\[ S_{\pi_i n_i \ldots p_i \sigma_i n_i \ldots} = \]
\[ \sum_{N=0}^{\infty} \frac{(-i)^N}{N!} \int d^4 x_1 \ldots d^4 x_N \left( \phi_1 \ldots \phi_N \right) a(x_1, \sigma_1, n_1) \ldots a(x_N, \sigma_N, n_N) \]
\[ \times \int \frac{d^4 p}{(2\pi)^4} \frac{d^4 q}{(2\pi)^4} \frac{d^4 r}{(2\pi)^4} \frac{d^4 s}{(2\pi)^4} \sigma_N a^+(p, \sigma, n) a^+(p', \sigma', n') \phi_0 \]

Here \[ \text{H}_N(x) = \sum_i g_i \text{H}_i(x) \] (normally ordered)

\[ \psi^\pm(x) = \frac{1}{\sqrt{2m}} \int \frac{d^3 p}{(2\pi)^3} \left[ \psi_0(p, \sigma, n) e^{ipx} + \psi_0(p, \sigma, n) e^{-ipx} \right] \]

\[ p^0 = \sqrt{p^2 + m^2} \]

Scalar fields \[ \psi(p, \sigma, n) = \frac{1}{\sqrt{2m}} \], spin or fields (5.35-36)

\[ \psi^+ \] will be called a "field adjoint."

We now move the \( a^\dagger \)'s to the right and the \( a \)'s to the left. So apart from the (anti)commutator relations

\[ a(x, \sigma, n) a^+(p', \sigma', n') = \pm a^+(p', \sigma', n') a(x, \sigma, n) + \delta_{(p', \sigma', n')} \delta_{(x, \sigma, n)} \]

we get no scattering.

We must also keep in mind that sign changes

\[ a(x, \sigma, n) a(p', \sigma', n') = \pm a(p', \sigma', n') a(x, \sigma, n) \]
\[ a^+(p, \sigma, n) a^+(p', \sigma', n') = \pm a^+(p', \sigma', n') a^+(p, \sigma, n) \]

We get zero because \[ a(p, \sigma, n) \phi_0 = \phi_0 a^+(p, \sigma, n) = 0 \].
So we are just left with bunches of (anti)commutators of pairs of \( a \) and \( a^\dagger \)'s.

\[ a(p', o', m') \text{ with } a^\dagger(p, o, m) \]

in \( \Psi_e^+(x) \) in \( \mathcal{H}_e^+(x) \) yields

\[
[a(p', o', m'), \Psi_e^+(x)]^+ = \frac{x_e(p', o', m') e^{i p' x}}{(2\pi)^{3/2}}
\]

\[ \Psi_e(x) \text{ in } \mathcal{H}_e(x) \text{ gives } \]

\[
[a(p', o', m'), \Psi_e(x)]^+ = \frac{\sqrt{2} |p'(o', m')| e^{i p' x}}{(2\pi)^{3/2}}
\]

\[ a^\dagger(p, o, m) \text{ with } a(p', o', m') \]

in \( \Psi_e(x) \) in \( \mathcal{H}_e(x) \) gives

\[
[a^\dagger(p, o, m), \Psi_e(x)]^+ = \frac{\sqrt{2} e(p, o, m') e^{i p x}}{(2\pi)^{3/2}}
\]
d) \( \text{initial } a^+(pom^c) \text{ with } a(p\tilde{\omega}p^c) \)

\[
\begin{aligned}
\Psi^+ \left( x \right) \text{ in } \mathcal{H} \left( x \right) \\
\left[ \Psi_k^+ \left( x \right), a^+(pom^c) \right]_+ = \frac{\Psi_k \left( pom^c \right) e^{ipx}}{(2\pi)^{3/2}}
\end{aligned}
\]

\[
\begin{aligned}
pom^c
\end{aligned}
\]

e) \text{Find } a(p\tilde{\omega}p^c) \text{ or } a(p\tilde{\omega}p^c) \\
\text{with initial } a^+(pom) \text{ or } a^+(pom^c) \\
gives \\
\left[ a \left( p'\omega'p' \right), a^+ \left( pom \right) \right]_+ = \delta \left( p' \cdot p \right) \delta_{\omega'\omega} \delta \left( p' \right)
\]
or
\[
\begin{aligned}
pom
\end{aligned}
\]

\[
\begin{aligned}
\left[ a \left( p'\omega'p' \right), a^+ \left( pom^c \right) \right]_+ = \delta \left( p' \cdot p \right) \delta_{\omega'\omega} \delta \left( p' \right)
\end{aligned}
\]
f) Pairing of field $\psi_0(x)$
in $\mathcal{H}_1(x)$ with field adjoint $\psi_0^+$
in $\mathcal{H}_2(y)$ gives

$$\Theta(x^0; y^0) \left[ \psi_0^{(+)}(x), \psi_0^+(y) \right]_T$$

$$\pm \Theta(y^0; x^0) \left[ \psi_m^{(-)}(y), \psi_e^{(-)}(x) \right]_T$$

$$\equiv -i \Delta_{gm}(x, y)$$

Here $\Theta(\alpha) = \begin{cases} 1 & \alpha > 0 \\ 0 & \alpha < 0 \end{cases}$

If $x^0 > y^0$, then

$$\mathcal{H}(x) \cdot \mathcal{H}(y) \sim \psi_e(x) \cdot \psi_m(y) \sim \psi_e^{(+)}(x) \cdot \psi_m^{(+)}(y)$$
because $\psi_e(x), \psi_m(y)$ already is normally ordered.

If $y^0 > x^0$, then

$$\mathcal{H}(y) \cdot \mathcal{H}(x) \sim \psi_m^{(+)}(y) \cdot \psi_e^{(+)}(x)$$
because $\psi_m^{(+)}(y) \cdot \psi_e^{(+)}(x)$ is normally ordered.

We'll get to the extra plus-minus sign later. See later.

$-i \Delta_{gm}(x, y)$ is the propagator.

Arrows point the way the particle moves through time and opposite to how the anti-particle goes.
 Rules:

(i) Draw all Feynman diagrams containing $N_i_i$ vertices of each type $i$ with the required incoming and outgoing lines for the initial and final particles and with any number of internal lines as long as each vertex has its proper number of lines.

Each vertex is $\circ_i \times$

Each line is labelled at its vertex $\frac{i}{\times}$

Each external line is labelled also where it enters or leaves the diagram.

(ii) Each vertex gives $-i$ from the $T(e^{-i \int d x H(x)})$

Each line gives its factor $(a) - (f)$.

(iii) Integrate over $x_1, x_2,$

(iv) Add up results
Note that the \( \frac{1}{N!} \) is cancelled because there are \( N! \) ways of labelling the vertices \( x_1, x_2, \ldots, x_N \). Each Feynman diagram represents \( N! \) identical diagrams.

Some diagrams do present additional combinatoric factors, as we'll see.

v) Say \( q_{\phi_i}(x) = \phi(x)^M \). Say each \( \phi(x) \) is paired with \( \phi(x) \) in \( M \) different vertices or with initial or final particles. There is an extra \( M! \).

E.g. if \( q_{\phi_i}(x) = \phi(x)^4 \), then

\[
\begin{array}{c}
\phi \\
\phi^4
\end{array}
\]

There are 4 \( \phi \)'s that can absorb \( p_1 \), then 3 \( \phi \)'s that can absorb \( p_2 \), then 2 that can generate \( p' \). So there is a factor of \( 4! \) here.

For this reason one often writes

\[
q_{\phi_i}(x) = \frac{1}{4!} \phi(x)^4.
\]
Similarly \( N_1(x) = (\phi^+(x))^2(\phi(x))^2 \)

would generate an extra factor \( 1 \cdot (2)! = 4 \).

So one might write \( N_1(x) = \frac{\phi^+ x^2}{4} \phi(x) \).

But \( N_1 = 4 \cdot \phi^3 \) and \( N_2 = \frac{\phi^+ x^3}{3!} \),

in the diagram

there would only be an extra \( 3! \) generated

because the \( \phi^3 \)'s in \( N_1 \) and the \( \phi^+ \)'s in \( N_2 \)
don't carry labels like \( p_1 \) or \( p_2 \) in the

previous example. So the factors are

\[
\frac{3!}{3!3!} = \frac{1}{3!} = \frac{1}{6}.
\]
Another common exception is \( q^4 = n^4 + 4 \).

In this case there are 3 equivalent diagrams

\[
\begin{align*}
1 & \quad 2 & \quad 3 \\
3 & \quad 1 & \quad 2 \\
\end{align*}
\]

and 3 more

\[
\begin{align*}
1 & \quad 2 & \quad 3 \\
2 & \quad 3 & \quad 1 \\
\end{align*}
\]

So in fact the factors are not

\[
\frac{3!}{3!} = 1 \quad \text{but} \quad \frac{2!}{3!} = \frac{1}{3}.
\]

In general instead of

\[
\frac{N!}{N!} \quad \text{one has} \quad \frac{(N-1)!}{N!} = \frac{1}{N}
\]

for the loop with \( N \) vertices.

And one has

\[
\frac{5!}{6!} = \frac{1}{6}
\]
A simple example: For a Hermitian scalar field $\phi = \phi^\dagger$ with 

$$\mathcal{H} = \frac{g}{\sqrt{3!}} \phi^3$$

consider $1 \pm 2 \rightarrow 1' \pm 2'$

$$\langle 1'2'1\rangle \Psi \rightarrow \langle 1'2'1\rangle$$

$$= \langle 1'2'1 \rangle - \frac{i}{2} \frac{g^2}{\sqrt{3!}} \int d^4x \phi^3(x) \phi^3(y) \int \frac{d^4p}{(2\pi)^3} \frac{e^{ipx}}{2\pi} \left( a(p) + a^\dagger(p) \right)$$

no scattering does not contribute so

$$S(1'2',12) = -\frac{g^2}{2!} \langle 1'2'1 \rangle a(1') \sum \frac{d^4p}{(2\pi)^3} \phi^3(x) \phi^3(y) a(1) a^\dagger(1) a^\dagger(1) a(1)$$

Now $\phi$ can be absorbed at $x$ or $y$. Pick $x$ and cancel the $2!$. The connected diagrams are

\[
\begin{align*}
\phi(x) &= \int \frac{d^3p}{(2\pi)^3} e^{ipx} \left( a(p) + a^\dagger(p) \right) \\
&= \sum \phi(x) \in [\phi(x) a(1)] \\
&= \phi(x) \in [\phi(x) a^\dagger(2)] \\
&= \phi(x) \in [a(1') \phi(y)] \\
&= 2 \phi(y) \in [a(1') \phi(y)]
\end{align*}
\]