Chapter 11

Functionals

11.1 Functionals

A **functional** $G[f]$ is a map from a space of functions to a set of numbers. For instance, the **action** functional $S[q]$ for a particle in one dimension maps the coordinate $q(t)$, which is a function of the time t, into a number—the action of the process. If the particle of mass m is moving slowly and freely, then for the interval (t_1, t_2) its action is

$$S[q] = \int_{t_1}^{t_2} dt \, \frac{m}{2} \left(\frac{dq(t)}{dt} \right)^2 . \tag{11.1}$$

If the particle is moving in a potential $V(q(t))$, then its action is

$$S[q] = \int_{t_1}^{t_2} dt \left[\frac{m}{2} \left(\frac{dq(t)}{dt} \right)^2 - V(q(t)) \right] . \tag{11.2}$$

11.2 Functional Derivatives

A **functional derivative** is a functional

$$\delta G[f][h] = \frac{d}{d\epsilon} G[f + \epsilon h] \bigg|_{\epsilon=0} \tag{11.3}$$

of a functional. For instance, if $G[f]$ is the functional

$$G[f] = \int dx \, f^n(x) \tag{11.4}$$

then its functional derivative is the functional that maps the pair of functions f, h to the number

$$\begin{aligned} \delta G[f][h] &= \frac{d}{d\epsilon} G[f + \epsilon h]\bigg|_{\epsilon=0} \\ &= \frac{d}{d\epsilon} \int dx \, (f(x) + \epsilon h(x))^n \bigg|_{\epsilon=0} \\ &= \int dx \, n f^{n-1}(x) h(x). \end{aligned} \tag{11.5}$$

Physicists often use the less elaborate notation

$$\frac{\delta G[f]}{\delta f(y)} = \delta G[f][\delta(x-y)] \tag{11.6}$$

in which the second function $h(x)$ is $\delta(x-y)$. Thus, in the preceding example

$$\frac{\delta G[f]}{\delta f(y)} = \int dx \, n f^{n-1}(x) \delta(x-y) = n f^{n-1}(y). \tag{11.7}$$

Functional derivatives of functionals that involve powers of derivatives also are easily dealt with. Suppose that the functional involves the square of the derivative $f'(x)$

$$G[f] = \int dx \, (f'(x))^2 . \tag{11.8}$$

Then its functional derivative is

$$\begin{aligned} \delta G[f][h] &= \frac{d}{d\epsilon} G[f + \epsilon h]\bigg|_{\epsilon=0} \\ &= \frac{d}{d\epsilon} \int dx \, (f'(x) + \epsilon h'(x))^2 \bigg|_{\epsilon=0} \\ &= \int dx \, 2 f'(x) h'(x) = -2 \int dx \, f''(x) h(x) \end{aligned} \tag{11.9}$$

in which we have integrated by parts and used suitable boundary conditions to drop the surface terms. In physics notation, we have

$$\frac{\delta G[f]}{\delta f(y)} = -2 \int dx \, f''(x) \delta(x-y) = -2 f''(y). \tag{11.10}$$

Let's now compute the functional derivative of the action (11.2), which involves the square of the time-derivative $\dot{q}(t)$ and the potential energy $V(q(t))$

$$
\begin{aligned}
\delta S[q][h] &= \frac{d}{d\epsilon} S[q+\epsilon h]\bigg|_{\epsilon=0} \\
&= \frac{d}{d\epsilon}\int dt \left[\frac{m}{2}\left(\dot{q}(t)+\epsilon\dot{h}(t)\right)^2 - V(q(t)+\epsilon h(t))\right]\bigg|_{\epsilon=0} \\
&= \int dt \left[m\dot{q}(t)\dot{h}(t) - V'(q(t))h(t)\right] \\
&= \int dt \left[-m\ddot{q}(t) - V'(q(t))\right] h(t) \qquad (11.11)
\end{aligned}
$$

where we once again have integrated by parts and used suitable boundary conditions to drop the surface terms. In physics notation, this is

$$\frac{\delta S[q]}{\delta q(t')} = \int dt \left[-m\ddot{q}(t) - V'(q(t))\right]\delta(t-t') = -m\ddot{q}(t') - V'(q(t')). \qquad (11.12)$$

In these terms, the stationarity of the action $S[q]$ is the vanishing of its functional derivative

$$\frac{\delta S[q]}{\delta q(t)} = 0 \qquad (11.13)$$

which is Lagrange's equation of motion

$$m\ddot{q} = -V'(q). \qquad (11.14)$$

Here's a shortcut to the functional derivative in the notation of physics

$$\frac{\delta G[f]}{\delta f(y)} = \frac{d}{d\epsilon} G[f+\epsilon\delta_y]\bigg|_{\epsilon=0} \qquad (11.15)$$

in which the function $h(x)$ has been replaced by $\delta_y(x) = \delta(x-y)$.