

# Spontaneous Emission

Let's compute the lifetime of the  $2p$  state of atomic hydrogen. The initial state is

$$|i\rangle = |2, 1, m\rangle \quad 1$$

and the final state is the ground state  $1s$

$$|f, k\rangle = |1, 0, 0, k, n\rangle \quad 2$$

and a single photon of wave number  $k$  and polarization  $r$ .

As before

$$\langle f, k, r | i, t \rangle_E = -\frac{i}{\hbar} \int_0^t \langle 100, k, n | e^{-iH_0 t'/\hbar} \left( -\frac{e}{mc} \mathbf{p} \cdot \mathbf{A}(\mathbf{x}, 0) \right) e^{iH_0 t'/\hbar} | 2, 1, m \rangle dt' \quad 3$$

in which

$$H_0 = H_0^{AT} + H_0^{EM} \quad 4$$

The field  $\mathbf{A}(\mathbf{x}, 0)$  is

$$\vec{A}(\mathbf{x}, 0) = \sum_{\mathbf{k}, r} \left( \frac{\hbar c^2}{V \omega_k} \right)^{1/2} \left[ e_{\mathbf{r}}(\mathbf{k}) a_{\mathbf{r}}(\mathbf{k}) e^{i\mathbf{k} \cdot \mathbf{x}} + e_{\mathbf{r}}^*(\mathbf{k}) a_{\mathbf{r}}^\dagger(\mathbf{k}) e^{-i\mathbf{k} \cdot \mathbf{x}} \right] \quad (5)$$

$$S_0 \langle f, k, r | i, t \rangle_E = \frac{i e}{\hbar m c} \int_0^t e^{i(E_f + \hbar \omega - E_i) t'/\hbar} \langle 100, k, n | \vec{p} \cdot \vec{e}_{\mathbf{r}}^* a_{\mathbf{r}}^\dagger(\mathbf{k}) | 2, 1, m \rangle dt' \left( \frac{\hbar c^2}{V \omega} \right)^{1/2} \quad (6)$$

Now

$$\langle k' | a_{k'}^\dagger(k) | 0 \rangle = \sqrt{0+1} \delta_{k'k} \delta_{k_3 k'_3} \quad (7)$$

so

$$\langle f | K | i, t \rangle_I = \frac{e}{\hbar mc} \left( \frac{\hbar c^2}{V \omega} \right)^{\frac{1}{2}} \langle 100 | p \cdot e^* e^{-i\vec{k} \cdot \vec{x}} | 21m \rangle \times \left( \frac{e^{i(\omega_2 + \omega)t} - 1}{\omega_2 + \omega} \right)$$

So the transition rate is

$$\Lambda_{i \rightarrow f} = \frac{2\pi}{\hbar} \left( \frac{e}{mc} \right)^2 \frac{\hbar c^2}{V \omega} |\langle 100 | p \cdot e^* e^{-i\vec{k} \cdot \vec{x}} | 21m \rangle|^2 \delta(E_1 - E_2 + \hbar\omega) \quad (9)$$

Now since

$$|k \cdot x| \sim \frac{2\pi a_0}{\lambda} \approx \frac{6 \cdot \frac{1}{20}}{300} \approx 10^{-3} \quad (10)$$

the dipole approximation is excellent

$$\langle 100 | p \cdot e^* e^{-i\vec{k} \cdot \vec{x}} | 21m \rangle \approx \langle 100 | p \cdot e^* | 21m \rangle$$

And as before

$$\begin{aligned} \langle 100 | \vec{p} | 21m \rangle &= -\frac{m}{i\hbar} \langle 100 | [\vec{x}, H_0^{AT}] | 21m \rangle \\ &= i m \omega_{12} \langle 100 | \vec{x} | 21m \rangle. \end{aligned}$$

Since the ground state  $|100\rangle$  is spherically symmetric, we may choose any value of  $m$ . Let's take  $m=0$ . Then only the  $z$  component of  $\vec{x}$  survives.

$$\langle 100 | x_i | 210 \rangle = \delta_{i3} \langle 100 | z | 210 \rangle. \quad 13$$

This matrix element is worked out in special problem 8.2:

$$\langle 100 | z | 210 \rangle = \frac{2^7 \sqrt{2}}{3^5} a_0 \quad 14$$

where  $a_0 = \hbar^2 / mc^2$  is the Bohr radius, about half an Angstrom.

So

$$\hat{W}_{1 \rightarrow 2} = \frac{2\pi}{\hbar} \left( \frac{e}{mc} \right)^2 \frac{hc^2}{\omega V} |m\omega_{12} e^{i\vec{k}\cdot\vec{r}}|_3 \langle 100 | z | 210 \rangle^2 \times \delta(\hbar\omega_{12} - \hbar\omega) \quad 15$$

$$= \frac{2\pi}{\hbar} \left( \frac{e}{mc} \right)^2 \frac{hc^2}{\omega V} m^2 \omega_{12}^2 \frac{2^{15}}{3^{10}} a_0^2 |e(\vec{k}, \omega)|_3^2 \delta(\hbar(\omega_{12} - \omega)) \quad 16$$

$$= 4\pi^2 \frac{e^2}{\omega V} \omega \frac{2^{15}}{3^{10}} a_0^2 |e(\vec{k}, \omega)|_3^2 \delta(\hbar(\omega_{12} - \omega)). \quad 17$$

Now the polarization vectors  $e(k, \nu)$  for  $\nu=1, 2$  span the subspace perpendicular to  $\hat{k}$   
 $S_0$

$$\sum_{\nu=1}^2 e(k, \nu) e^\dagger(k, \nu) = 1 - \hat{k} \hat{k}^T \quad 18$$

and so

$$\sum_{\nu=1}^2 |e(k, \nu)_3|^2 = \left( \sum_{\nu=1}^2 e(k, \nu) e^\dagger(k, \nu) \right)_{33} \quad 19$$

$$= \delta_{33} - \hat{k}_3 \hat{k}_3 = 1 - \hat{k}_3^2 \quad 20$$

So if we use coordinates in which  $\hat{k}$  has polar angle  $\theta$  and  $\phi$ , then

$$\sum_{\nu=1}^2 |e(k, \nu)_3|^2 = 1 - \cos^2 \theta. \quad 21$$

Now we must integrate over final states, which amounts to integrating over final photon momenta and summing over photon polarization:

$$\vec{k} = \frac{2\pi}{L} \vec{n} \quad 22$$

$$\vec{n} = \frac{L}{2\pi} \vec{k} \quad 23$$

$$\sum_{\vec{n}} = \int \left( \frac{L}{2\pi} \right)^3 d^3 k = V \int \frac{d^3 k}{(2\pi)^3} \quad 24$$

The total rate  $w$  then is

$$w = \frac{2^{17}}{3^{10}} \pi^2 e^2 \left( \frac{d^2 q_0}{(2\pi)^3} \omega \right) \sum_{r=1}^2 |e(k, \omega)_r|^2 \delta(\hbar(\omega_2 - \omega)) \quad 25$$

$$= \pi^2 \frac{2^{17}}{3^{10}} e^2 a_0^2 2\pi \int_{-1}^1 \frac{d \cos \theta}{(2\pi)^3} \int_0^\infty d\omega (1 - \cos^2 \theta) \hbar^2 \delta(\hbar c(\omega_2 - \hbar)) \quad 26$$

$$= \pi^3 \frac{2^{18}}{3^{10}} \frac{e^2 a_0^2}{(2\pi)^3} \int_{-1}^1 dx (1 - x^2) \frac{\omega_{21}}{\hbar c} \left( \frac{\omega_{21}}{c} \right)^2 \quad 27$$

$$= \frac{2^{19}}{3^{10}} e^2 a_0^2 \left[ x - \frac{x^3}{3} \right]_{-1}^1 \frac{\omega_{21}^3}{\hbar c^3} \quad 28$$

$$= \frac{2^{19}}{3^{10}} e^2 a_0^2 \left( 2 - \frac{2}{3} \right) \frac{\omega_{21}^3}{\hbar c^3} \quad 29$$

$$= \frac{2^{17}}{3^{11}} e^2 a_0^2 \frac{\omega_{21}^3}{\hbar c^3} \quad 30$$

Now  $\hbar \omega_{21} = \frac{1}{2} m c^2 \alpha^2 \left( \frac{1}{1^2} - \frac{1}{2^2} \right)$

$$= \frac{3}{8} m c^2 \alpha^2 \quad 31$$

So twice  $a_0 = \hbar^2 / m e^2$ ,

$$W = \frac{2^{17}}{3^{11}} \left(\frac{3}{8}\right)^3 \left(\frac{m c^2 a^2}{\hbar}\right)^3 \frac{e^2 a_0^2}{\hbar c^3} \quad 32$$

$$= \frac{2^8}{3^8} \left(\frac{m c^2 a^2}{\hbar}\right)^3 \frac{e^2}{\hbar c^3} \left(\frac{\hbar^2}{m e^2}\right)^2 \quad 33$$

$$= \left(\frac{2}{3}\right)^8 \frac{m c^3 a^6}{\hbar^4} \frac{e^2 \hbar^4}{e^4} = \left(\frac{2}{3}\right)^8 m a^6 \frac{c^3}{e^2} \quad 34$$

$$= \left(\frac{2}{3}\right)^8 \frac{m c^2 a^6}{\hbar} \frac{\hbar c}{e^2} = \left(\frac{2}{3}\right)^8 \frac{m c^2}{\hbar} a^5 \quad 35$$

The lifetime  $\tau$  is  $W^{-1}$

$$\tau = \frac{1}{W} = \left(\frac{3}{2}\right)^8 \frac{\hbar}{m c^2} a^{-5} \quad 36$$

Numerically

$$\left. \begin{aligned} m c^2 &= 0.511 \text{ MeV} \\ \hbar &= 6.582 \times 10^{-22} \text{ MeVs} \\ \alpha &= 1/137.036 \end{aligned} \right\} 37$$

so

$$\tau = \frac{25.63 \cdot 6.582 \times 10^{-22} \text{ MeVs}}{0.511 \text{ MeV}} (137.036)^5 = 1.60 \times 10^{-9} \text{ s} \quad 38$$

is the life-time of the  $2p$  state of H.

The  $2p \rightarrow 1s$  transition is said to be an "allowed" transition because it proceeds thru the 1 part of

$$e^{-i\mathbf{k}\cdot\mathbf{x}} = 1 - i\mathbf{k}\cdot\mathbf{x} + \frac{(-i\mathbf{k}\cdot\mathbf{x})^2}{2!} + \dots \quad 39$$

But the  $2s \rightarrow 1s$  transition must go thru a higher term in this expansion. It is therefore suppressed by (at least) a factor of

$$\begin{aligned} |\mathbf{k}\cdot\mathbf{x}|^2 &\sim \left( \frac{\omega}{c} a_0 \right)^2 \\ &= \left( \frac{\hbar\omega}{\hbar c} \frac{\hbar^2}{m c^2} \right)^2 = \left( \frac{\hbar\omega}{m c^2} \frac{\hbar c}{e^2} \right)^2 \\ &= \left( \frac{13}{24} \frac{m c^2 \alpha^2}{m c^2} \frac{1}{\alpha} \right)^2 = \left( \frac{3}{8} \alpha \right)^2 \\ &= \left( \frac{3}{8 \cdot 137} \right)^2 = 7.5 \times 10^{-6} \end{aligned}$$

since  $a_0 = \hbar^2 / m e^2$  and  $\alpha = e^2 / \hbar c = 1/137$ .  
Transitions that proceed via this  $\mathbf{k}\cdot\mathbf{x}$  term are called "magnetic-dipole" or "electric-quadrupole" transitions, depending upon whether the anti-symmetric or symmetric part of  $\mathbf{e}\cdot\mathbf{p}\mathbf{k}\cdot\mathbf{x}$  is effective.