nu(y)\, e^{-ipy}\right)|0\rangle\, d^4x\, d^4y$$

in which the minus sign built into T cancels another minus sign.

So

$$S_1 = -e^2 \langle 0 | \int \bar{u}' \gamma_\mu v' e^{i(p'+q')x} T(A^\mu(x) A^\nu(y)) e^{-i(p+q)y} \bar{v} \gamma_\nu u | 0 \rangle d^4x\, d^4y$$

But now our formula for the photon propagator (P) gives

$$S_1 = -e^2 \int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik(x-y)} (-i\eta^{\mu\nu})}{k^2 + i\epsilon} \bar{u}' \gamma_\mu v' \bar{v} \gamma_\nu u\, e^{i(p'+q')x - i(p+q)y} d^4x\, d^4y$$

$$= +ie^2 \int \frac{d^4k}{(2\pi)^4} \bar{u}' \gamma_\mu v' \bar{v} \gamma^\mu \frac{u\, e^{iy(k-p-q) + ix(p'+q'-k)}}{k^2 + i\epsilon} d^4x\, d^4y$$

$$= +ie^2 \int d^4k\, \bar{u}' \gamma_\mu v' \bar{v} \gamma^\mu \frac{u\, e^{ix(p'+q'-k)}}{k^2 + i\epsilon} \delta^4(k-p-q)\, d^4x$$

$$= +ie^2 \int \bar{u}' \gamma_\mu v' \frac{\bar{v} \gamma^\mu u\, e^{ix(p'+q'-p-q)}}{(p+q)^2 + i\epsilon} d^4x$$

$$= \frac{+ie^2 (2\pi)^4 \delta^{(4)}(p'+q'-p-q)}{(p+q)^2} \bar{u}' \gamma_\mu v' \bar{v} \gamma^\mu u$$

In the other diagrams, we make b^{+}, at x, as in S_1, but we stop b^+ at x too.

$$S_2 = \frac{+e^2\sqrt{2E's}}{\sqrt{2E_p 2E_{q'} 2E_q}} \langle 0|a' \int T\left(\bar{v}\gamma_\mu v' e^{i(q'-q)x} A^\mu(x)\, \bar{\Psi}(y)\gamma_\nu u A^\nu(y) e^{-ipy}\right)|0\rangle d^4x\, d^4y$$

$$= +e^2 \langle 0| \int \bar{v}\gamma_\mu v' e^{i(q'-q)x}\, T(A^\mu(x) A^\nu(y))\, e^{iy(p'-p)}\, \bar{u}'\gamma_\nu u |0\rangle d^4x\, d^4y$$

$$= +e^2 \int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik(x-y)}(-i\eta^{\mu\nu})}{k^2+i\epsilon} \bar{v}\gamma_\mu v' \bar{u}'\gamma_\nu u\, e^{i(q'-q)x + iy(p'-p)}\, d^4x\, d^4y$$

$$= -ie^2 \int \frac{d^4k}{(2\pi)^4} \frac{\bar{v}\gamma_\mu v' \bar{u}'\gamma^\mu u}{k^2+i\epsilon} e^{iy(k+p'-p) + ix(q'-q-k)}\, d^4x\, d^4y$$

$$= -ie^2 \int \frac{\bar{v}\gamma_\mu v' \bar{u}'\gamma^\mu u}{k^2+i\epsilon}\, \delta^4(k+p'-p)\, e^{ix(q'-q-k)}\, d^4k\, d^4x$$

$$= -ie^2 \int \frac{\bar{v}\gamma_\mu v' \bar{u}'\gamma^\mu u}{(p-p')^2+i\epsilon}\, e^{ix(q'-q-p+p')}\, d^4x$$

$$= \frac{-ie^2 (2\pi)^4 \delta^4(p'+q'-p-q)}{(p-p')^2}\, \bar{v}\gamma_\mu v' \bar{u}'\gamma^\mu u .$$

So the total amplitude is

$$S = S_2 + S_1 = ie^2 (2\pi)^4 \delta^4(p'+q'-p-q)$$
$$\times \left[-\frac{\bar{v}\gamma_\mu v' \bar{u}'\gamma^\mu u}{(p-p')^2} + \frac{\bar{u}'\gamma_\mu v' \;\bar{v}\gamma^\mu u}{(p+q)^2} \right].$$