Identical Particles

As far as we know, all electrons are exactly the same, apart from the orientation of their spins. All helium-4 atoms in their ground states are the same. Similar remarks apply to other "elementary" particles, atoms, and nuclei.

The hamiltonian and other physical operators therefore must be symmetric in the labels of the particles. For example,

$$H = \frac{\vec{p}_1^{\,2}}{2m} + \frac{\vec{p}_2^{\,2}}{2m} + \frac{q^2}{4\pi\epsilon_0}\frac{1}{|\vec{r}_1 - \vec{r}_2|}.$$

H does not change when we interchange 1 and 2. Note that $m_1 = m_2 = m$ because the particles are <u>identical</u>.

Another example is for two spin-$\frac{1}{2}$ particles

$$H = g\,(S_1 + S_2)^2,$$

The $s=1$ state with $S_3=0$ is

$$\frac{1}{\sqrt{2}}\left(|+-\rangle + |-+\rangle \right) = |1,0\rangle$$

while the $S=0$, $S_3=0$ state $|0,0\rangle$ is

$$|00\rangle = \frac{1}{\sqrt{2}}\left(|+-\rangle - |-+\rangle \right).$$

So $\quad \frac{1}{\sqrt{2}}\left(|1,0\rangle + |00\rangle \right) = |+-\rangle$

and $\quad \frac{1}{\sqrt{2}}\left(|10\rangle - |00\rangle \right) = |-+\rangle.$

If $H = g\vec{S}^2$, what is the energy of the state

$|\psi\rangle = \alpha|+-\rangle + \beta|-+\rangle \qquad$ with $\quad |\alpha|^2 + |\beta|^2 = 1$?

We note that

$$|\psi\rangle = \frac{\alpha}{\sqrt{2}}\left(|10\rangle + |00\rangle \right) + \frac{\beta}{2}\left(|10\rangle - |00\rangle \right)$$

So

$$\langle\psi|H|\psi\rangle = \frac{1}{2}\left[\alpha^*\left(\langle 10| + \langle 00| \right) + \beta^*\left(\langle 10| - \langle 00| \right) \right]$$

$$g\vec{S}^2 \left[\alpha\left(|10\rangle + |00\rangle \right) + \beta\left(|10\rangle - |00\rangle \right) \right]$$

$$\langle\psi|H|\psi\rangle = \frac{g}{2}\left[\alpha^*\left(\langle 10| + \langle 00|\right) + \beta^*\left(\langle 10| - \langle 00|\right)\right]$$

$$2\hbar^2(\alpha+\beta)|10\rangle$$

$$= g\hbar^2\left[\alpha^*(\alpha+\beta) + \beta^*(\alpha+\beta)\right]$$

$$= g\hbar^2|\alpha+\beta|^2.$$

Enforcing normalization, we get

$$\langle\psi|H|\psi\rangle = g\hbar^2\frac{|\alpha+\beta|^2}{|\alpha|^2+|\beta|^2}.$$

The states

$$|\psi\rangle = \frac{\alpha|+-\rangle + \beta|-+\rangle}{|\alpha|^2+|\beta|^2}$$

differ in energy! These states are said to exhibit "<u>exchange</u> <u>degeneracy</u>."

Note that they are not degenerate.

What is the physical state $|\psi\rangle$?

The answer depends upon the spin of the elementary particle:

If the spin is $s = \frac{2n+1}{2}\hbar$

where n is an integer, then the particle is a <u>fermion</u> and the state $|\psi\rangle$ must be anti-symmetric under the interchange of the two labels.

If the spin is $s = n\hbar$ where n is an integer, then the state $|\psi\rangle$ must be <u>symmetric</u> under the interchange of the labels.

In our example, $s = \frac{\hbar}{2}$, the electron is a fermion, and so $\beta = -\alpha$ and

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|+-\rangle - |-+\rangle\right) = |00\rangle.$$

Among all the exchange-degenerate states, this is the one of lowest energy

$$\langle\psi|H|\psi\rangle = 0.$$

In the simple examples I have worked, the spin-statistics connection amounts to the following, more physical rule:

Use physical operations to symmetrize the state; the minus signs will appear automatically and correctly.

Thus as an example if we start with the state

$$|\psi_0\rangle = |+-\rangle$$

then a rotation by π about the x-axis will invert both spins

$$e^{-i\frac{\vec{S}\cdot\hat{x}\pi}{\hbar}} = e^{-i\frac{\hbar}{2}\sigma_x\frac{\pi}{\hbar}} = e^{-i\frac{\pi}{2}\sigma_x}$$

$$= \cos\frac{\pi}{2} + i\sigma_x \sin\left(-\frac{\pi}{2}\right) = -i\sigma_x$$

So

$$e^{-i(S_1+S_2)\cdot\hat{x}\frac{\pi}{\hbar}} = e^{-i\frac{\pi}{2}(\sigma_{x1}+\sigma_{x2})}$$

and so

$$e^{-i(S_1+S_2)\cdot\frac{\hat{x}\pi}{\hbar}} |+-\rangle$$

$$= e^{-i\frac{\pi}{2}\sigma_{x1}}|+\rangle \; e^{-i\frac{\pi}{2}\sigma_{x2}}|-\rangle$$

$$= (-i)^2 |-\rangle\,|+\rangle = -|-\rangle|+\rangle = -|-+\rangle.$$

So the physical state is

$$|\psi\rangle = |\psi_0\rangle + e^{-i\frac{\vec{S}\cdot\hat{x}\pi}{\hbar}}|\psi_0\rangle$$

$$= |+-\rangle - |-+\rangle$$

and after normalization

$$|\psi\rangle_{\text{physical}} = \frac{1}{\sqrt{2}}\left(|+-\rangle - |-+\rangle\right).$$